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New Lindley Half Cauchy Distribution: Theory and Applications

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***Abstract**: In this paper, we have defined a new two-parameter new Lindley half Cauchy (NLHC) distribution using Lindley-G family of distribution which accommodates increasing, decreasing and a variety of monotone failure rates. The statistical properties of the proposed distribution such as probability density function, cumulative distribution function, quantile, the measure of skewness and kurtosis are presented. We have briefly described the three well-known estimation methods namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods. All the computations are performed in R software. By using the maximum likelihood method, we have constructed the asymptotic confidence interval for the model parameters. We verify empirically the potentiality of the new distribution in modeling a real data set.*

***Keywords**: Estimation, Generalized Rayleigh (GR) distribution, Half-Cauchy distribution, Lindley distribution.*

I. INTRODUCTION

The one parameter Lindley (L) distribution was presented by (Lindley, 1958) [17] in the context of Bayesian statistics, as a counterexample to fiducial statistics. In recent years, many studies have been focused to obtain various modified forms of the baseline distribution using the Lindley-G family presented by Zografos and Balakrishnan (2009) [32] with more flexible density and hazard rate functions. A detailed study on the Lindley distribution was done by (Ghitany et al., 2008) [8]. A three-parameter generalized Lindley distribution was presented by (Zakerzadeh & Dolati, 2009) [30].

Consider a random variable X follows Lindley distribution with parameter θ and its probability density function (PDF) is given by

$$f(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\ x>0,\theta>0 \tag{1.1}$$

And its cumulative distribution function (CDF) is

$$F(x)=1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x};\ x>0,\theta>0 \tag{1.2}$$

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A two-parameter Lindley distribution was presented by Shanker & Mishra (2013) [27]. Gupta and Singh (2013) [11] investigated the estimation of the parameters using hybrid censored data. Ashour & Eltehiwy (2015)[2] has introduced the exponentiated power Lindley distribution. The estimation of the model parameters for censored samples by (Krishna and Kumar, 2011) [14], Zeghdoudi & Nedjar (2016) [31] has introduced the Gamma Lindley distribution and its application, Reyes et al. (2019) [24] has introduced the Slash Lindley-Weibull Distribution, and Hassan & Nassr (2019) [12] has created the Power Lindley-G family of distributions. Ieren et al. (2020) [13] has introduced the Odd Lindley-Rayleigh distribution and its properties and applications to simulated and real-life datasets. The half-Cauchy distribution is a derivative of the Cauchy distribution obtained by reflecting the curve on the origin so that only positive values can be observed. As a heavy-tailed distribution, the half-Cauchy distribution has been used as an alternate to model dispersal distances (Shaw, 1995) [28], since it can predict more common long-distance dispersal events. Additionally, Paradis et al. (2002) [21] utilised the half-Cauchy distribution to model ringing data on two species of tits in Britain and Ireland. For more information about half-Cauchy distribution, readers can go through (Okagbue et al. 2018) [20]. Let T be a random variable which follows the half-Cauchy then its cumulative distribution function (CDF) and probability density function (PDF) respectively are defined as,

$$G(t)=\frac{2}{\pi}\arctan\left(\frac{t}{\lambda}\right),\quad \lambda>0,\ t>0 \tag{1.3}$$

And

$$g(t)=\frac{2}{\pi}\left(\frac{\lambda}{\lambda^2+t^2}\right),\quad \lambda>0,\ t>0 \tag{1.4}$$

Last few decades there are several modifications of the half-Cauchy distribution have been made by many researchers as a baseline distribution. Cordeiro & Lemonte (2011) [5] has introduced the beta-half-Cauchy distribution, and Polson & Scott (2012) [22] have made an extensive study on the half-Cauchy prior for a global scale parameter. The Kumaraswamy-half-Cauchy distribution was introduced by (Ghosh, 2014) [10]. Alzaatreh et al. (2016)[1] has introduced the gamma half-Cauchy distribution. Cordeiro et al (2017) [6] has created the generalized odd half-Cauchy family of distributions. Hence we are motivated to introduce Lindley half-Cauchy distribution. The article is organized as follows. In Section 2, the proposed new Lindley half Cauchy distribution is derived and we obtain some properties of the NLHC distribution such as probability density function, reliability function, hazard rate function, quantile function, and skewness and kurtosis.

