Half Logistic Exponential Extension Distribution with Properties and Applications

Arun Kumar Chaudhary, Vijay Kumar

Abstract: ***In this article, we have introduced a new distribution based on type I half logistic-G family and exponential extension as a base distribution known as Half Logistic Exponential Extension (HLEE) distribution. The statistical properties of this model are also explored, such as the behavior of probability density, hazard rate, and quantile functions are investigated. The Maximum likelihood estimation (MLE) method is used to estimate model parameters. For the potentiality of the proposed model we have compared the goodness of fit with some others models. We have proven the importance and flexibility of the new distribution in modeling with real data applications empirically.***

Keywords: ***Estimation, Exponential extension, Half-logistic exponential extension distribution, MLE.***

I. INTRODUCTION

For the study of survival data in various fields of applied sciences, the probability distributions are frequently used. Several lifetime distributions are introduced to model such types of data, but existing distributions do not always offer a better fit. Hence most of the studies are focused on generalizing distributions and investigating their flexibility and applicability. Generally, these new family of continuous distributions provides a better fit as compared to usual classical distributions and are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. The exponential distribution plays an important role in analyses of lifetime or survival data, in part because of their convenient statistical theory, their important 'lack of memory' property, and their constant hazard rates. The generalizations of the exponential distribution were introduced by different researchers; some of them are generalized exponential (Gupta and Kundu, 2001) [11], beta exponential (Nadarajah and Kotz, 2006) [22], beta generalized exponential (Barreto-Souza et al., 2010)[6], Kumaraswamy exponential (Cordeiro and de Castro, 2011) [7], Nadarajah & Haghighi (2011) have presented an extension of the exponential distribution [21], gamma exponentiated exponential by (Ristic and Balakrishnan, 2012) [23], Transmuted exponentiated exponential distribution by (Merovci, 2013) [18], the exponentiated exponential geometric (Louzada et al., 2014) [17] and Kumaraswamy transmuted exponential (Afify et al., 2016) distributions [2]. Lemonte (2013) has introduced a new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function [16]. Gomez et al. (2014) have presented a new extension of the exponential distribution. [10] Recently, Hassan et al. (2018) have introduced the Alpha power transformed extended exponential distribution [12]. Almarashi et al. (2019) have presented a new extension of exponential distribution with statistical properties and applications [3]. Abdulkabir & Ipinyomi (2020) have introduced the Type II half logistic exponentiated exponential distribution [1]. (Balakrishnan, 1985) introduced the half logistic distribution is a member of the family of logistic distributions which have the following cumulative distribution function (CDF) [5]

$$F(t;\lambda)=\frac{1-e^{-\lambda t}}{1+e^{-\lambda t}} \quad ;t>0,\lambda>0$$

and its corresponding PDF is

$$f(t;\lambda)=\frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2} \quad ;t>0,\lambda>0$$

Cordeiro et al. (2015) introduced the cumulative distribution function (CDF) and the probability density function (PDF) of type I half logistic-G family, which are respectively given by

[8] $$F(y)=\frac{1-\left(1-G(y)\right)^{\lambda}}{1+\left(1-G(y)\right)^{\lambda}}, \quad y,\lambda>0 \tag{1.1}$$

and

$$f(y)=\frac{2\lambda g(y)\left(1-G(y)\right)^{\lambda-1}}{\left[1+\left(1-G(y)\right)^{\lambda}\right]^2}, \quad y,\lambda>0. \tag{1.2}$$

where G(y) and g(y) are CDF and PDF of baseline distribution. The chief purpose of this study is to introduce a more flexible model by adding just one extra parameter to the Exponential Extension distribution (Kumar, 2010) to achieve a better fit to real data [13]. We discuss the statistical properties of the HLEE distribution and its applicability. The arrangements of the contents of the proposed study are as follows. In Section 2, we present the new half-logistic exponential extension distribution and its various mathematical and statistical properties. We have employed two well-known estimation methods to estimate the model parameters, namely the maximum likelihood estimation (MLE), and we have calculated the model parameters, and associated confidence intervals using the observed information

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matrix are discussed in Section 3. In Section 4, a real data set has been analyzed to explore the applications and suitability of the proposed distribution. In this section, we present the ML estimators of the parameters and approximate confidence intervals and also displayed the graph of profile log-likelihood of the ML parameters. Finally, in Section 5, we present concluding remarks.

