Generalized Tribonacci Function and Tribonacci Numbers

Krishna Kumar Sharma

Abstract: *In the language of mathematics, sequence is considered to be list of numbers arranged in a particular way. A lot of sequences have been minutely studied till date. One of the most conspicuous among them is Fibonacci sequence. It is the sequence, which can be found by adding two previous terms, where the initial conditions are 0 and 1. In a similar manner, Tribonacci sequence is also obtained by adding three previous consecutive terms. In this research paper, we introduce Tribonacci function* $\phi: \mathbb{R} \to \mathbb{R}$ *with period s (positive integer) such that*

$$\phi(y+3s) = \phi(y+2s) + \phi(y+s) + \phi(y), \forall\, y \in \mathbb{R}$$

We construct some of the interesting properties, using induction technique, ϕ *– odd function and* ϕ *- even function for Tribonacci function with period s. In the present research article we also show that* $lim_{y\to\infty} \frac{\phi(y+s)}{\phi(y)}$ *exists.*

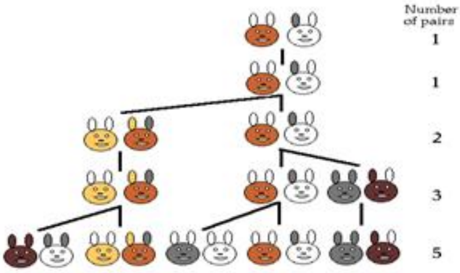
Keywords: *Fibonacci Numbers, Tribonacci Function, Tribonacci Numbers.*

I. INTRODUCTION

The Fibonacci numbers [4, 7] were first innovated by Leonardo Pisano. With passage of time his nickname came to be known as Fibonacci. The Fibonacci sequence has a pattern in which each term is the summation of the two previous numbers.

$$F_{n+1} = F_n + F_{n-1}, n \geq 1, F_1 = F_2 = 1 \qquad (1.1)$$

This popular mathematical sequence is commonly known as the Fibonacci sequence. It is famous for having many enigmatic patterns within it. One of the most well-known experiments dealing with the Fibonacci sequence is the experiment done on rabbits, which is known as "The Rabbit Problem".



We easily can see the growth of rabbits at the start of each month 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 144…

Fibonacci sequence is omnipresent in nature too, inform of pretty flowers. On the stigma of a sunflower, the seeds are packed in a specific manner so that they follow the Fibonacci sequence. This spiral obstructs the seed of the sunflower from flowing out, thus helping them to survive. The petals of innumerable flowers and many other plants can also be connected to the Fibonacci sequence in the form that they give birth to new petals.

II. PRELIMINARIES

In the research article [3], J S Hahn et al. defined the Fibonacci function on real field.

$$\phi(y+2) = \phi(y+1) + \phi(y) \qquad (2.1)$$

In this article, they also obtained the limit value of Fibonacci function. That is approximately1.618.Recently, Sharma [5] developed the generalized Fibonacci function using Binet's formula and induction technique. In this research paper, the relation between generalized Fibonacci function and generalized Fibonacci numbers was constructed and he also proposed the concept of generalized Fibonacci functions with period s using the definition of ϕ -even and ϕ -odd functions.

III. TRIBONACCI SEQUENCE

Definition 3.1: Tribonacci sequence is a generalization of the Fibonacci sequence, in which each term is the sum of three previous terms.

$$Z_n = Z_{n-1} + Z_{n-2} + Z_{n-3}, \forall\, n \geq 3, Z_0 = 0, Z_1 = 0, Z_2 = 1$$

First few terms of Tribonacci sequence are
0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274…

Definition 3.2: Binet form of Tribonacci Sequence

$$Z_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\alpha^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

Where α, β, γ are the roots of $y^3 - y^2 - y - 1 = 0$, α is also called the Tribonacci constant and its value is

$$\alpha = \frac{1+\sqrt[3]{19+\sqrt[3]{33}}+\sqrt[3]{19-\sqrt[3]{33}}}{3} = 1.8392$$

IV. TRIBONACCI FUNCTION

Definition 4.1: A function $\mathbf{f}: \mathbb{R} \to \mathbb{R}$ is said to be Tribonacci function if

$$\mathbf{f}(y+3) = \mathbf{f}(y+2) + \mathbf{f}(y+1) + \mathbf{f}(y), \forall y \in \mathbb{R} \qquad (4.1)$$

