Fuzzy $\alpha - \psi^*$-Irreducible Spaces

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Abstract: In this paper, the concept of $\psi^*$ operator on a family of fuzzy $\alpha$-open sets in a fuzzy topological space is introduced. Also, the concepts of fuzzy $\alpha - \psi^*$-irreducible spaces, fuzzy $\alpha - \psi^*$-generic sets and fuzzy $\alpha - \psi^*$-quasi-Sober spaces are initiated and some properties are discussed.

Keywords : operator on $Fac(X, \tau)$, fuzzy $\alpha - \psi^*$-generic sets , fuzzy $\alpha - \psi^*$-irreducible spaces, fuzzy $\alpha - \psi^*$-quasi-Sober spaces

I. INTRODUCTION

L.A. Zadeh[8] initiated fuzzy set in 1965. In 1968, Chang [2] characterized fuzzy topological space. Njastad [5] introduced $\alpha$-open sets. In the same sprit Bin Shahna [1] defined fuzzy $\alpha$-open sets and fuzzy $\alpha$-closed sets. The idea of an irreducible or hyperconnected topological space has been studied by T. Thompson. In this paper, the concept of $\psi^*$ operator on a family of fuzzy $\alpha$-open sets in a fts is introduced. Also, the concepts of fuzzy $\alpha - \psi^*$-irreducible spaces, fuzzy $\alpha - \psi^*$-generic sets and fuzzy $\alpha - \psi^*$-quasi-Sober spaces are initiated and some properties are discussed.

II. PRELIMINARIES

This section contains basic definitions and preliminary results needed for this paper.

Definition 2.1 [4] Let $X$ be a topological space.

(i) $X$ is irreducible, if $X \neq \emptyset$, and whenever $X = Z_1 \cup Z_2$ with $Z_i$ closed, $X = Z_1$ or $X = Z_2$.

(ii) $Z \subset X$ is an irreducible component of $X$ if $Z$ is a maximal irreducible subset of $X$.

Definition 2.2 [3] A subset $F \neq \emptyset$ of a topological space is irreducible if, $F \subseteq A \cup B$ where $A$ and $B$ then $F \subseteq A$ or $F \subseteq B$.

Definition 2.3 [4] A topological space $X$ is said to be quasi-sober if for every irreducible closed subset has a generic point.

Definition 2.4 [6] A fuzzy set $\mu_X$ is quasi-coincident with the fuzzy set $\mu_Y$ if $\exists x \in X$ such that $\mu_X(x) + \mu_Y(x) > 1$.

Definition 2.5 [7] A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called fuzzy dense if there exists no fuzzy closed set $\mu$ in $(X, T)$ such that $\lambda \preceq \mu$ and $cl(\lambda) = cl(\mu)$.

III. FUZZY $\alpha - \psi^*$-IRREDUCIBLE SPACES

Throughout this paper, fuzzy topological space is shortly denoted by fts. Then $Fac(X, \tau)$, $Fac(X, \tau)$ and $FP$ (X) denote set of all fuzzy $\alpha$-open sets, fuzzy $\alpha$-closed sets in $(X, \tau)$ and fuzzy points over $X$ respectively.

Definition 3.1 Let $(X, \tau)$ be a fts. A fuzzy operator $\psi^*$: $Fac(X, \tau) \rightarrow I^X$ is defined as, if for each $\mu \in Fac(O, X, \tau)$ with $\mu \neq 0_X$, $F \cap (\mu) \leq \psi^*(\mu)$ and $\psi^*(0_X) = 0_X$.

Remark 3.1 It is easy to check that some examples of fuzzy operators on $Fac(O, X, \tau)$ are the well known fuzzy operators viz. $F \cap$, $F \cup(F \cap)$, $F \cap(F \cap)$, $F \cap(F \cap(F \cap))$ and $F \cap(F \cap(F \cap(F \cap)))$.