II. THE HALF LOGISTIC EXPONENTIAL EXTENSION (HLEE) DISTRIBUTION

The proposed new distribution is developed by using the type I half logistic-G family (1.1) and (1.2), and baseline distribution is the exponential extension (Kumar, 2010) [13]. The choice of the baseline distribution as the exponential extension is that it is more flexible, and the hazard rate function can be increasing, decreasing and constant. The two-parameter exponential extension distribution has one shape and one scale parameter. The random variable X follows exponential extension distribution with the shape and scale parameters as α > 0 and λ > 0 respectively, if it has the following probability density function (PDF)

$$g(x)=\alpha\left(1+\frac{\lambda}{x}\right)e^{-\lambda/x}\exp\left\{-\alpha\, x e^{-\lambda/x}\right\} \quad ;x>0,\alpha>0,\lambda>0 \tag{2.1}$$

and its cumulative density function (CDF) is

$$G(x)=1-\exp\left\{-\alpha\, x e^{-\lambda/x}\right\} \quad ;x>0,\alpha>0,\lambda>0 \tag{2.2}$$

By substituting (2.1) and (2.2) in (1.1) and (1.2), we get the CDF and PDF of new distribution half logistic exponential extension (HLEE) respectively as,

$$F(x)=\frac{1-\exp\left\{-\alpha\,\lambda x e^{-\beta/x}\right\}}{1+\exp\left\{-\alpha\,\lambda x e^{-\beta/x}\right\}} \quad ;x>0,\alpha>0,\beta>0,\lambda>0 \tag{2.3}$$

$$f(x)=\frac{2\alpha\lambda\left(1+(\beta/x)\right)e^{-\beta/x}\exp\left\{-\alpha\,\lambda x e^{-\beta/x}\right\}}{\left\{1+\exp\left(-\alpha\,\lambda x e^{-\beta/x}\right)\right\}^2} \quad ;x>0,\alpha>0,\beta>0,\lambda>0 \tag{2.4}$$

The hazard rate function (HRF)

Suppose that t be survival time of a component or item and we want the probability that it will not survive for an additional time *dt*, then hazard rate function is,

$$h(t)=\lim_{dt\to 0}\frac{pr(t\le T<t+dt)}{dt.R(t)}=\frac{f(t)}{R(t)}=\frac{f(t)}{1-F(t)};\ 0<t<\infty$$

where R(t) is a reliability function.

Hence let, $X\sim \text{HLEE}(\alpha,\beta,\lambda)$ then its hazard rate function is

$$h(x)=\frac{\alpha\lambda\left(1+(\beta/x)\right)e^{-\beta/x}}{\left\{1+\exp\left(-\alpha\,\lambda x e^{-\beta/x}\right)\right\}};x>0,\alpha>0,\beta>0,\lambda>0 \tag{2.5}$$

We have plotted the graph of the probability density function and hazard function of HLEE distribution in Figure 1. It is found that the shapes of the HLEE density are arc, negative-skewed, positive-skewed and symmetrical. The hazard rate function (HRF) for the HLEE distribution is also flexible due to its various shapes, such as monotonically increasing, decreasing, and constant.