We illustrated the different methods for estimating the model parameters namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods In Section 3. The estimation of the model parameters for uncensored data is discussed and we have studied the potentiality of the proposed model by comparing it with some well-known distributions namely weighted Lindley, Chen, generalized Rayleigh and Lindley in Section 4.
Concluding remarks are presented in Section 5 we draw some conclusions about the data contained in the article.

II. THE NEW LINDLEY HALF-CAUCHY (NLHC) DISTRIBUTION

Ristic and Balakrishnan (2011) [25] has introduced another generalized family of distribution which is defined as

$$F(x) = 1 - \int_0^{-\ln[G(x)]} r(t)\,dt \qquad (2.1)$$

Using $r(t)$ as PDF of Lindley distribution (1.1) and the baseline distribution G(x) as CDF of half-Cauchy distribution (1.3) then the CDF of Lindley half-Cauchy II is obtained as,

$$F(x) = \left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\};\ \theta,\lambda>0,\ x>0 \qquad (2.2)$$

and its corresponding PDF can be expressed as,

$$f(x) = \frac{2}{\pi}\left(\frac{\theta^2}{1+\theta}\right)\left(\frac{\lambda}{\lambda^2+x^2}\right)\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta-1}\left\{1-\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\};\ \theta,\lambda>0,\ x>0 \qquad (2.3)$$

The survival function of new Lindley half Cauchy distribution is

$$S(x) = 1 - F(x)$$

$$= 1-\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\};\ \theta,\lambda>0,\ x>0 \qquad (2.4)$$

And its hazard rate function can be expressed as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\frac{2}{\pi}\left(\frac{\theta^2}{1+\theta}\right)\left(\frac{\lambda}{\lambda^2+x^2}\right)\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta-1}\left\{1-\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\}}{1-\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\}};\ \theta,\lambda>0,\ x>0 \qquad (2.5)$$

In Figure 1 we have displayed plots for the PDF and hazard function of new Lindley half Cauchy for several parameter values. Figure 1 shows that the PDF has various shapes while hazard function has very flexible shapes, such as decreasing, increasing, constant and inverted bathtub.

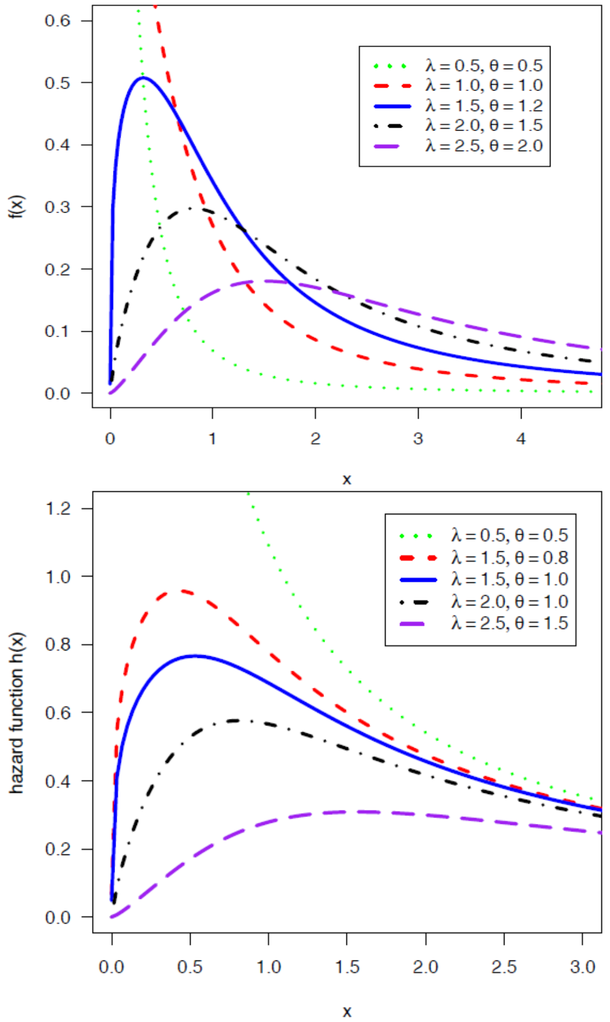

Figure 1. Graph of PDF (upper panel) and hazard function (lower panel) for different values of λ and θ.