Definition 4.2: Tribonacci function with period s

Let s be positive integer, a function $\phi: \mathbb{R} \to \mathbb{R}$ is called Tribonacci function with period s, if

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$$\phi(y+3s) = \phi(y+2s) + \phi(y+s) + \phi(y), y \in \mathbb{R} \quad (4.2)$$

Example 4.3: let ϕ (y) = $k^{\frac{y}{s}}$ be Tribonacci function with period $s \in N$

When $k > 0$, it is given that

$$k^{\frac{y}{s}+3} = k^{\frac{y}{s}+2} + k^{\frac{y}{s}+1} + k^{\frac{y}{s}}$$

$$\Rightarrow k^3 = k^2 + k + 1$$

$$\Rightarrow k^3 - k^2 - k - 1 = 0$$

$$\Rightarrow k = \frac{1 + \sqrt[3]{19 + \sqrt[3]{33}} + \sqrt[3]{19 - \sqrt[3]{33}}}{3}$$

$$= 1.8392 = \alpha$$

Hence ϕ (y) = $(\alpha)^{\frac{y}{s}}$

Proposition 4.4: let $\phi: \mathbb{R} \to \mathbb{R}$ be Tribonacci function with period $s \in N$, suppose that if ϕ is differentiable, then ϕ' is also a Tribonacci function with period s.

Let $y \in \mathbb{R}$, since $\phi(y+3s) = \phi(y+2s) + \phi(y+s) + \phi(y)$

$$\Rightarrow \phi'(y+3s) = \phi'(y+2s) + \phi'(y+s) + \phi'(y)$$

Proposition 4.5: If $\phi : \mathbb{R} \to \mathbb{R}$ be Tribonacci function with period $k \in N$ and we define $g_t(y) = \phi(y+t)$, $\forall\, y \in \mathbb{R}, t \in \mathbb{R}$ then $g_t(y)$ is also a Tribonacci function with period s.

Let $y \in \mathbb{R}$, then $g_t(y+3s) = \phi(y+3s+t)$

$$= \phi(y+2s+t) + \phi(y+s+t) + \phi(y+t)$$

$$= g_t(y+2s) + g_t(y+s) + g_t(y)$$

Example 4.6: Let $s \in N$ and $t \in R$, we define $g_t: \mathbb{R} \to \mathbb{R}$ by $g_t(y) = \alpha^{\frac{(y+t)}{s}}, \forall y \in \mathbb{R}$, then g_t is a Tribonacci function with period s.

Theorem 4.7: Let $\phi(y)$ be Tribonacci function and let $\{Z_n\}, \{Z'_n\}$ and $\{Z''_n\}$ be sequence of Tribonacci numbers with $Z_0 = 0, Z_1 = 1, Z_2 = 2, Z_3 = 3$ and $Z'_0 = 0, Z'_1 = 0, Z'_2 = 1, Z'_3 = 1$ and $Z''_0 = 1, Z''_1 = -1, Z''_2 = 1, Z''_3 = 1$,then

$\phi(y+ns) = Z'_n\phi(y+2s) + Z_{n-2}\phi(y+s) + Z''_n\phi(y)$, for any $y \in \mathbb{R}$ and an integer $n \geq 3$ such that

$Z_n = Z'_n + Z'_{n+1}$ and $Z''_n = Z'_{n-1}$

Proof: if n = 3, then

$$\phi(y+3s) = \phi(y+2s) + \phi(y+s) + \phi(y)$$

$$= \phi(y+2s) + Z_1\phi(y+s) + Z''_3\phi(y)$$

If n = 4, then

$$\phi(y+4s) = \phi(y+3s) + \phi(y+2s) + \phi(y+s)$$

$$= Z'_3\phi(y+2s) + Z_1\phi(y+s) + Z''_3\phi(y) + \phi(y+2s) + 0.\phi(y+s) + 0.\phi(y) + 0\phi(y+2s) + \phi(y+s) + 0\phi(y)$$

$$= (Z'_3 + Z'_2 + Z'_1)\phi(y+2s) + (Z_1 + Z'_0 + Z'_{-1})\phi(y+s) + (Z''_3 + Z''_2 + Z''_1)\phi(y)$$