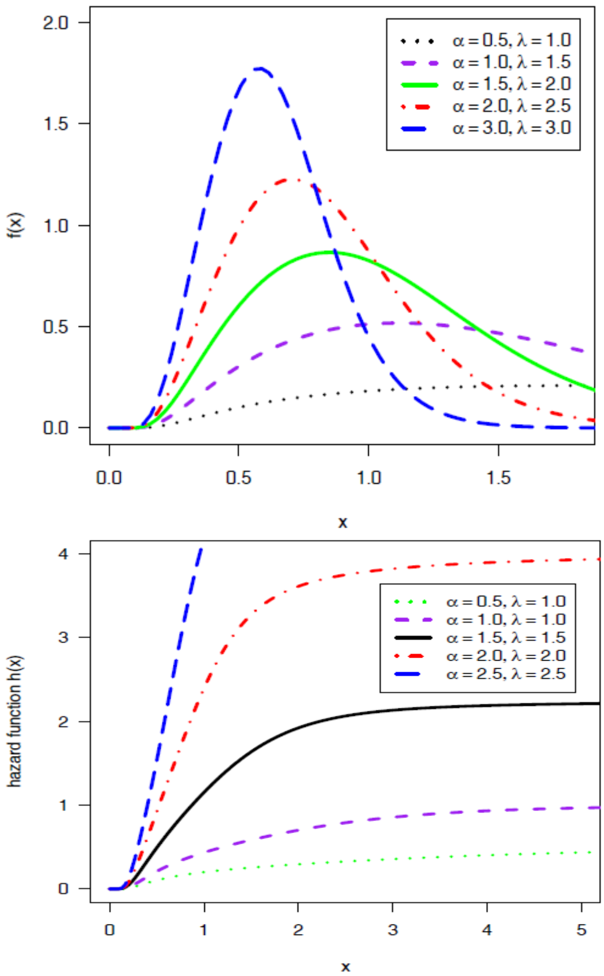


Figure 1. Plots of PDF (upper panel) and hazard function (lower panel) for β = 1 and different values of α and λ.

Quantile function of HLEE distribution:

In statistics and probability, the quantile function, associated with a probability distribution of a random variable, stipulates the value of the random variable such that the probability of the variable being less than or equal to that value equals the given probability. It is also termed the percent-point function or inverse cumulative distribution function**.** The definition of the p^{th} quantile is the real solution of the following equation

$$Q(p)=F^{-1}(p)$$

the quantile function is obtained by inverting CDF (2.3) as

$$\log(x)-\frac{\beta}{x}-\log\left\{-\frac{1}{\alpha\lambda}\log\left(\frac{1-p}{1+p}\right)\right\}=0,\quad 0<p<1 \tag{2.6}$$

Generation of the random numbers:

For the generation of the random numbers of the HLEE distribution, we suppose simulating values of random variable X with the CDF (2.3). Let U symbolize a uniform random variable in (0, 1), then the simulated values of X are obtained by

$$\ln x+\frac{\beta}{x}-\ln\left[-\frac{1}{\alpha\lambda}\ln\left(\frac{1-u}{1+u}\right)\right]=0\ ; \quad 0<u<1$$

Skewness and Kurtosis of HLEE distribution

The coefficient of skewness and kurtosis are important measures of dispersion in descriptive statistics. These measures are used mostly in data analysis to study the shape of the distribution or data set. The Bowley's coefficient of skewness based on quartiles is,

$$S_k\left(Bowley\right)=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},\ \text{and}$$

The coefficient of kurtosis based on octiles given by (Moors, 1988) is [19]

$$K_u\left(Moors\right)=\frac{Q(0.875)-Q(0.625)+Q(0.375)-Q(0.125)}{Q(3/4)-Q(1/4)}$$

III. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

A branch of statistics which estimates the values of unknown parameters centered on measured empirical data that has a random component is known as estimation theory. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements. In this study, we have discussed the maximum likelihood method of estimation.

In this section, we have discussed the maximum likelihood estimators (MLE's) of the HLEE distribution. Consider $\underline{x}=(x_1,\ldots,x_n)$ be a random sample of size 'n' from HLEE(α, β, λ) then the log density $l(\alpha,\lambda,\theta/\underline{x})$ can be written as,

$$\ell(\alpha,\beta,\lambda\,|\,x)=\log(2\alpha\lambda)+\log\left(1+\frac{\beta}{x}\right)-\frac{\beta}{x}-\alpha\,\lambda x e^{-\beta/x}-2\log\left[1+\exp\left(-\alpha\,\lambda x e^{-\beta/x}\right)\right]$$

And the corresponding likelihood function is

$$L(\alpha,\beta\,|\,\underline{x})=2\alpha\lambda\prod_{i=1}^{n}\frac{\{1+(\beta/x_i)\}e^{-\beta/x_i}\exp\left\{-\alpha\,\lambda x_i\,e^{-\beta/x_i}\right\}}{\left\{1+\exp\left(-\alpha\,\lambda x_i\,e^{-\beta/x_i}\right)\right\}^2}$$