Definition 3.2 Let $(X, \tau)$ be a fts and $\psi^*$ be a fuzzy operator on $Fac(O, X, \tau)$. Then any $\mu \in Fac(O, X, \tau)$ is called fuzzy $\alpha - \psi^*$-open if $\mu \preceq \psi^*(\mu)$. Then $1_X - \mu$ is called fuzzy $\alpha - \psi^*$-closed.

Notation 3.1 The family of all fuzzy $\alpha - \psi^*$-open (resp. fuzzy $\alpha - \psi^*$-closed) sets in $(X, \tau)$ is notated by $F\alpha - \psi^*O(X, \tau)$ (resp. $F\alpha - \psi^*C(X, \tau)$).

Definition 3.3 For any $\mu \in I^X$ in a fts $(X, \tau)$ and $\psi^*$ be a fuzzy operator on $Fac(O, X, \tau)$, the fuzzy $\alpha - \psi^*$-interior of $\mu$ (briefly, $F\alpha - \psi^*int(\mu)$) is defined by $F\alpha - \psi^*int(\mu) = \cap\{\sigma: \sigma \leq \mu \text{ and } \sigma \in F\alpha - \psi^*O(X, \tau)\}$.

Definition 3.4 For any $\mu \in I^X$ in a fts $(X, \tau)$ and $\psi^*$ be a fuzzy operator on $Fac(O, X, \tau)$, the fuzzy $\alpha - \psi^*$-closure of $\mu$ (briefly, $F\alpha - \psi^*cl(\mu)$) is defined by $F\alpha - \psi^*cl(\mu) = \cup\{\sigma: \sigma \geq \mu \text{ and } \sigma \in F\alpha - \psi^*C(X, \tau)\}$.

Definition 3.5 Any fts $(X, \tau)$ is said to be a fuzzy $\alpha - \psi^*$-irreducible space, where $\psi^*$ is a fuzzy operator on $Fac(O, X, \tau)$ if, for any $\mu_1, \mu_2 \in F\alpha - \psi^*O(X, \tau)$ where $\mu_1 \neq 0_X, \mu_2 \neq 0_X$ and $\mu_1 \cup \mu_2$.

Definition 3.6 Any $\lambda \in I^X$ in a fts $(X, \tau)$ is said to be fuzzy $\alpha - \psi^*$-irreducible,
where \( \psi \) is a fuzzy operator on \( FaO(X, r) \), if \( \lambda \neq 0_X \) and \( \lambda \leq (\mu_1 \vee \mu_2) \) where \( \mu_1, \mu_2 \in Fa\alpha - \psi \ast C(X, r) \), then either \( \lambda \leq \mu_1 \) or \( \lambda \leq \mu_2 \). Then the set of all fuzzy \( \alpha - \psi \ast \)-irreducible sets is noted by \( Fa\alpha - \psi \ast I(X, r) \).

**Definition 3.7** Any \( \lambda \in I^X \) is called a fuzzy \( \alpha - \psi \ast \)-maximal irreducible set of \( ftS (X, r) \), where \( \psi \ast \) is a fuzzy operator on \( FaO(X, r) \), if there is no \( \mu \in Fa\alpha - \psi \ast I(X, r) \) such that \( \mu > \lambda \). Then collection of all fuzzy \( \alpha - \psi \ast \)-maximal irreducible sets is denoted by \( Fa\alpha - \psi \ast MI(X, r) \).

