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Approximate Solution of the Fuzzy Triangular Initial Value Problem with Different Fractional Operator

Moath Ali Alshorman, Nurnadiah Zamri, Asia Khalaf Albzeirat, Mohammed Ali

Abstract: This study aims to conduct a comparison regarding the process of solving the fuzzy triangular initial value problem (FTIVP). The series solution of this problem is acquired through the reproducing kernel theory (RKT), although there have been past studies on FTIVP, there is no specialist study to compare solutions for the definition of different fractional operator. The comparisons where located through the difference in the use of an operator in the process of solution by using Riemann-Liouville integral operator (RLIO) and then by using Caputo fractional derivative operator (CFDO). Algorithm was presented to validate the method of solution and to view the effect of changing the operators on the solution behaviour in the two cases. During this comparison, the effectiveness of RKT was cleared and the notion of difference between using RLIO and CFDO were fixedly identified. Applications: The results identified in this research pronounced active difference in the behavior of errors, CDFO variations, and the behavior of error in favour of RLIO.

Index Terms:: Fuzzy Triangular Initial Value Problem, Riemann-Liouville Integral, Caputo Fractional Derivative, Reproducing Kernel Theory

I. INTRODUCTION

The fuzzy fractional differential has been used for prescribing a lot of models in many fields like engineering, physics, chemistry and in the interpretation of many phenomena [1-5] in our lives. Considering the significance of this kind of equations which explain these applications and phenomena we find concern in the modernization and development of mathematical methods to solve it [6-13].

This reserch follows the general form of the FTIVP for fractional differential equation with Caputo fractional derivative:

$$D_{c,a}^{\alpha} y(z) = g(z, y(z)),\ y(z_0) = (a,b,c) = z_F \qquad (1)$$

Where $D_{c,[a,z]}^{\alpha} y(z)$ in the sense of Caputo derivatives, $0 < \alpha \le 1, z > a$ and $y(z_0) = (a,b,c)$ is a fuzzy initial value.

We organized this paper as follows. Primarily started by a simple preface about the FIVP. After that general and basic definitions of fractional calculus, few fuzzy definitions and few reproducing kernel definitions implicated are given in section 2. Algorithm of solution is given in Section 3. numerical example is shown in Section 4. Finally the conclusion of this research are presented in section 5.

II. GENERAL MAIN DEINITIONS

Key definitions will be reviewed in this section of the study, the fractional operator, fuzzy concept and reproducing kernel.

Definition 1. The fractional integer operator by Riemann - Liouville of order $\alpha > 0$ [14-18] are defined as next form

i. The left fractional integral for Riemann-Liouville are defined where $\alpha > 0$ is defined as next form

$$I_{RL,[a,z]}^{\alpha} y(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z-\tau)^{\alpha-1} y(\tau) d\tau \qquad (2)$$

ii. The right fractional integral for Riemann-Liouville is defined where $\alpha > 0$ is defined as next form

$$I_{RL,[z,b]}^{\alpha} y(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (\tau - z)^{\alpha-1} y(\tau) d\tau \qquad (3)$$

Definition 2. Caputo sense [19-21] was defined the fractional derivatives as the below form

i. The left fractional derivatives for Caputo where $\alpha \in (0,\ 1)$ is defined as the below form

$$D_{c,[a,z]}^{\alpha} y(z) = \frac{(1)}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_a^z (z-\tau)^{\lceil \alpha \rceil - \alpha - 1} y^{(\lceil \alpha \rceil)}(\tau)\, d\tau. \qquad (4)$$

The right fractional derivatives for Caputo where $\alpha \in (0,\ 1)$ is defined as the below form

$$D_{c,[z,b]}^{\alpha} y(z) = \frac{(1)}{\Gamma(\lceil \alpha \rceil - \alpha)} \int_z^b (\tau - z)^{\lceil \alpha \rceil - \alpha - 1} y^{(\lceil \alpha \rceil)}(\tau)\, d\tau. \qquad (5)$$

Where $y^{(\lceil \alpha \rceil)}(\zeta) = \dfrac{dy^{\lceil \alpha \rceil}(\tau)}{d\tau^{\lceil \alpha \rceil}}$ and $\lfloor \alpha \rfloor \le \alpha < \lceil \alpha \rceil, \alpha \in Z^+$.

Where as the symbol $\lceil\ \rceil$ mention for the closest integer number more than α and $\lfloor\ \rfloor$ mention for the closest integer number lees than α.

Some basic properties of RFIO and CFDO are listed in the following:

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Moath Ali Alshorman, Faculty of Informatics and Computing, University Sultan Zainal Abidin, Besut Campus, 22200 Besut, Terengganu, Malaysia.

Nurnadiah Zamri, Faculty of Informatics and Computing, University Sultan Zainal Abidin, Besut Campus, 22200 Besut, Terengganu, Malaysia.

Asia Khalaf Albzeirat Department of Mathematics, Mutah University, Mutah, Jordan.