The log-likelihood function of HLEE:

$$\ell(\alpha,\beta,\lambda\,|\,\underline{x})=n\log(2\alpha\lambda)+\sum_{i=1}^{n}\log\left(1+\frac{\beta}{x_i}\right)-\beta\sum_{i=1}^{n}\frac{1}{x_i}-\alpha\,\lambda\sum_{i=1}^{n}x_i\,e^{-\beta/x_i}$$
$$-2\sum_{i=1}^{n}\log\left[1+\exp\left(-\alpha\,\lambda x_i\,e^{-\beta/x_i}\right)\right] \tag{3.1}$$

By differentiating (3.1) with respect to unknown parameters α, β and λ, we get

$$\frac{\partial\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\alpha}=\frac{n}{\alpha}-\lambda\sum_{i=1}^{n}x_i\,e^{-\beta/x_i}+2\lambda\sum_{i=1}^{n}\frac{x_i\,e^{-\beta/x_i}}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]}$$

$$\frac{\partial\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\beta}=-\sum_{i=1}^{n}\frac{1}{x_i}+\sum_{i=1}^{n}\frac{1}{(x_i+\beta)}+\alpha\,\lambda\sum_{i=1}^{n}e^{-\beta/x_i}-2\alpha\lambda\sum_{i=1}^{n}\frac{e^{-\beta/x_i}}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]}$$

$$\frac{\partial\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\lambda}=\frac{n}{\lambda}-\alpha\sum_{i=1}^{n}x_i e^{-\beta/x_i}-2\alpha\sum_{i=1}^{n}\frac{x_i\,e^{-\beta/x_i}}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]} \tag{3.2}$$

After equating to zero and solving equations (3.2) for the unknown parameters (α, β, λ), we will get the ML estimators of the HLEE distribution. But, it is difficult to solve (3.2) manually, so by using computer software, one can solve these equations. Let us represent the parameter vector by $\underline{\Theta}=(\alpha,\beta,\lambda)$ and the corresponding MLE of $\underline{\Theta}$ as $\hat{\underline{\Theta}}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$, then the asymptotic normality results in, $\left(\hat{\underline{\Theta}}-\underline{\Theta}\right)\rightarrow N_3\left[0,\left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$ where $I\left(\underline{\Theta}\right)$ is the Fisher's information matrix given by,

$$I\left(\underline{\Theta}\right)=-\begin{pmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right)\\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right)\\ E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right)\end{pmatrix}$$

Again differentiating (3.2) with respect to (α, β, λ) we obtain the following equations as,

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\alpha^2}=-\frac{n}{\alpha^2}-2\lambda^2\sum_{i=1}^{n}\frac{x_i\,e^{-\beta/x_i}\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}-\beta/x_i\right)}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]^2}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\beta^2}=-\sum_{i=1}^{n}\frac{x_i}{(\beta+x_i)^2}+\alpha\lambda\sum_{i=1}^{n}\frac{e^{-\beta/x_i}}{x_i}-2\alpha\lambda\sum_{i=1}^{n}\frac{e^{-2\beta/x_i}\left[\exp(\alpha\,\lambda\,e^{-\beta/x_i})(e^{\beta/x_i}-\alpha\lambda)-e^{\beta/x_i}\right]}{x_i\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\lambda^2}=-\frac{n}{\lambda^2}+2\alpha^2\sum_{i=1}^{n}\frac{x_i\,e^{-\beta/x_i}\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}-\beta/x_i\right)}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]^2}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\alpha\partial\beta}=\lambda\sum_{i=1}^{n}e^{-\beta/x_i}+2\alpha\lambda\sum_{i=1}^{n}\frac{e^{-2\beta/x_i}\left[\exp(\alpha\,\lambda\,e^{-\beta/x_i})(e^{\beta/x_i}-\alpha\lambda)-e^{\beta/x_i}\right]}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\alpha\partial\lambda}=-\sum_{i=1}^{n}x_i e^{-\beta/x_i}+2\sum_{i=1}^{n}\frac{x_i\,e^{-2\beta/x_i}\left[\exp(\alpha\,\lambda\,e^{-\beta/x_i})(\alpha\lambda-e^{\beta/x_i})-e^{\beta/x_i}\right]}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]^2}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda\,|\,x)}{\partial\beta\partial\lambda}=\alpha\sum_{i=1}^{n}e^{-\beta/x_i}-2\alpha\sum_{i=1}^{n}\frac{e^{-2\beta/x_i}\left[\exp(\alpha\,\lambda\,e^{-\beta/x_i})(\alpha\lambda-e^{\beta/x_i})-e^{\beta/x_i}\right]}{\left[1-\exp\left(\alpha\,\lambda\,e^{-\beta/x_i}\right)\right]^2}$$