Quantile function of LH-C distribution

The quantile function of NLHC (λ,θ) can be obtained as

$$Q(u) = F^{-1}(u)$$

Hence the quantile function can be written as,

$$u-1+\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\} = 0\ ;\ 0<u<1 \qquad (2.6)$$

Also the random numbers can be generated for the NLHC(λ, θ) distribution, for this let, simulating values of random variable X with the CDF (2.2). Let V represent a uniform random variable in (0, 1), then the simulated values of X are obtained by setting,

$$v-1+\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{\lambda}\right)\right]\right\} = 0\ ;\ 0<v<1 \qquad (2.7)$$

and solving for x.

Skewness and Kurtosis:

Skewness and Kurtosis are mostly used in data analysis to study the nature of the distribution or data set. Skewness and Kurtosis based on quantile function are

$$Skewness(B)=\frac{Q_3+Q_1-2Q_2}{Q_3-Q_1},$$

where Q_1, Q_2 and Q_3 is the first quartile, median and third quartile respectively. And the Coefficient of kurtosis based on octiles given by (Moors, 1988) [19] is

$$K_u(M)=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}$$

III. METHOD OF ESTIMATION

In statistics, the estimation theory is an important branch that involves estimating the values of unknown parameters founded on measured empirical data that contains a random component. The parameters define an underlying physical setting in such a way that their value influences the distribution of the measured data.

An estimator tries to approximate the unknown parameters by means of the measurements. In this study, we have employed three well-known estimation procedures namely the maximum likelihood (MLE), ordinary least squares (LSE), and Cramer-von Mises (CVM) estimators.

3.1. Maximum Likelihood Estimation (MLE):

The maximum likelihood method is the most commonly used method of parameter estimation (Casella & Berger, 1990) [3]. Let $\underline{x}$ = (x_1,…..,x_n) be a random sample of size 'n' from NLHC(λ, θ), then the likelihood function is defined as

$$L(\lambda,\theta;x)=\prod_{i=1}^{n} f(x;\lambda,\theta)$$

$$=\frac{2}{\pi}\left(\frac{\theta^2}{1+\theta}\right)\prod_{i=1}^{n}\left(\frac{\lambda}{\lambda^2+x_i^2}\right)\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right\}^{\theta-1}\left\{1-\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right]\right\};\ \theta,\lambda>0,\ x>0$$

The Likelihood density function is

$$l=2n\ln\theta-n\ln(1+\theta)+n\ln 2-n\ln\pi+n\ln\lambda-\sum_{i=1}^{n}\ln(\lambda^2+x_i^2)+(\theta-1)\sum_{i=1}^{n}\ln\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right\}$$
$$+\sum_{i=1}^{n}\ln\left\{1-\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right]\right\} \tag{3.1.1}$$

Differentiating the Likelihood density function (3.1.1) with respect to λ and θ we get,

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}-\frac{2n\lambda}{\lambda^2+x^2}-\sum_{i=1}^{n}\frac{\lambda}{(\lambda^2+x_i^n)\tan^{-1}\left(\frac{x_i}{\lambda}\right)}\left[(\theta-1)-\left\{1-\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right]\right\}^{-1}\right] \tag{3.1.2}$$

$$\frac{\partial l}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{1+\theta}+\sum_{i=1}^{n}\ln\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x_i}{\lambda}\right)\right\} \tag{3.1.3}$$

By equating (3.1.2) and (3.1.3) to zero and solving them for λ and θ we get ML estimates $\hat{\lambda}$ and $\hat{\theta}$ but manually we could not solve them, so one can use computer package like R Mathematica, etc. Let MLE of parameter space $\underline{\Phi}=(\lambda,\theta)$ as $\hat{\underline{\Phi}}=(\hat{\lambda},\hat{\theta})$ then the asymptotic multivariate normal $\left(\hat{\underline{\Phi}}-\underline{\Phi}\right)\rightarrow N_2\left[0,\left(I\left(\underline{\Phi}\right)\right)^{-1}\right]$ distribution can be utilised to construct approximate confidence interval with the level of significance α for each parameter λ and θ, where $I\left(\underline{\Phi}\right)$ is the Fisher's information matrix defined as

$$I\left(\underline{\Phi}\right)=-\begin{bmatrix} I_{11} & I_{12}\\ I_{21} & I_{22}\end{bmatrix}$$

Where $I_{11}=E\left(\frac{\partial^2 l}{\partial\lambda^2}\right)$ $I_{12}=E\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right)$,

$I_{21}=E\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right)$ and $I_{22}=E\left(\frac{\partial^2 l}{\partial\theta^2}\right)$

Let $O(\hat{\underline{\Phi}})$ be the observed Fisher information matrix which is used to estimate the information matrix $I\left(\underline{\Phi}\right)$ given by

$$O(\hat{\underline{\Phi}})=-\begin{pmatrix}\frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\theta}\\ \frac{\partial^2 l}{\partial\lambda\partial\theta} & \frac{\partial^2 l}{\partial\theta^2}\end{pmatrix}_{|(\hat{\lambda},\hat{\theta})}=-H\left(\underline{\Phi}\right)_{|(\underline{\Phi}=\hat{\underline{\Phi}})} \tag{3.1.4}$$

where H signifies the Hessian matrix.

