Intuitionistic Fuzzy Normal Operator on IFH - Space

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***Abstract:** In this article, we define Intuitionistic Fuzzy Normal Operator operating on an IFH-Space. An operator $\mathcal{S}$ is an intuitionistic fuzzy normal operator if $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$ i.e. $\mathcal{S}$ commutes with its intuitionistic fuzzy adjoint.*

***Keywords:** IFH-space, Intuitionistic fuzzy Adjoint operator (IFA-Operator), Intuitionistic Fuzzy Self-Adjoint operator (IFSA-Operator), Intuitionistic Fuzzy Normal operator (IFN-Operator).*

I. INTRODUCTION

Let IFB($\mathcal{H}$) be the set of all IF-Bounded Linear Operators on IFH-Space $\mathcal{H}$. Park [5] first studied the concept of Intuitionistic Fuzzy Metric Spaces. Later on, Intuitionistic Fuzzy Metric and Norm have been defined by Saadati [6]. Then Goudarzi et al. [4] in 2009, introduced Intuitionistic Fuzzy Inner Product Space (IFIP-space). Majumdhar and Samanta [9] defined IFIP-space in 2011. In 2018, Radharamani et al. [1], [2] have given the definition and properties of Intuitionistic Fuzzy Hilbert Space (IFH-Space) $\mathcal{H}$ as a triplet $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ and also the concept of intuitionistic fuzzy adjoint and self-adjoint operators (IFA and IFSA-operators) in IFH-space. If $\mathcal{S} \in$ IFB($\mathcal{H}$), $\ni$ $\langle \mathcal{S}x, y\rangle = \langle x, \mathcal{S}^*y\rangle, \forall\, x, y \in \mathcal{H}$. Also $\mathcal{S}$ is an IFSA-operator if $\mathcal{S} = \mathcal{S}^*$.

Now we introduced intuitionistic fuzzy normal operator on $\mathcal{H}$, if $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$. Here we establish some theorems and an example for intuitionistic fuzzy normal operator like addition and multiplication of intuitionistic fuzzy normal operator. $\mathcal{S}$ is intuitionistic fuzzy normal $\Leftrightarrow \|\mathcal{S}^*u\| = \|\mathcal{S}u\|$. $\mathcal{S}$ is intuitionistic fuzzy normal $\Leftrightarrow$ its real and imaginary parts commute. We will discuss these in detail.

II. PRELIMINARIES

Definition 2.1: [4]

A continuous t – norm $\mathcal{T}$ is called continuous t – representable iff $\exists$ a continuous t – norm * and a continuous t – conorm $\diamond$ on the interval [0,1] such that for all $x = (x_1, x_2),\ y = (y_1, y_2) \in L^*$, $\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2)$.

Definition 2.2: [4]

Let $\mu: \mathcal{V}^2 \times (0, +\infty) \to [0,1]$ and $\vartheta: \mathcal{V}^2 \times (0, +\infty) \to [0,1]$ be Fuzzy sets, such that $\mu(x,y,t) + \vartheta(x,y,t) \leq 1, \forall\, x, y \in \mathcal{V}\ \&\ t > 0$.

An Intuitionistic Fuzzy Inner Product Space (IFIP-Space) is a triplet $(\mathcal{V}, \mathcal{F}_{\mu,v}, \mathcal{T})$, where $\mathcal{V}$ is a real Vector Space, $\mathcal{T}$ is a continuous t -representable and $\mathcal{F}_{\mu,v}$ is an Intuitionistic Fuzzy set on $\mathcal{V}^2 \times \mathbb{R}$ satisfying the following conditions for all $x,\ y,\ z \in \mathcal{V}$ and s, $r,\ t \in \mathbb{R}$:

(IFI - 1) $\mathcal{F}_{\mu,v}(x,\ y,\ 0) = 0$ and $\mathcal{F}_{\mu,v}(x,\ x,\ t) > 0$, for every $t > 0$.

(IFI - 2) $\mathcal{F}_{\mu,v}(x,\ y,\ t) = \mathcal{F}_{\mu,v}(y,\ x,\ t)$.