$$= Z'_4\phi(y+2s) + Z_2\phi(y+s) + Z''_4\phi(y)$$

If n = 5, then

$$\phi(y+5s) = \phi(y+4s) + \phi(y+3s) + \phi(y+2s)$$

$$= Z'_4\phi(y+2s) + Z_2\phi(y+s) + Z''_4\phi(y) +$$

$$Z'_3\,\phi(y+2s) + Z_1\phi(y+s) + T''_3\phi(y) + \phi(y+2s)$$

$$= (Z'_4 + Z'_3 + Z'_2)\phi(y+2s) + (\mathbf{Z}_2 + Z_1 + Z_0)\phi(y+s) + (Z''_4 + Z''_3 + Z''_2)\phi(y)$$

$$= Z'_5\phi(y+2s) + Z_3\phi(y+s) + \mathbf{Z}''_5\phi(y)$$

Now we assume that it is true for n = n + 1and n + 2, then

$\phi\{y+(n+3)s\} = \phi\{y+(n+2)s\} + \phi\{y+(n+1)s\} + \phi\{y+ns\}$

$= Z'_{n+2}\phi(y+2s) + Z_n\phi(y+s) + Z''_{n+2}\phi(y) + Z'_{n+1}\phi(y+$ 2s+Zn−1ϕy+s+Zn+1′′ϕ(y) + $Z'_n\phi(y+2s) + Z_{n-2}\phi(y+s) + Z''_n\phi(y)$

$$= [Z'_{n+2} + Z'_{n+1} + Z'_n]\phi(y+2s) + [Z_n + Z_{n-1} + \mathbf{Z}_{n-2}]\phi(y+s) + [Z''_{n+2} + Z''_{n+1} + Z''_n]\phi(y)$$

$$= Z'_{n+3}\phi(y+2s) + Z_{n+1}\phi(y+s) + Z''_{n+3}\phi(y)$$

It proves the theorem.

Corollary4.8: Let $\{Z'_n\}$ be the sequence of Tribonacci numbers with $Z'_1 = 0, Z'_2 = 1, Z'_3 = 1$ and m >1 is the root of equation $y^3 - y^2 - y - 1 = 0$,then

$$m^n = \mathbf{Z'_n}m^2 + \{Z'_{n-2} + Z'_{n-1}\}m + Z'_{n-1}.$$

By previous result $\phi(y+ns) = Z'_n\phi(y+2s) + Z_{n-2}\phi(y+$ s+Zn′′ϕ(y) where Zn′′=Zn−1′

Then $\phi(y+ns) = Z'_n\phi(y+2s) + Z_{n-2}\phi(y+s) + Z'_{n-1}\phi(y)$

Here $\phi(y)$ is Tribonacci function with period s, hence

$$m^{\frac{y+ns}{s}} = Z'_n m^{\frac{y+2s}{s}} + \{Z'_{n-2} + Z'_{n-1}\}m^{\frac{y+s}{s}} + Z'_{n-1}m^{\frac{y}{s}}$$

$$\Rightarrow m^n = Z'_n m^2 + \{Z'_{n-2} + Z'_{n-1}\}m + Z'_{n-1}.$$

Example 4.9: Let $\{R_n\}$,$\{S_n\}$ and $\{T_n\}$ be Tribonacci sequences, we define a function $\phi(y) = R_{[y]} + S_{[y]}t + T_{[y]}\,t^2$

Where $t = y - [y] \in [0,1]$,then

$$\phi(y+3s) = R_{[y]+3s} + S_{[y]+3s}t + T_{[y]+3s}\,t^2$$

$$= [R_{[y]+2s} + R_{[y]+s} + R_{[y]}] + [S_{[y]+2s} + S_{[y]+s} + S_{[y]}]t + [T_{[y]+2s} + T_{[y]+s} + T_{[y]}]t^2$$

$$= [R_{[y]+2s} + S_{[y]+2s}t + T_{[y]+2s}t^2] + [R_{[y]+s} + S_{[y]+s}t + T_{[y]+s}t^2] + [R_{[y]} + S_{[y]}t + T_{[y]}t^2]$$

$= \phi(y+2s) + \phi(y+s) + \phi(y)$

Hence ϕ is a Tribonacci function with period s.

Proposition 4.10: let ψ be Tribonacci function with period s. Define $f(y) = \psi(y+t+t^2)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}$

$$f(y+3s) = \psi(y+3s+t+t^2)$$

$$= \psi(y+2s+t+t^2) + \psi(y+s+t+t^2) + \psi(y+t+t^2)$$

$$= f(y+2s) + f(y+s) + f(y)$$

It shows that f is also a Tribonacci function.