In practice, we don't know $\underline{\Theta}$;hence it is useless that the MLE has an asymptotic variance $\left(I\left(\underline{\Theta}\right)\right)^{-1}$. Hence, we approximate the asymptotic variance by substituting in the estimated value of the sparameters. The observed fisher information matrix $O(\hat{\underline{\Theta}})$ is used as an estimate of the information matrix $I\left(\underline{\Theta}\right)$ given by

$$O(\widehat{\Theta}) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \alpha \partial \lambda} & \frac{\partial^2 l}{\partial \beta \partial \lambda} & \frac{\partial^2 l}{\partial \lambda^2} \end{pmatrix}_{|(\hat{\alpha}, \hat{\beta}, \hat{\lambda})} = -H\left(\underline{\Theta}\right)_{|(\underline{\Theta} = \hat{\underline{\Theta}})}$$

Where H is the Hessian matrix.

The Newton-Raphson algorithm produces the observed information matrix to maximize the likelihood. Therefore, the variance-covariance matrix is given by,

$$\left[-H\left(\underline{\Theta}\right)_{|(\underline{\Theta} = \hat{\underline{\Theta}})}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix} \tag{3.3}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β, λ can be constructed as,

$$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\alpha})}\ ,\ \hat{\beta} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\beta})}$$
$$\text{and } \hat{\lambda} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\lambda})},$$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

IV. DATA ANALYSIS

In the data analysis section, we present a real data set for the illustration of the suitability and applicability of the proposed probabilistic model. The data represent the waiting times (minute) of 100 bank customers Ghitany et al. (2008) [9]. The data are presented below,

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5

The HLEE model is used to fit this data set. We have started the iterative procedure by maximizing the log-likelihood function given in equation (3.1) directly with an initial guess for α = 0.5, β = 0.1, λ = 1.1 far away from the solution. We have used *optim*() function in R (Schmuller, J. 2017) [24] with option BFGS method (Venables et al., 2020) [25]. The iterative process stopped after 45 iterations. We obtain $\hat{\alpha}$ = 0.1260 and $\hat{\beta}$ = 1.9684 and $\hat{\lambda}$ = 1.3437 and the corresponding log-likelihood value is LL = -317.0963. An estimate of the variance-covariance matrix, using equation (3.3) is

$$\begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 0.1822 & 0.0160 & -1.9409 \\ 0.0160 & 0.4713 & -0.1224 \\ -1.9409 & -0.1223 & 20.6905 \end{pmatrix}$$

By using the equation (3.1) we have computed the MLEs and their corresponding standard errors. In Table 1 we have presented the MLE's with their standard errors for α, β and λ.

Table 1

MLE and SE for α, β and λ.

Parameter	MLE	SE	t-value	Pr(>t)
Alpha	0.1260	0.4269	0.295	0.76793
Beta	1.9684	0.6865	2.867	0.00414
Lambda	1.3437	4.5487	0.295	0.76768

In Figure 2 we have plotted the graph of profile log-likelihood functions of α, β and λ (Kumar & Ligges, 2011) [14]. It is verified that the maximum likelihood estimators are unique.

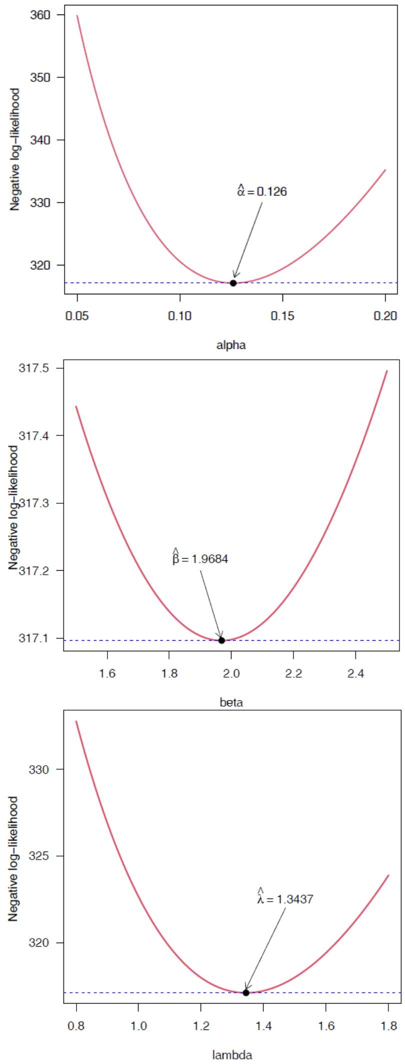


Figure 2. Profile log-likelihood functions of α, λ and θ.