(IFI - 3) $\mathcal{F}_{\mu,v}(x,\ x,\ t) \neq H(t)$ for some $t \in \mathbb{R}$ iff $x \neq 0$,

where $H(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$

(IFI - 4) For any $\alpha \in \mathbb{R}$,

$$\mathcal{F}_{\mu,v}(\alpha x, y, t) = \begin{cases} \mathcal{F}_{\mu,v}\left(x, y, \frac{t}{\alpha}\right), & \alpha > 0 \\ H(t), & \alpha = 0 \\ \mathcal{N}_s\left(\mathcal{F}_{\mu,v}\left(x, y, \frac{t}{\alpha}\right)\right), & \alpha < 0 \end{cases}$$

(IFI - 5) $\sup\left\{\mathcal{T}\left(\mathcal{F}_{\mu,v}(x,\ z,\ s), \mathcal{F}_{\mu,v}(y,\ z,\ r)\right)\right\} = \mathcal{F}_{\mu,v}(x + y,\ y,\ t)$.

(IFI - 6) $\mathcal{F}_{\mu,v}\ (x,\ y,\ \cdot): \mathbb{R} \to [0,1]$ is Continuous on $\mathbb{R} \setminus \{0\}$.

(IFI - 7) $\lim_{t\to 0} \mathcal{F}_{\mu,v}\ (x,\ y,\ t) = 1$.

Note 2.3: [4]

(i) Here the standard negator

$$\mathcal{N}_s(x) = 1 - x, \qquad \forall\ x \in [0,1]$$

(ii) By putting $\langle x,\ y\rangle = \mathcal{F}_{\mu,v}(x, y, \cdot)$, it is very simple to show that the Intuitionistic Fuzzy Inner Product acts quite similarly as the Ordinary Inner Product.

(iii) Schwarz inequality:

$\mathcal{F}_{\mu,v}(x, y, ts) \geq \mathcal{T}(\mathcal{F}_{\mu,v}\ (x, x,\ t^2)\ \mathcal{F}_{\mu,v}\ (y, y,\ s^2))$,
for $x.y \in \mathcal{V}$ and s, $t > 0$.

(iv) A sequence $\{x_n\} \in \mathcal{V}$ is called τ-convergent to $x \in \mathcal{V}$, if for any given $\epsilon > 0$ and $\lambda > 0$, $\exists\, N^0 \in \mathbb{Z}^+$, $N^0 = N^0(\epsilon, \lambda)$, $\ni$ $\mathcal{P}(x_n - x, \epsilon) > \mathcal{N}_s(\lambda)$, whenever $n > N^0$.

(v) Let $\oint(x)$ be a continuous linear functional on $\mathcal{V}$. Then it is said to be $\tau_{\mathcal{F}_{\mu,v}}$ - continuous, if $x_n \xrightarrow{\tau_{\mathcal{F}_{\mu,v}}} x \Longrightarrow \oint(x_n) \xrightarrow{\tau_{\mathcal{F}_{\mu,v}}} \oint(x)$, for any $\{x_n\},\ x \in \mathcal{V}$.

Theorem 2.4: [2]

Let $(\mathcal{V},\ \mathcal{F}_{\mu,v},\ \mathcal{T})$ be an IFIP-Space, where $\mathcal{T}$ is a continuous t - representable for every $x, y \in \mathcal{V}$, sup $\{t \in$ *ℝ: Fμ,v x,y,t<1* $< \infty$. Define *., ,.:* $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$ by *x, y =*sup *t∈ℝ: Fμ,vx,y,t<1*. Then *(V, ·,·)* is an IFIP-space, so that $(\mathcal{V}, \mathcal{P}_{\mu,v})$ is a normed space, where $\mathcal{P}_{\mu,v}(x,t) = \langle x, x\rangle^{1/2}\ \forall\ x \in \mathcal{V}$.

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Definition 2.5: [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFIP-Space with IP: $\langle x, y\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}, \forall x, y \in \mathcal{V}$. If $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ is complete in the norm $\mathcal{P}_{\mu,\nu}$, then $\mathcal{V}$ is an Intuitionistic Fuzzy Hilbert Space (IFH-Space).