By using the Newton-Raphson algorithm to maximize the likelihood and we can obtain the variance-covariance matrix as,

$$\left[-H\left(\underline{\Phi}\right)_{|(\underline{\Phi}=\hat{\underline{\Phi}})}\right]^{-1}=\begin{pmatrix}\mathrm{var}(\hat{\lambda}) & \mathrm{cov}(\hat{\lambda},\hat{\theta})\\ \mathrm{cov}(\hat{\lambda},\hat{\theta}) & \mathrm{var}(\hat{\theta})\end{pmatrix} \tag{3.1.5}$$

Hence, approximate 100(1-α) % confidence intervals for λ and θ can be constructed from the asymptotic normality of MLEs as,

$$\hat{\lambda}\pm Z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\lambda})}\ \text{ and }\ \hat{\theta}\pm Z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})}$$

where $Z_{\alpha/2}$ represents the upper percentile of standard normal variate.

3.2. Method of Least-Square Estimation (LSE)

Swain et al. (1988) [29] has introduced the ordinary least square estimators and weighted least square estimators to estimate the parameters of the Beta distribution. The least-square estimators of the unknown parameters λ and θ of NH-C II distribution can be calculated by minimizing,

$$\Psi\left(X;\lambda,\theta\right)=\sum_{j=1}^{n}\left[G(X_j)-\frac{j}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters λ and θ.

Let the ordered random variables be $X_{(1)}<X_{(2)}<\ldots\ldots<X_{(n)}$ $G(X_{(j)})$ and denotes the cumulative distribution function, where $\{X_1,X_2,\ \ldots\ldots,X_n\}$ denotes a random sample of size n from a CDF G(.). Thus, the least square estimators of λ and θ say $\hat{\lambda}$ and $\hat{\theta}$ respectively, can be attained by minimizing

$$\Psi(X;\lambda,\theta)=\sum_{j=1}^{n}\left[\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)\right]\right\}-\frac{j}{n+1}\right]^2;\ \theta,\lambda>0,\ x>0 \qquad (3.2.2)$$

with respect to λ and θ.

By solving the following two nonlinear equations equating to zero, we get the least square estimators as,

Let $V\left(x_j\right)=\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)$

$$\frac{\partial\Psi}{\partial\lambda}=2\sum_{j=1}^{n}\left[\left\{V\left(x_j\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[V\left(x_j\right)\right]\right\}-\frac{j}{n+1}\right]$$
$$\left\{\left[\frac{2\theta V\left(x_j\right)}{\pi}\frac{x_j}{\lambda^2+x_j^2}\right]\left[1-\left(\frac{\theta}{1+\theta}\right)\ln V\left(x_j\right)\right]+\left[V\left(x_j\right)\right]^{\theta}\left[\frac{2\theta}{\pi(1+\theta)V\left(x_j\right)}\frac{x_j}{\lambda^2+x_j^2}\right]\right\}$$

$$\frac{\partial\Psi}{\partial\theta}=2\sum_{j=1}^{n}\left[\left\{V\left(x_j\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[V\left(x_j\right)\right]\right\}-\frac{j}{n+1}\right]$$
$$\left\{\left\{V\left(x_j\right)\right\}^{\theta}\ln\left[V\left(x_j\right)\right]\left[1-\left(\frac{\theta}{1+\theta}\right)\ln V\left(x_j\right)\right]+\left[V\left(x_j\right)\right]^{\theta}\left[\frac{\ln V\left(x_j\right)}{(1+\theta)^2}\right]\right\}$$