**Definition 3.8** Let \( \psi \ast \) be a fuzzy operator on \( FaO(X_1, r_1) \) and \( FaO(X_2, r_2) \) in a ftss \( (X_1, r_1) \) and \( (X_2, r_2) \) respectively. Any function \( f : (X_1, r_1) \rightarrow (X_2, r_2) \) is said to be a fuzzy \( \alpha - \psi \ast \)-continuous function if for every \( \mu \in Fa\alpha - \psi \ast O(X_2, r_2) \),
\[
f^{-1}(\mu) \in Fa\alpha - \psi \ast O(X_1, r_1) .
\]

**Proposition 3.1** Let \( \psi \ast \) be a fuzzy operator on \( FaO(X_1, r_1) \) and \( FaO(X_2, r_2) \) in a ftss \( (X_1, r_1) \) and \( (X_2, r_2) \) respectively. Let \( f : (X_1, r_1) \rightarrow (X_2, r_2) \) be a bijective and fuzzy \( \alpha - \psi \ast \)-continuous function. If \( \lambda \in Fa\alpha - \psi \ast I(X_1, r_1) \) then \( f(\lambda) \in Fa\alpha - \psi \ast I(X_2, r_2) \).

**Proof.** Let \( \mu_1, \mu_2 \in Fa\alpha - \psi \ast C(X_2, r_2) \) such that \( f(\lambda) \leq (\mu_1 \vee \mu_2) \). Then
\[
f^{-1}(f(\lambda)) \leq f^{-1}((\mu_1 \vee \mu_2)) ,
\]
which implies that \( \lambda \leq f^{-1}(\mu_1 \vee \mu_2) \). Then \( f^{-1}(f(\lambda)) = \lambda \), since \( f \) is one-one. Hence \( f(\lambda) \leq \mu_1 \vee \mu_2 \) and hence \( f(\lambda) \leq \mu_2 \) as \( f \) is onto. Therefore \( f(\lambda) \in Fa\alpha - \psi \ast I(X_2, r_2) \).

**Remark 3.2** For any \( \lambda \in I^X \) in a ftss \( (X, r) \) and \( \psi \ast \) be a fuzzy operator on \( FaO(X, r) \), \( \lambda \leq Fa\alpha - \psi \ast cl(\lambda) \).

**Proposition 3.2** If \( \lambda \in Fa\alpha - \psi \ast I(X, r) \) in ftss \( (X, r) \) and \( \psi \ast \) be a fuzzy operator on \( FaO(X, r) \), then \( Fa\alpha - \psi \ast cl(\lambda) \in Fa\alpha - \psi \ast I(X, r) \).

**Proof.** Assume that \( \lambda \in I^X \) and \( Fa\alpha - \psi \ast cl(\lambda) \leq \mu_1 \vee \mu_2 \) where \( \mu_1, \mu_2 \in Fa\alpha - \psi \ast C(X, r) \). Since \( \lambda \leq Fa\alpha - \psi \ast cl(\lambda) \) and \( \lambda \in Fa\alpha - \psi \ast I(X, r) \), \( \lambda \leq \mu_1 \) or \( \lambda \leq \mu_2 \). Then \( Fa\alpha - \psi \ast cl(\lambda) \leq \mu_1 \) or \( Fa\alpha - \psi \ast cl(\lambda) \leq \mu_2 \).

Therefore \( Fa\alpha - \psi \ast cl(\lambda) \in Fa\alpha - \psi \ast I(X, r) \).

**Proposition 3.3** If \( \lambda \in Fa\alpha - \psi \ast MI(X, r) \), then \( \lambda \in Fa\alpha - \psi \ast C(X, r) \).

**Proof.** Let \( \lambda \in Fa\alpha - \psi \ast MI(X, r) \). Then, there is no \( \mu \in Fa\alpha - \psi \ast I(X, r) \) such that \( \mu > \lambda \). By Proposition 3.2, \( Fa\alpha - \psi \ast cl(\lambda) \in Fa\alpha - \psi \ast I(X, r) \).

For any \( \lambda \in I^X \) and \( \psi \ast \) be a fuzzy operator on \( FaO(X, r) \) in ftss \( (X, r) \). If \( \lambda \in I^X \) be a fuzzy \( \alpha - \psi \ast \)-irreducible closed set. Any fuzzy set \( \mu \in I^X \) with \( \mu \leq \lambda \), is said to be a fuzzy \( \alpha - \psi \ast \)-generic set of \( \lambda \) if \( Fa\alpha - \psi \ast cl(\mu) = \lambda \).