Theorem 2.6:[2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-Space with IP: $\langle x, y\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}, \forall x, y \in \mathcal{V}$. A sequence $\{x_n\}$ on $\mathcal{V}$ is $\tau_{\mathcal{F}_{\mu,\nu}}$-convergent (i.e. $x_n \xrightarrow{\tau_{\mathcal{F}_{\mu,\nu}}} x$) if $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$.

Proof:

Since $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$, we have

$$\lim_{n\to\infty} \mathcal{P}_{\mu,\nu}(x_n - x, \epsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} \langle x_n - x, x_n - x\rangle = 0$$

$$\Rightarrow \sup\{t \in \mathbb{R}^+: \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) < 1\} = 0 \quad \dots (2.1)$$

Hence for any $\epsilon > o$ & $0 < \lambda < 1$, we have

$$\sup\{t \in \mathbb{R}^+: \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) < 1\}$$

$$= \sup\{t \in [0, \epsilon]: \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) < 1\} + \sup\{t \in (\epsilon, \infty): \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) < 1\} \quad \dots (2.2)$$

$$\geq \epsilon \sup\{t \in (\epsilon, \infty): \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) < 1\}$$

$$= \epsilon\,(1 - \mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, \epsilon)$$

From equations (2.1) & (2.2), there exists (μ, ν) $(\epsilon, \lambda) \in \mathbb{R}^+$, if $n > N$, then $\mathcal{F}_{\mu,\nu}(x_n - x, x_n - x, t) >$

$1 - \lambda$.

So that, $x_n \xrightarrow{\tau_{\mathcal{F}_{\mu,\nu}}} x$.

Theorem 2.7: [2] (Riesz Theorem)

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-Space. For any $\tau_{\mathcal{F}_{\mu,\nu}}$ – continuous linear functional $\mathscr{f}$, a unique vector $y \in \mathcal{V}$, such that $\forall x \in \mathcal{V}$, we have $\mathscr{f}(x) = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}$.

Proof:

Continuous linear functional $\mathscr{f}$ is $\tau_{\mathcal{F}_{\mu,\nu}}$-continuous if $x_n \xrightarrow{\tau_{\mathcal{F}_{\mu,\nu}}} x$ implies $\mathscr{f}(x_n) \xrightarrow{\tau_{\mathcal{F}_{\mu,\nu}}} \mathscr{f}(x)$ for any $\{x_n\} \in \mathcal{V}$.

If $x_n \xrightarrow{\mathcal{P}_{\mu,\nu}} x$, then by theorem (2.6), $x_n \xrightarrow{\tau_{\mathcal{F}_{\mu,\nu}}} x$. So $\mathscr{f}$ is continuous on IFH-Space like an ordinary Hilbert Space.

Therefore, by Riesz Representation Theorem, it is proved.

Theorem 2.8: [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFIP-Space, where $\mathcal{T}$ is continuous t-representable and sup $\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) <$ *1 < ∞, ∀ x, y∈V*. Then sup *t ∈ ℝ: Fμ,ν x+y, z, t < 1*=sup*t ∈ ℝ: Fμ,ν x,z,t<1* + sup *t∈ℝ : Fμ,ν y,z,t<1 ∀x,y∈V.*

Theorem 2.9: (IFA-operator in IFH-space) [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-Space and let $\mathcal{S} \in$ IFB$(\mathcal{V})$. Then there exists unique $\mathcal{S}^* \in$ IFB$(\mathcal{V}) \ni \langle \mathcal{S}x, y\rangle = \langle x, \mathcal{S}^* y\rangle\, \forall\, x, y \in \mathcal{V}$.

Definition 2.10: (IFSA-operator) [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-Space with IP: $\langle x, y\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}, \forall x, y \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB$(\mathcal{V})$. Then $\mathcal{S}$ is Intuitionistic Fuzzy Self-Adjoint Operator, if $\mathcal{S} = \mathcal{S}^*$, where $\mathcal{S}^*$ is Intuitionistic Fuzzy Self-Adjoint of $\mathcal{S}$.