Theorem 4.11: let $\{U_n\}$ be the Tribonacci sequence, then

$$U_{[y+ns]} = Z'_n U_{[y]+2s} + \{Z'_{n-2} + Z'_{n-1}\}U_{[y]+s} + Z'_{n-1}U_{[y]}$$

$$U_{[y+ns]-s} = Z'_n U_{[y]+s} + \{Z'_{n-2} + Z'_{n-1}\}U_{[y]} + Z'_{n-1}U_{[y]-s}$$

$$U_{[y+ns]-2s} = Z'_n U_{[y]} + \{Z'_{n-2} + Z'_{n-1}\}U_{[y]-s} + Z'_{n-1}U_{[y]-2s}$$

IV. ODD FUNCTION AND EVEN FUNCTION WITH PERIOD s

In 2013, Sroysang [1] defined odd function and even function in his way.

Definition 5.1: let s be a positive integer and $\psi: \mathbb{R} \to \mathbb{R}$, if $\psi h \equiv 0$, where $h: \mathbb{R} \to \mathbb{R}$ is continuous, then $h = 0$.the function ψ is said to be an odd function with period s if $\psi(y+s) = -\psi(y), \forall y \in \mathbb{R}$ and even function if $\psi(y+s)=\psi(y)$

Example 5.2: If $\psi(y) = \sin\pi y, \forall\, y \in \mathbb{R}$

$$\psi(y+s) = \sin\pi(y+s)$$

$$= -\sin\pi y$$

$$= -\psi(y)$$

This implies that $\psi(y) = \sin\pi y$ is an odd function with period s.

Example5.3: If $\psi(y) = y - [y]$, then $\psi(y)h(y) \equiv 0 \Rightarrow h(y) \equiv 0$

We have $\psi(y+s) = (y+s) - [y+s]$

$$= (y+s) - \{[y]+s\}$$

$$= y - [y]$$

$$= \psi(y)$$

It implies that $\psi(y) = y - [y]$ is an even function.

Theorem 5.4: Let ϕ (y) = $\psi(y)g(y)$ be a function, where $\psi(y)$ is ϕ - even function with period s and g(x) is continuous function. Then ϕ (y) is a Tribonacci function with period s, iff g(y) is Tribonacci function with period s.

Proof: Let ϕ (y) is Tribonacci function with period s, then

$$\psi(y)g(y+3s) = \psi(y+3s)g(y+3s) = \phi(y+3s)$$

$$= \phi(y+2s) + \phi(y+s) + \phi(y)$$

Now $\psi(y)g(y+3s) = \psi(y)\{g(y+2s) + g(y+s) + g(y)\}$

Implies that

$$\psi(y)\{g(y+3s) - g(y+2s) - g(y+s) - g(y)\} \equiv 0$$

and $g(y+3s) - g(y+2s) - g(y+s) - g(y) \equiv 0$

$\Rightarrow g(y+3s) = g(y+2s) + g(y+s) + g(y)$

It shows that g(y) is Tribonacci function with period s.

Conversely, now assume that g(y) is Tribonacci function with period s, then

$$g(y+3s) = g(y+2s) + g(y+s) + g(y)$$

$$\Rightarrow \phi(y+3s) = \psi(y)g(y+3s)$$

$$= \psi(y)\{g(y+2s) + g(y+s) + g(y)\}$$

$$= \psi(y+2s)g(y+2s) + \psi(y+s)g(y+s) + \psi(y)g(y)$$

$$= \phi(y+2s) + \phi(y+s) + \phi(y)$$

$\Rightarrow$ ϕ is also a Tribonacci function with period s.

Theorem 5.5: If ϕ (y) is a Tribonacci function with period s, then $\lim_{\phi\to\infty} \frac{\phi(y+s)}{\phi(y)}$ exists.