3.3. Method of Cramer-Von-Mises (CVM)

Macdonald (1971) [18] has developed one of the important estimation method called Cramér-von-Mises type minimum distance estimators. It provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of λ and θ are attained by minimizing the function,

$$CVM\left(\lambda,\theta\right)=\frac{1}{12p}+\sum_{j=1}^{p}\left[F\left(x_{j:p}\,|\,\lambda,\theta\right)-\frac{2j-1}{2p}\right]^2 \qquad (3.3.1)$$

$$=\frac{1}{12p}+\sum_{j=1}^{p}\left[\left\{\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)\right]\right\}-\frac{2j-1}{2p}\right]^2$$

Equating to zero and solving the following two nonlinear equations simultaneously, we get the CVM estimators of λ and θ as,

Let $M\left(x_j\right)=\frac{2}{\pi}\tan^{-1}\left(\frac{x_j}{\lambda}\right)$

$$\frac{\partial C}{\partial\lambda}=\frac{4\theta}{\pi(1+\theta)}\sum_{j=1}^{n}\left[\left\{M\left(x_j\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[M\left(x_j\right)\right]\right\}-\frac{2j-1}{2p}\right]\left\{\frac{M\left(x_j\right)x_j}{\lambda^2+x_j^2}\right\}$$
$$\left\{\left[1+\theta-\theta\ln M\left(x_j\right)\right]+\left[M\left(x_j\right)\right]^{\theta-2}\right\}$$

$$\frac{\partial C}{\partial\theta}=\frac{2}{(1+\theta)^2}\sum_{j=1}^{n}\left[\left\{M\left(x_j\right)\right\}^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left[M\left(x_j\right)\right]\right\}-\frac{2j-1}{2p}\right]\left\{M\left(x_j\right)\right\}^{\theta}\ln\left[M\left(x_j\right)\right]$$
$$\left\{2+\theta-\theta\ln M\left(x_j\right)\right\}.$$

IV. APPLICATION TO A REAL DATASET

In this segment, we illustrate the usefulness of our purposed model using a well-known real data set. The data presented below represents time interval between failures (in thousands of hours) of secondary reactor pumps (Suprawhardana, et.al, 1999): [26]

0.062, 0.070, 0.101, 0.150, 0.199, 0.273, 0.347, 0.358, 0.402, 0.491, 0.605, 0.614, 0.746, 0.954, 1.060, 1.359, 1.921, 2.160, 3.465, 4.082, 4.992, 5.320, 6.560

In Figure 2 we have displayed the Contour plot and the fitted CDF with empirical distribution function (EDF) (Kumar & Ligges, 2011) [15].

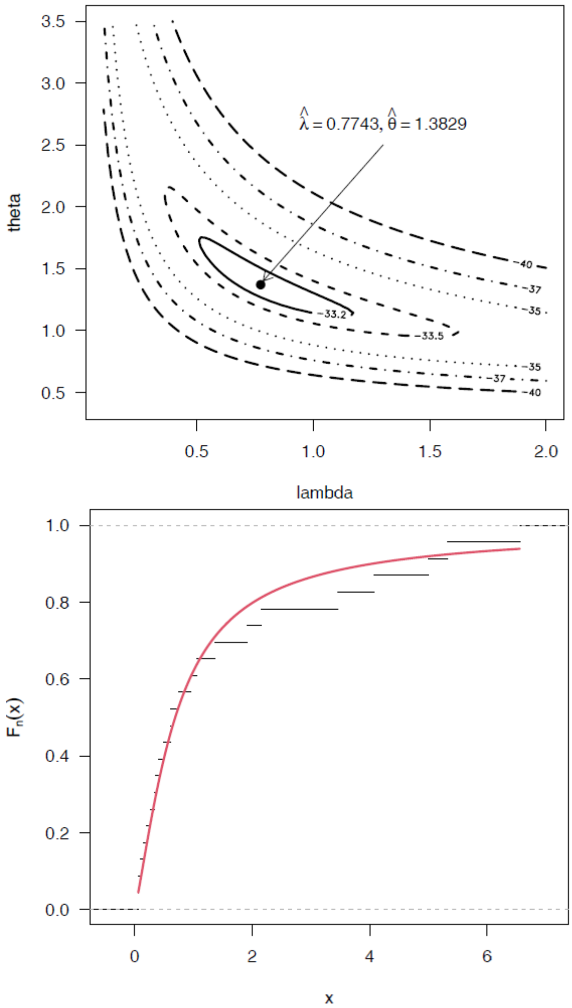

Figure 2. Contour plot (upper panel) and the fitted CDF with empirical distribution function (lower panel).