Theorem 2.11: [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-Space with IP: $\langle x, y\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}, \forall x, y \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB$(\mathcal{V})$. Then $\mathcal{S}$ is Intuitionistic Fuzzy Self-Adjoint Operator.

Theorem 2.12: [2]

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFIP-Space with IP: $\langle x, y\rangle = \sup\{t \in \mathbb{R}, \mathcal{F}_{\mu,\nu}(x, y, t) < 1\}, \forall x, y \in \mathcal{V}$ and let $\mathcal{S}^*$ be the intuitionistic fuzzy adjoint operator. Then

(i) $(\mathcal{S}^*)^* = \mathcal{S}$

(ii) $(\beta\mathcal{S})^* = \beta\mathcal{S}^*$

(iii) $(\beta\mathcal{S}_1 + \gamma\mathcal{S}_2)^* = \beta{\mathcal{S}_1}^* + \gamma{\mathcal{S}_2}^*$ where β, γ are scalars.

(iv) $(\mathcal{S}_1\mathcal{S}_2)^* = {\mathcal{S}_2}^*{\mathcal{S}_1}^*$.

III. MAIN RESULTS

In this section, we introduced the definition of Intuitionistic Fuzzy Normal Operator in IFH-Space and also explain some elementary properties of IFN-Operator in IFH-Space in detail.

Definition 3.1: (Intuitionistic Fuzzy Normal Operator)

Let $(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-space with an IP: $\langle u, v\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\}, \forall\, u, v \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB$(\mathcal{V})$. Then $\mathcal{S}$ is an Intuitionistic Fuzzy Normal Operator if it commutes with its IF-Adjoint. i.e. $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$.

Remark 3.2:

1. It is obvious that every IFSA-operator is an IFN-operator.
2. If $\mathcal{S}$ is intuitionistic fuzzy normal and α is a scalar, then $\alpha\mathcal{S}$ is also intuitionistic fuzzy normal.
3. The limit $\mathcal{S}$ of any intuitionistic fuzzy convergent sequence $\{\mathcal{S}_k\}$ of intuitionistic fuzzy normal operators is intuitionistic fuzzy normal.

   Proof:

   We know that ${\mathcal{S}_k}^* \to \mathcal{S}_k$. So

$$\mathcal{P}_{\mu,\nu}((\mathcal{S}\mathcal{S}^* - \mathcal{S}^*\mathcal{S})u, t) \leq \mathcal{P}_{\mu,\nu}((\mathcal{S}\mathcal{S}^* - \mathcal{S}_k{\mathcal{S}_k}^*)u, t) + \mathcal{P}_{\mu,\nu}((\mathcal{S}_k{\mathcal{S}_k}^* - {\mathcal{S}_k}^*\mathcal{S}_k)u, t) + \mathcal{P}_{\mu,\nu}(({\mathcal{S}_k}^*\mathcal{S}_k - \mathcal{S}^*\mathcal{S})u, t) \longrightarrow 0$$

Which implies that $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$.

Theorem 3.3:

If $\mathcal{S}_1$and $\mathcal{S}_2$ are IFN-operators on $\left(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T}\right)$ with the property that either commutes with IF-adjoint of the other, then $\mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{S}_1\mathcal{S}_2$ are IFN-operators.

Proof:

It is luminous by taking IF-adjoints that $\mathcal{S}_1\mathcal{S}_2{}^* = \mathcal{S}_2{}^*\mathcal{S}_1 \Leftrightarrow \mathcal{S}_2\mathcal{S}_1{}^* = \mathcal{S}_1{}^*\mathcal{S}_2$.
So, the assumption implies that each operator commutes with intuitionistic fuzzy adjoint of the other.