Proof: From the result

$$\phi(y+ns) = Z'_n\phi(y+2s) + Z_{n-2}\phi(y+s) + Z''_n\phi(y)$$

$$= Z'_n f(y+2s) + \{Z'_{n-2} + Z'_{n-1}\}\phi(y+s) + Z'_{n-1}\phi(y)$$

For any $y \in R$, $\exists\, n \in Z$ such that $r = y + ns$

$$\lim_{r\to\infty} \frac{\phi(r+s)}{\phi(r)} = \lim_{n\to\infty} \frac{\phi(y+ns+s)}{\phi(y+ns)}$$

$$= \lim_{n\to\infty} \frac{\phi\{(y+(n+1)s\}}{\phi(y+ns)}$$

$$= \lim_{n\to\infty} \frac{Z'_{n+1}\phi(y+2s) + \{Z'_{n-1} + Z'_n\}\phi(y+s) + \mathbf{Z}'_n\phi(y)}{Z'_n\phi(y+2s) + \{Z'_{n-2} + Z'_{n-1}\}\phi(y+s) + Z'_{n-1}\phi(y)}$$

$$= \lim_{n\to\infty} \frac{Z'_n}{Z'_{n-1}} \left[\frac{\frac{Z'_{n+1}}{Z'_n}\phi(y+2s) + \{\frac{Z'_{n-1}}{Z'_n} + 1\}\phi(y+s) + \phi(y)}{\frac{Z'_n}{Z'_{n-1}}\phi(y+2s) + \{\frac{Z'_{n-2}}{Z'_{n-1}} + 1\}\phi(y+s) + \phi(y)} \right]$$

$$= \alpha \left[\frac{\alpha\phi(y+2s) + \left\{\frac{1}{\alpha} + 1\right\}\phi(y+s) + \phi(y)}{\alpha\phi(y+2s) + \left\{\frac{1}{\alpha} + 1\right\}\phi(y+s) + \phi(y)} \right] = \alpha$$

Where $\alpha = \lim_{n\to\infty} \frac{Z_{n+1}}{Z_n}$ is a root of equation $y^3 - y^2 - y - 1 = 0$,for which $1 < \alpha < 2$.

We consider ϕ (y) > 0, ϕ (y + s) > 0, and ϕ (y + 2 s) > 0

We change $\lim_{y\to\infty} \frac{\phi(y+s)}{\phi(y)} \to \frac{\phi(\varepsilon+2n+s)}{\phi(\varepsilon+2n)}$,any real number y can be written in the form of $\varepsilon + 2n$ for some $\varepsilon \in R$ and $n \in N$

Then $\frac{\phi\{\varepsilon+2(n+s)+s\}}{\phi\{\varepsilon+2(n+s)\}} = \frac{\phi(\varepsilon+2n+3s)}{\phi(\varepsilon+2n+2s)}$

$$= \frac{\phi(\varepsilon+2n+2s)+\phi(\varepsilon+2n+s)+\phi(\varepsilon+2n)}{\phi(\varepsilon+2n+2s)}$$

$= 1+\frac{\phi(\varepsilon+2n+s)+\phi(\varepsilon+2n)}{\phi(\varepsilon+2n+2s)} < 2$, Because $\frac{\phi(\varepsilon+2n+s)+\phi(\varepsilon+2n)}{\phi(\varepsilon+2n+2s)} <$
1

Now we take two consecutive sequences

$$\frac{\phi\{\varepsilon+2(n+1)+s)+s\}}{\phi\{\varepsilon+2(n+1)+s)\}} - \frac{\phi\{\varepsilon+2(n+s)+s\}}{\phi\{\varepsilon+2(n+s)\}}$$

$$= \frac{\phi(\varepsilon+2n+2+3s)}{\phi(\varepsilon+2n+2+2s)} - \frac{\phi(\varepsilon+2n+3s)}{\phi(\varepsilon+2n+2s)}$$

$$= \frac{\phi(\varepsilon+2n+2+3s)\phi(\varepsilon+2n+2s)-\phi(\varepsilon+2n+3s)\phi(\varepsilon+2n+2+2s)}{\phi(\varepsilon+2n+2+2s)\phi(\varepsilon+2n+2s)}$$
≥ 0

$\Rightarrow$ Numerator is non - negative.

$\Rightarrow$ This given sequence is increasing monotonically.

By Monotone convergence theorem

We can conclude that $\lim_{n\to\infty}\frac{\phi\{\varepsilon+2(n+s)+s\}}{\phi\{\varepsilon+2(n+s)\}}$ exists.

$\Rightarrow \lim_{x\to\infty}\frac{\phi(y+s)}{\phi(y)}$ exists.

V. CONCLUSION

In this research note, several properties of Tribonacci Function have been constructed. Induction method was the important technique to obtain these results. These results provides the new direction for further research. This research can also be framed for higher orders in future.

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