The MLEs are calculated by using the log-likelihood function (3.1.1) directly using optim() function (Dalgaard, 2008) [7] in R software (R Core Team, 2020) [23]. By using the maximum likelihood estimation method for the above data set, we have obtained $\hat{\lambda}$ = 0.7743 and $\hat{\theta}$ = 1.3829 and its corresponding Log-Likelihood value is -33.0591. In Table 1 we have presented the MLE's with their standard errors (SE) and 95% confidence interval for λ and θ.

Table 1

MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
Lambda	0.7743	0.3059	(0.1747, 1.9619)
Theta	1.3829	0.5899	(0.2267, 2.5391)

Hence the Hessian variance-covariance matrix is obtained as,

$$\begin{pmatrix} \operatorname{var}(\hat{\lambda}) & \operatorname{cov}(\hat{\lambda},\hat{\theta}) \\ \operatorname{cov}(\hat{\lambda},\hat{\theta}) & \operatorname{var}(\hat{\theta}) \end{pmatrix} = \begin{pmatrix} 0.3671 & -0.3307 \\ -0.3307 & 0.3480 \end{pmatrix}$$

In Figure 3, the Profile log-likelihood functions of parameters λ and θ are demonstrated. It can be concluded that the estimated parameters using the MLE method are unique.

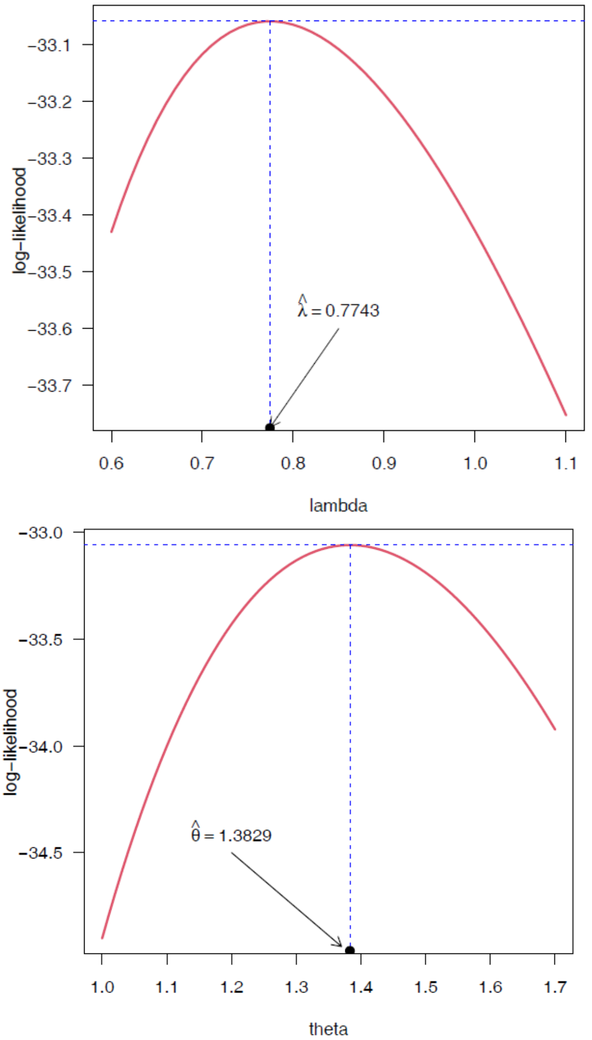

Figure 3. Plots of the Profile log-likelihood functions of the parameters λ and θ

For the comparison purpose we use negative log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), and Corrected Akaike Information criterion (AICC), which are used to select the best model among several models. The expressions to compute AIC, BIC, and AICC are given below:

I. $$AIC = -2l(\hat{\theta}) + 2d$$

II. $$BIC = -2l(\hat{\theta}) + d\log(n)$$

III. $$AICC = AIC + \frac{2d(d+1)}{n-d-1}$$

Where d denotes the number of parameters and n denotes the size of the sample in the model under consideration. The neg ative log-likelihood value and the value of AIC, BIC, and AI CC are displayed in Table 2. We conclude that the proposed model produces a better fit.

Table 2 Estimated parameters, log-likelihood, AIC, BIC and AICC

Method	$\hat{\alpha}$	$\hat{\theta}$	-LL	AIC	BIC	AICC
MLE	0.7743	1.383	33.0591	70.1181	72.3891	70.6636
LSE	1.5155	0.9351	33.54645	71.0929	73.3639	71.6383
CVE	1.1887	1.0645	33.2704	70.5408	72.8118	71.0862

In Figure 4 we have presented the P-P plot (empirical distribution function against theoretical distribution function) and Q-Q plot (empirical quantiles against theoretical quantiles).