(a) We first prove that $\mathcal{S}_1 + \mathcal{S}_2$ is an IFN-operator as follows:

$$(\mathcal{S}_1 + \mathcal{S}_2)(\mathcal{S}_1 + \mathcal{S}_2)^* = (\mathcal{S}_1 + \mathcal{S}_2)(\mathcal{S}_1{}^* + \mathcal{S}_2{}^*) = \mathcal{S}_1\mathcal{S}_1{}^* + \mathcal{S}_1\mathcal{S}_2{}^* + \mathcal{S}_2\mathcal{S}_1{}^* + \mathcal{S}_2\mathcal{S}_2{}^* \quad \ldots (3.1)$$

$$\& \ (\mathcal{S}_1 + \mathcal{S}_2)^*(\mathcal{S}_1 + \mathcal{S}_2) = (\mathcal{S}_1{}^* + \mathcal{S}_2{}^*)(\mathcal{S}_1 + \mathcal{S}_2) = \mathcal{S}_1{}^*\mathcal{S}_1 + \mathcal{S}_1{}^*\mathcal{S}_2 + \mathcal{S}_2{}^*\mathcal{S}_1 + \mathcal{S}_2{}^*\mathcal{S}_2 \quad \ldots (3.2)$$

∴ from (3.1) and (3.2),

$$(\mathcal{S}_1 + \mathcal{S}_2)(\mathcal{S}_1 + \mathcal{S}_2)^* = (\mathcal{S}_1 + \mathcal{S}_2)^*(\mathcal{S}_1 + \mathcal{S}_2)$$

Thus $\mathcal{S}_1 + \mathcal{S}_2$ is an IFN-Operator.

(b) Next, we will show that $\mathcal{S}_1\mathcal{S}_2$ is an IFN-operator.

$$\begin{aligned}(\mathcal{S}_1\mathcal{S}_2)(\mathcal{S}_1\mathcal{S}_2)^* &= \mathcal{S}_1\mathcal{S}_2\mathcal{S}_2{}^*\mathcal{S}_1{}^* \\ &= \mathcal{S}_1\mathcal{S}_2{}^*\mathcal{S}_2\mathcal{S}_1{}^* \\ &= \mathcal{S}_2{}^*\mathcal{S}_1\mathcal{S}_1{}^*\mathcal{S}_2 \\ &= \mathcal{S}_2{}^*\mathcal{S}_1{}^*\mathcal{S}_1\mathcal{S}_2 \\ &= (\mathcal{S}_1\mathcal{S}_2)^*(\mathcal{S}_1\mathcal{S}_2)\end{aligned}$$

Thus $\mathcal{S}_1\mathcal{S}_2$ is an IFN-operator.

Theorem 3.4:

Let $\left(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T}\right)$ be an IFH-space with IP:
$\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\right\}, \forall\, u, v \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB($\mathcal{V}$). $\mathcal{S}$ is Intuitionistic Fuzzy Normal iff $\mathcal{P}_{\mu,\nu}(\mathcal{S}^*u, t) = \mathcal{P}_{\mu,\nu}(\mathcal{S}\mathrm{u}, t)\ \forall\, u \in \mathcal{V}$.

Proof:

Let $\mathcal{P}_{\mu,\nu}(\mathcal{S}^*u, t) = \mathcal{P}_{\mu,\nu}(\mathcal{S}\mathrm{u}, t)$

$\Leftrightarrow \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}^*u, t) = \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}\mathrm{u}, t)$

$\Leftrightarrow \langle\mathcal{S}^*u, \mathcal{S}^*u\rangle = \langle\mathcal{S}\mathrm{u}, \mathcal{S}\mathrm{u}\rangle$

$\Leftrightarrow \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}^*u, \mathcal{S}^*u, t) < 1\} = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}u, \mathcal{S}\mathrm{u}, t) < 1\}$

$\Leftrightarrow \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}\mathcal{S}^*u, u, t) < 1\} = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}^*\mathcal{S}u, \mathrm{u}, t) < 1\}$

$\Leftrightarrow \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}\mathcal{S}^*u, u, t) < 1\} - \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}^*\mathcal{S}u, \mathrm{u}, t) < 1\} = 0$

$\Leftrightarrow \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}((\mathcal{S}\mathcal{S}^* - \mathcal{S}^*\mathcal{S})u, u, t) < 1\} = 0$

$\Leftrightarrow \langle(\mathcal{S}\mathcal{S}^* - \mathcal{S}^*\mathcal{S})u, u\rangle = 0$

$\Leftrightarrow (\mathcal{S}\mathcal{S}^* - \mathcal{S}^*\mathcal{S})u = 0$

$\Leftrightarrow \mathcal{S}\mathcal{S}^* - \mathcal{S}^*\mathcal{S} = 0$

$\Leftrightarrow \mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$ i.e. $\mathcal{S}$ is Intuitionistic Fuzzy Normal.