**IV. FUZZY \( \alpha - \psi \ast \)-QUASI-SOBER SPACE**

**Definition 4.1** Let \( \psi \ast \) be a fuzzy operator on \( FaO(X, r) \) in ftss \( (X, r) \). Let \( \lambda \in I^X \) be a fuzzy \( \alpha - \psi \ast \)-irreducible closed set. Any fuzzy set \( \mu \in I^X \) with \( \mu \leq \lambda \), is said to be a fuzzy \( \alpha - \psi \ast \)-generic set of \( \lambda \) if \( Fa\alpha - \psi \ast cl(\mu) = \lambda \).
Definition 4.2 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). Then \((X, \tau)\) is said to be a fuzzy \(\alpha - \psi^*\) quasi-Sober, if for every fuzzy \(\alpha - \psi^*\) irreducible closed set there exists a fuzzy \(\alpha - \psi^*\) generic set.

Remark 4.1 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). Let \( Y \subset X \) and \((Y, \tau_Y)\) be a fuzzy subspace of \((X, \tau)\). Then \( \psi \) is a fuzzy operator on \( \text{FaoO}(Y, \tau_Y) \).

Definition 4.3 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). If \( \lambda, \mu \in I^X \) with \( \lambda \mu \) there exists a \( \delta \in \text{FaoO} - \psi^* C(X, \tau) \) such that either \( \mu \leq \delta \), \( \lambda \delta \) or \( \lambda \leq \delta \), \( \mu \delta \) then \((X, \tau)\) is called fuzzy \(\alpha - \psi^*\) Kolmogorov.

Proposition 4.1 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). Let \( Y \subset X \) and \((Y, \tau_Y)\) be a fuzzy subspace of \((X, \tau)\). If \((X, \tau)\) is fuzzy \(\alpha - \psi^*\) Kolmogorov, then \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) Kolmogorov.

Proof. Let \( \lambda, \mu \in I^X \) with \( \lambda \mu \). Then there exists a \( \delta \in \text{FaoO} - \psi^* C(X, \tau) \) such that \( \mu \leq \delta \), \( \lambda \delta \) or \( \lambda \leq \delta \), \( \mu \delta \). Let \( \lambda^I \), \( \mu^I \) \( \in I^Y \) with \( \lambda^I \), \( \mu^I \). Since \( \delta \in \text{FaoO} - \psi^* C(X, \tau) \), \( \delta^I \), \( \mu^I \in \text{FaoO} - \psi^* C(Y, \tau_Y) \). Also, since \( \lambda^I \leq \lambda \), \( \lambda \leq \delta \), \( \mu^I \leq \mu, \mu \leq \delta \), \( \lambda^I \leq \delta^I \), \( \mu^I \leq \delta^I \) with \( \mu^I \), \( \delta^I \). Hence \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) Kolmogorov.

Notation 4.1 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \sigma) \) and \( \text{FaoO}(Y, \sigma) \) where \((X, \tau)\) and \((Y, \sigma)\) are any two fss. For any \( \lambda, \mu \in I^X \), \( \delta \in I^X \), \( \text{FaoO} - \psi^* \) with respect to \((X, \tau)\) and \((Y, \sigma)\) are denoted by \( \text{FaoO} - \psi^* \) and \( \text{FaoO} - \psi^* \) respectively. Then the collection of all fuzzy \(\alpha - \psi^*\) irreducible closed in \((X, \tau)\) id denoted by \( \text{FaoO} - \psi^* IC(X, \tau) \).

Proposition 4.2 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). Let \( Y \subset X \) and \((Y, \tau_Y)\) be a fuzzy \(\alpha - \psi^*\) closed subspace of \((X, \tau)\). If \((X, \tau)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober, then \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober.