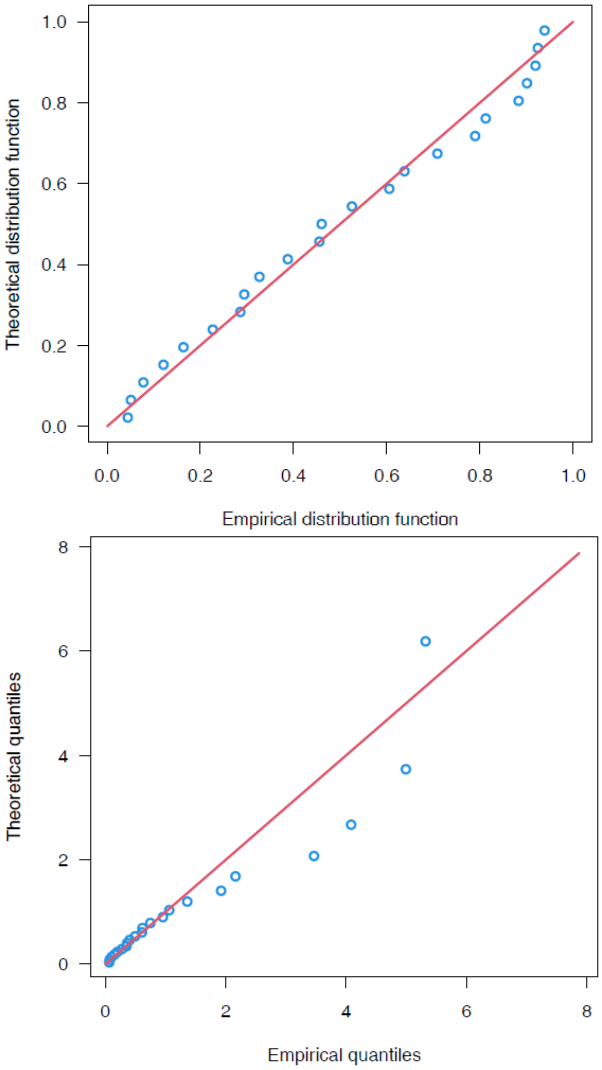

Figure 4. The graph of the P-P plot (upper panel) and Q-Q plot (lower panel)

To illustrate the goodness of fit of the new Lindley half Cauchy distribution, we have taken some well known distribution for comparison purpose which are listed blew,

I. Weighted Lindley distribution (W-Lindley):

The weighted Lindley distribution has presented by (Ghitany et al., 2011) [9] whose PDF is

$$f(t) = \frac{\theta^{\alpha+1}}{(\alpha+\theta)\Gamma(\alpha)} t^{\alpha-1}(1+t)e^{-\theta t} \quad ; t \geq 0, \alpha > 0, \theta > 0.$$

II. Chen distribution:

Chen (2000) [4] has introduced Chain distribution having probability density function (PDF) as

$$f(x;\lambda,\theta)=\lambda\beta x^{\theta-1}e^{x\theta}\exp\left\{\lambda\left(1-e^{x^{\theta}}\right)\right\} \quad ;(\lambda,\theta)>0, x>0$$

III. Generalized Rayleigh (GR) distribution

The Generalized Rayleigh (GR) distribution has presented by (Kundu & Raqab, 2005) [16] having the probability density function,

$$f_{GR}\ (x;\alpha,\lambda)=2\,\alpha\,\lambda^{2}\,x\,e^{-(\lambda x)^{2}}\left\{1-e^{-(\lambda x)^{2}}\right\}^{\alpha-1} \quad ;(\alpha,\lambda)>0, x>0$$

Here α and λ are the shape and scale parameters respectively.