Theorem 3.5:

Let $\left(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T}\right)$ be an IFH-space with IP:
$\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\right\}, \forall\, u, v \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB($\mathcal{V}$) be an Intuitionistic Fuzzy Normal Operator. Then $\mathcal{P}_{\mu,\nu}(\mathcal{S}^2u, t) = \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}\mathrm{u}, t)$.

Proof:

Let $\mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}^2\mathrm{u}, t) = \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}\mathcal{S}\mathrm{u}, t)$

$$\begin{aligned}&= \langle\mathcal{S}\mathcal{S}\mathrm{u}, \mathcal{S}\mathcal{S}\mathrm{u}\rangle \\ &= \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}\mathcal{S}u, \mathcal{S}\mathcal{S}\mathrm{u}, t) < 1\} \\ &= \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathcal{S}^*\mathcal{S}u, \mathcal{S}^*\mathcal{S}u, t) < 1\} \\ &= \langle\mathcal{S}^*\mathcal{S}u, \mathcal{S}^*\mathcal{S}u\rangle \\ &= \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}^*\mathcal{S}u, t)\end{aligned}$$

$\Rightarrow \mathcal{P}_{\mu,\nu}(\mathcal{S}^2u, t) = \mathcal{P}_{\mu,\nu}(\mathcal{S}^*\mathcal{S}u, t) \quad \ldots (3.3)$

By known result, $\mathcal{P}_{\mu,\nu}(\mathcal{S}^*\mathcal{S}u, t) = \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}\mathrm{u}, t) \ldots (3.4)$

Therefore from (3.3) & (3.4), $\mathcal{P}_{\mu,\nu}(\mathcal{S}^2u, t) = \mathcal{P}_{\mu,\nu}{}^2(\mathcal{S}\mathrm{u}, t)$.

Remark 3.6:

Any complex number z can be expressed uniquely as $z = a + ib$ where a & b are real numbers and that these real numbers are called real and imaginary parts of z. i.e. $a = \frac{z+\bar{z}}{2}$ & $b = \frac{z-\bar{z}}{2i}$.

The correlation between general operators and complex numbers and between IFSA-operators and real numbers suggests that for an arbitrary operator $\mathcal{S} \in$ IFB($\mathcal{V}$), we form

$$T_1 = \frac{\mathcal{S} + \mathcal{S}^*}{2} \ \& \ T_2 = \frac{\mathcal{S} - \mathcal{S}^*}{2i}$$

T_1 and T_2 are clearly IFSA-operators and they have the property that

$$\mathcal{S} = T_1 + iT_2$$

The uniqueness of this expression for $\mathcal{S}$ follows at once

$$\mathcal{S}^* = T_1 - iT_2$$

The operators T_1 and T_2 are called real part and imaginary part of $\mathcal{S}$.

Theorem 3.7:

Let $\left(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T}\right)$ be an IFH-space with IP:
$\langle u, v\rangle = \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\}, \forall\, u, v \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB($\mathcal{V}$). $\mathcal{S}$ is intuitionistic fuzzy normal iff its real and imaginary parts commute.

Proof:

Suppose that T_1 and T_2 are real and imaginary parts of $\mathcal{S}$. So that

$\mathcal{S} = T_1 + iT_2$ & $\mathcal{S}^* = T_1 - iT_2$

Then, $\mathcal{S}\mathcal{S}^* = (T_1 + iT_2)(T_1 - iT_2) = T_1{}^2 + T_2{}^2 + i(T_2T_1 - T_1T_2) \quad \ldots (3.5)$

$\mathcal{S}^*\mathcal{S} = (T_1 - iT_2)(T_1 + iT_2) = T_1{}^2 + T_2{}^2 + i(T_1T_2 - T_2T_1) \quad \ldots (3.6)$

It is clear that $T_1T_2 = T_2T_1$.
Then from (3.5) & (3.6), $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$.
Conversely, if $\mathcal{S}\mathcal{S}^* = \mathcal{S}^*\mathcal{S}$, then

$$T_1T_2 - T_2T_1 = T_2T_1 - T_1T_2$$

So, $2T_1T_2 = 2T_2T_1$
Implies that $T_1T_2 = T_2T_1$.