We have considered some probability models for the comparison of goodness of fit and potentiality of the observed distribution which are as follows.

A. Exponentiated power Lindley distribution (EPL):

The probability density function of EPL distribution (Ashour & Eltehiwy, 2015) is [4]

$$f(x;\alpha,\beta,\theta)=\frac{\alpha\beta\theta^2x^{\beta-1}}{(\theta+1)}\left(1+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right]^{\alpha-1};x>0$$

$$\alpha>0,\beta>0,\theta>0,$$

B. Weibull distribution:

The probability density function of Weibull (W) distribution is

$$f_W(x)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-(x/\lambda)^\theta};\ \lambda\theta>0,x\geq 0$$

C. Generalized Rayleigh distribution

The probability density function of Generalized Rayleigh (GR) distribution (Kundu & Raqab, 2005) is [15]

$$f_{GR}(x;\alpha,\lambda)=2\alpha\lambda^2 x\, e^{-(\lambda x)^2}\left\{1-e^{-(\lambda x)^2}\right\}^{\alpha-1}\quad;(\alpha,\lambda)>0,x>0$$

D. Gompertz distribution:

The probability density function of Gompertz distribution (Murthy et al., 2003) with parameters α and θ is [20]

$$f_{GZ}(x)=\theta\ e^{\alpha x}\exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha x}\right)\right\}\quad;x\geq 0,\theta>0,-\infty<\alpha<\infty.$$

Where λ and α are the scale and shape parameters, respectively.

For the test of goodness of fit and adequacy of the proposed model, Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) are calculated and presented in Table 2.

Table 2

Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
HLEE	317.0963	640.1925	648.0081	640.4425	643.3556
EPL	317.1008	640.2016	648.0171	640.4516	643.3646
Weibull	318.7307	641.4614	646.6717	641.5851	643.5701
GR	321.5182	647.0364	652.2467	647.1601	649.1451
GZ	323.9756	651.9512	657.1615	652.0749	654.0599

To evaluate the validity of the model, we calculate the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function using maximum likelihood estimators. To know more about the nature of the distribution, we have to plot Q-Q and P-P plots. In particular, the Q-Q plot is used widely; it provides more information about the lack-of-fit at the tails of the distribution. In Figure 3 we have plotted the graph of CDF and Q-Q of HLEE distribution. From Figure 3, it is proven that the HLEE model fits the data exactly.