IV. Lindley distribution:

The Lindley distribution (Lindley, 1958) [17] whose probability density function (PDF) of can be expressed as

$$f(x)=\frac{\lambda^{2}}{\lambda+1}(1+x)e^{-\lambda x} \quad ;x\geq 0,\ \lambda>0.$$

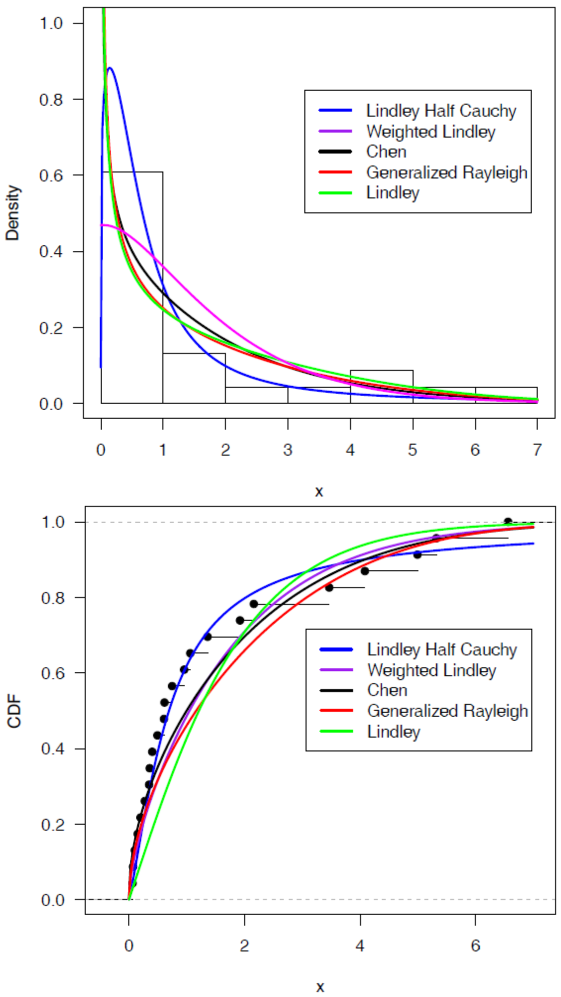

Figure 5. The Histogram and the PDF of fitted distributions (upper panel) and Empirical CDF with estimated CDF (lower panel).

In Table 3, we have reported the negative log-likelihood value, the AIC, BIC, CAIC and HQIC of the selected model. We conclude that the NLHC distribution provides a better fit to this data set then other selected distributions.

Table 3, Log-likelihood, AIC, BIC, CAIC and HQIC of the selected model

Model	-LL	AIC	BIC	CAIC	HQIC
NLHC	33.0591	70.1181	72.3891	70.7181	70.6893
Power Cauchy	33.1078	70.2160	72.4869	70.8160	70.7871
W- Lindley	33.6207	71.2414	73.5124	71.8414	71.8126
Chen	33.8402	71.6804	73.9514	72.2804	72.2515
Lindley	35.3054	72.6108	73.7463	72.8013	72.8963

We have reported the test statistics and their corresponding p-value of the NLHC and competing models in Table 4. The result shows that the NLHC model has the minimum value of the test statistic and higher *p*-value hence we conclude that the NLHC gets quite better fit and more reliable results from other alternatives.

Table 4

The *KS*, A^2, *and W* statistics and their corresponding *p*-value

Model	*KS(p-value)*	*W(p-value)*	A^2*(p-value)*
NLHC	0.1002(0.9572)	0.0337(0.9654)	0.2877(0.9467)
Power Cauchy	0.0952(0.9725)	0.0287(0.9823)	0.2552(0.9674)
W- Lindley	0.1606(0.5403)	0.1336(0.4462)	0.7583(0.5105)
Chen	0.1364(0.7356)	0.1024(0.5774)	0.6452(0.6045)
Lindley	0.2441(0.1084)	0.3826(0.07960)	2.2994(0.0638)

V. CONCLUDING REMARKS

In this study, we have presented a two-parameter new Lindley half Cauchy distribution. We have studied some structural properties of the NLHC distribution including the survival function, hazard rate function, quantile function, function for the random number generation and skewness and kurtosis. Three different well-known estimation methods are used to estimate the parameters of the proposed distribution namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods. Furthur we obtain the observed information matrix and we have constructed the asymptotic confidence interval for the maximum likelihood estimates. To compare the above-mentioned estimation method we have calculated the log-likelihood, AIC, BIC and AICC for each estimation method. It is found that MLE is the best among the others under consideration. We have studied the potentiality of the proposed model by comparing it with some well-known distributions namely weighted Lindley, Chen, generalized Rayleigh and Lindley and found the NLHC distribution fits quite better than other alternatives. We expect that the proposed model will entice wider applications in areas such as life science, engineering, hydrology, survival and lifetime data, economics and others.

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