Example 3.8:

Let $\left(\mathcal{V}, \mathcal{F}_{\mu,\nu}, \mathcal{T}\right)$ be an IFH-space with IP:
$\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(u, v, t) < 1\right\}, \forall\, u, v \in \mathcal{V}$ and let $\mathcal{S} \in$ IFB($\mathcal{V}$) be an arbitrary (intuitionistic fuzzy) operator and if γ & δ are scalars such that $|\gamma| = |\delta|$, show that $\gamma\mathcal{S} + \delta\mathcal{S}^*$ is intuitionistic fuzzy normal.

Proof:

From theorem 3.4, it is enough to prove $\mathcal{P}_{\mu,\nu}((\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u, t = \mathcal{P}_{\mu,\nu}((\gamma\mathcal{S} + \delta\mathcal{S}^*)u, t)$

Let us consider, $\mathcal{P}_{\mu,\nu}((\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u, t)$

$$\mathcal{P}_{\mu,\nu}{}^2((\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u, t) = \langle(\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u, (\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u\rangle$$

$$= \langle(\gamma\mathcal{S}^* + \delta(\mathcal{S}^*)^*)u, (\gamma\mathcal{S}^* + \delta(\mathcal{S}^*)^*)u\rangle$$

$$= \langle(\gamma\mathcal{S}^* + \delta\mathcal{S})u, (\gamma\mathcal{S}^* + \delta\mathcal{S})u\rangle$$

$$= \sup\{\, t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\left((\gamma\mathcal{S}^* + \delta\mathcal{S})\, u, (\gamma\mathcal{S}^* + \delta\mathcal{S})\, u, t\,\right) < 1\}$$

$$= \sup\{\, t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\, (\gamma\mathcal{S}^*\, u, \gamma\mathcal{S}^*\, u, t) < 1\} + \sup\{\, t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\, (\delta\mathcal{S}u, \delta\mathcal{S}u, t) < 1\}$$

$$= \sup\{\, t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\gamma\mathcal{S}u, \gamma\mathcal{S}u, t) < 1\} + \sup\{\, t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\, (\delta\mathcal{S}^*\, u, \delta\mathcal{S}^*\, u, t) < 1\} \quad [\text{Since } \mathcal{S}^* = \mathcal{S}]$$

$$= \sup\{t \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\left((\gamma\mathcal{S} + \delta\mathcal{S}^*)u, (\gamma\mathcal{S} + \delta\mathcal{S}^*)u, t\right) < 1\}$$

$$= \langle(\gamma\mathcal{S} + \delta\mathcal{S}^*)u, (\gamma\mathcal{S} + \delta\mathcal{S}^*)u\rangle$$

$$= \mathcal{P}_{\mu,\nu}{}^2((\gamma\mathcal{S} + \delta\mathcal{S}^*)u, t)$$

$$\Rightarrow \mathcal{P}_{\mu,\nu}((\gamma\mathcal{S} + \delta\mathcal{S}^*)^* u, t) = \mathcal{P}_{\mu,\nu}((\gamma\mathcal{S} + \delta\mathcal{S}^*)u, t)$$

Therefore $\gamma\mathcal{S} + \delta\mathcal{S}^*$ is intuitionistic fuzzy normal.

IV. CONCLUSION

Here we conclude that the idea of Intuitionistic Fuzzy Normal Operator (IFN-Operator) in IFH-Space is moderately new. We endeavoured to prove a few properties of Intuitionistic Fuzzy Normal Operator in Intuitionistic Fuzzy Hilbert Space. By the consequences of this paper analysts can maturate Intuitionistic Fuzzy Functional Analysis.

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