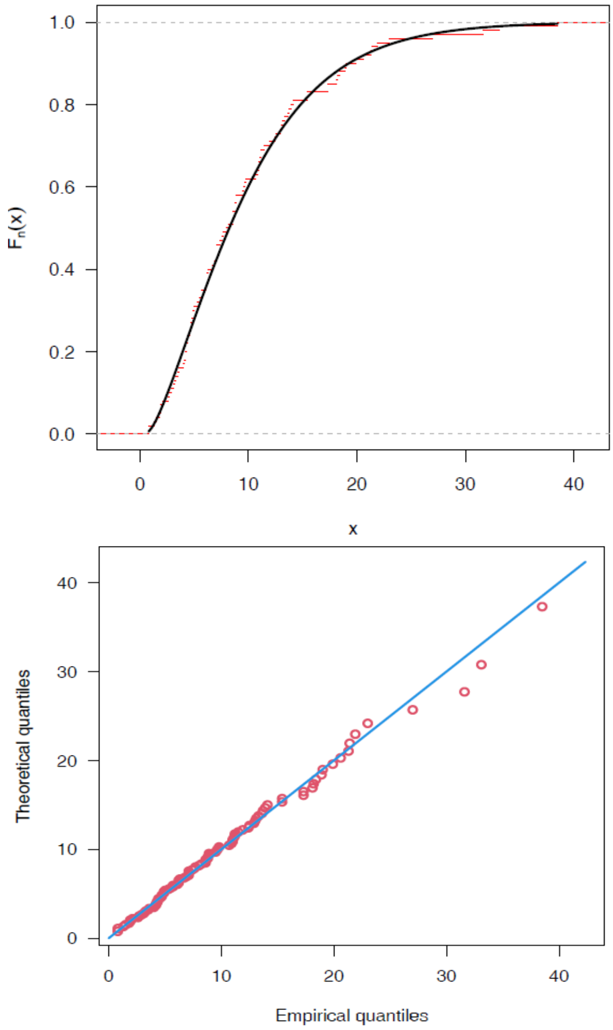
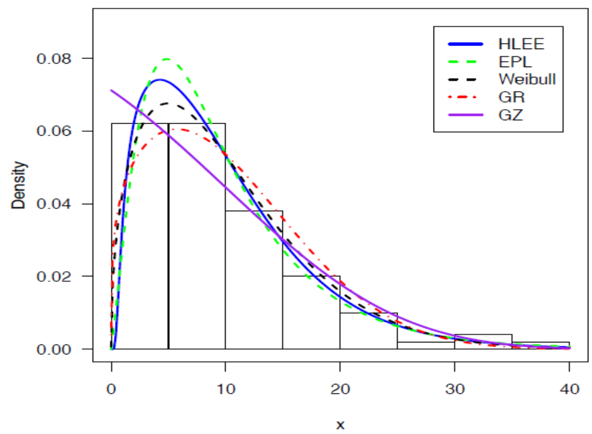


Figure 3. The graph of the CDF plot (upper panel) and Q-Q plot (lower panel).

The fitted density functions and distribution functions are displayed in Figure 4, which compares the distribution functions for the different models with the empirical distribution function that produces the same. Therefore, for the given data set illustrates the HLEE distribution gets better fit and more valid results from other competing models.



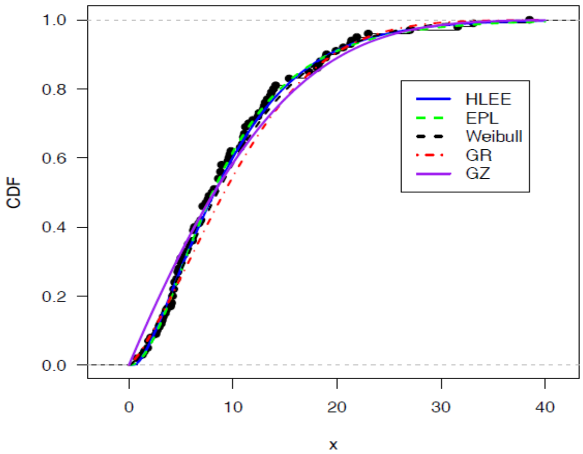


Figure 4. The Histogram and the PDF of fitted distributions (upper panel) and Empirical CDF with estimated CDF (lower panel).

We have reported the test statistics and their corresponding p -value of the HLEE distribution and competing models in Ta ble 4. The result shows that the HLEE distribution has the mi nimum value of the test statistic and higher *p*-value; hence we conclude that the HLEE distribution gets quite better fit and more consistent and reliable results from others taken for co mparison.

Table 4, The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*A^2(p-value)*	*W(p-value)*
HLEE	0.0441(0.9900)	0.0263(0.9871)	0.1854(0.9939)
EPL	0.0375(0.9989)	0.0178(0.9987)	0.1280(0.9996)
Weibull	0.0578(0.8920)	0.0611(0.8084)	0.4058(0.8426)
GR	0.0945(0.3337)	0.2043(0.2595)	1.0911(0.3126)
GZ	0.1060(0.2113)	0.2214(0.2297)	1.6275(0.1488)

V. CONCLUSION

We have proposed and studied a new three-parametric probabilistic model called as half logistic exponential extension distribution. Some statistical and mathematical properties of the derived model are investigated. The parameters of the proposed model are estimated by using the maximum likelihood estimation method. The potentiality of the proposed distribution is revealed by using a real dataset, where the proposed distribution provided a better fit in comparison with some other lifetime distributions. The applications have shown that the HLEE model is suitable for modeling a real dataset in the field of survival analysis. Hope the proposed distribution will be useful for practitioners in the area of probability theory and statistics.

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