Proof. Let \( \lambda \in \text{FaoO} - \psi^* IC(Y, \tau_Y) \). Then \( \lambda \in \text{FaoO} - \psi^* IC(X, \tau) \). Since \((X, \tau)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober, there exists a fuzzy \(\alpha - \psi^*\) generic set \( \mu \in I^X \) such that \( \text{FaoO} - \psi^* IC(\mu) \). Thus \( \text{FaoO} - \psi^* IC(\mu) \). This implies that \( \text{FaoO} - \psi^* IC(\mu) \). Therefore \( \text{FaoO} - \psi^* IC(\mu) \). Hence \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober.

Proposition 4.3 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X_1, \tau_1) \) and \( \text{FaoO}(X_2, \tau_2) \) where \((X_1, \tau_1)\) and \((X_2, \tau_2)\) are any two fss. Let \( f: (X_1, \tau_1) \rightarrow (X_2, \tau_2) \) be a injective and fuzzy \(\alpha - \psi^*\) continuous function. If \( \lambda \in \text{FaoO} - \psi^* IC(X_1, \tau_1) \), then \( \text{FaoO} - \psi^* IC(f(\lambda)) \in \text{FaoO} - \psi^* IC(X_2, \tau_2) \).

Proof. Let \( \lambda \in \text{FaoO} - \psi^* IC(X_1, \tau_1) \). Let \( \mu_1, \mu_2 \in \text{FaoO} - \psi^* C(X_2, \tau_2) \). Since \( \text{FaoO} - \psi^* cl(\lambda) \leq \mu_1 \land \mu_2 \). Since \( f(\lambda) \leq \text{FaoO} - \psi^* cl(f(\lambda)) \), \( f(\lambda) \leq \mu_1 \land \mu_2 \). As \( f \) is injective, \( \lambda \leq \text{FaoO} - \psi^* cl(f(\lambda)) \). Then \( \lambda \leq \text{FaoO} - \psi^* cl(f(\lambda)) \). Since \( f \) is a fuzzy \(\alpha - \psi^*\) continuous function, \( \text{FaoO} - \psi^* cl(f(\lambda)) \leq \mu_1 \land \mu_2 \). Hence \( \text{FaoO} - \psi^* cl(f(\lambda)) \in \text{FaoO} - \psi^* IC(X_2, \tau_2) \).

Proposition 4.4 Let \( \psi \) be a fuzzy operator on \( \text{FaoO}(X, \tau) \) in a fts \((X, \tau)\). Let \( Y \subset X \) and \((Y, \tau_Y)\) be a fuzzy \(\alpha - \psi^*\) open subspace of \((X, \tau)\). If \((X, \tau)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober, then \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober.

Proof. Let \( \mu \in \text{FaoO} - \psi^* IC(Y, \tau_Y) \). Then \( \mu \in \text{FaoO} - \psi^* IC(X, \tau) \). Since \((X, \tau)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober, there exists a fuzzy \(\alpha - \psi^*\) generic set \( \lambda \in I^X \) such that \( \text{FaoO} - \psi^* cl(\lambda) = \mu \). Then \( \text{FaoO} - \psi^* cl(\lambda) = \mu \). Since \( \mu \in \text{FaoO} - \psi^* IC(Y, \tau_Y) \), \( \text{FaoO} - \psi^* cl(\lambda) = \mu \) and \( \lambda \leq \mu \). Hence \((Y, \tau_Y)\) is fuzzy \(\alpha - \psi^*\) quasi-Sober.

V. CONCLUSION

In this paper, we explored \( \psi^* \) operator on a family of fuzzy \(\alpha\) open sets in a fuzzy topological spaces. We can extend \( \psi^* \) operator on fuzzy homotopy and fuzzy spectral spaces.

REFERENCES

AUTHORS PROFILE

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