Mohammed Ali Department of Mathematics, Jordan University of Science and Technology, Irbid 22110, Jordan.

i) $D^{\alpha}_{c,[a,z]} I^{\alpha}_{RL,[a,z]} y(z) = y(z)$ (6)

ii) $I^{\alpha}_{RL,[a,z]} D^{\alpha}_{c,[a,z]} y(z) = y(z) - \sum_{k=0}^{\lfloor \alpha \rfloor} y^{(k)}(0) \frac{z^k}{k!},$ (7)

$z > 0, \ \lfloor \alpha \rfloor \le \alpha < \lceil \alpha \rceil \in Z^{+}.$

We symbolize the real numbers group by R and the space of $n-$ dimensional fuzzy number by R^n_F where $u_F(z): R^n \to [0,1]$.

Definition 3. Where R is the group of real numbers and R^n_F the space of $n-$ dimensional fuzzy number, let $u_F(z) \in R^n_F$ and $r \in [0,1]$. The $r-$cut off $u_F(z)$ is the crisp set $[u_F(z)]$ that contains all elements with degree in $u_F(z)$ either one greater than or equal to α, that is;

$[u_F(z)]^r = \{z \in R : u_F(z) \ge r\}$

For fuzzy number $u_F(z)$, its $r-$cut is closed and bounded interval in R and we denoted them as;

$[u_F(z)]^r = [u_{1,1r}(z), u_{1,2r}(z)]$

Where, $u_{1,1r} = \min\{z : z \in [u_F(z)]^r\}$ and

$u_{1,2r} = \max\{z : z \in [u_F(z)]^r\}$ for each $r \in [0,1]$.

For more details, see [26-27].

Definition 4. Triangular fuzzy number (TFN):

$u_F(z) \in R_F, u_F$ is called TFN if its function has next form;

$$u_F(Z) = \begin{cases} 0, & z < a \\ \dfrac{z-a}{b-a}, & a \le z \le b \\ \dfrac{c-z}{c-b}, & b \le z \le c \\ 0, & z > c \end{cases} \quad (8)$$

And its $r-$cut is as follows:

$[u]^r = [a + r(b-a), c - r(c-b)]$, for $r \in [0,1]$. For more details, see [26-27]).

Definition 5. Functions under $r-$cut:

Let be a fuzzy $y(z)$ valued with function $y(z): [a,b] \to R_F$, where $y(z)$ bounded and continuous function openinterval (a,b), $y_F(z)$ can be expressed under $r-$cut representation and Caputo derivative as follows:

$$\left[D^{\alpha}_{c,[a,b]} y_F(z)\right]^r \quad (9)$$

Where D^{α}_{c} is the operation that active representation is confined to $y_F(z)$ and that leads to:

$$\left[D^{\alpha}_{c,[a,z]} y(z)\right]^r = \left[D^{\alpha}_{c,[a,z]} y_{1,1r}(z), D^{\alpha} y_{1,1r}(z)\right] \quad (10)$$

Through using Definition 3 and the attribute of the Caputo fractional derivative, that lead to;

$$\left[I^{\alpha}_{RL,[a,z]} D^{\alpha}_{c,[a,z]} y(z)\right]^r =$$
$$= \left[I^{\alpha}_{RL,[a,z]} D^{\alpha}_{c} y_{1,1r}(z), I^{\alpha}_{RL,[a,z]} D^{\alpha}_{c} y_{1,2r}(z)\right]$$
$$= \left[y_{1,1r}(z) - \sum_{k=0}^{m-1} y^{(k)}_{1,1r}(0^+) \frac{z^k}{k!}, y_{1,2r}(z) - \sum_{k=0}^{m-1} y^{(k)}_{1,2r}(0^+) \frac{z^k}{k!}\right] \quad (11)$$

Definition 6. Let HS is a Hilbert space and $U_y(z) \in HS$ $y(y) \in HS$ satisfies the following:

$\langle y(z), U_y(z) \rangle = y(y), \forall y \in Z$ then,

i) $U_y(z)$ is a reproducing kernel of HS.

ii) HS is a reproducing kernel Hilbert space (RKHS).

Definition 7. The definition of function space $FS^m_2[a,b]$, as follows:

$$FS^m_2[a,b] = \left\{ \begin{array}{l} u \ : \ u^{(i)} \text{ is absolutely continuous,} \\ i = 1,\ 2,\ ..,\ m\text{-}1,\ u^{(m)} \in L^2[a,\ b] \end{array} \right\}$$

The inner product in the function space FS^m_2[a, b][28] for any functions $u(z),\ v(z) \in FS^m_2[a,\ b]$: is determined generally as;

$$\langle u,\ v \rangle_{FS^m_2[a,\ b]} = \sum_{i=0}^{m\text{-}1} u^{(i)}(a)\, v^{(i)}(a) + \int_a^b u^{(m)}\, v^{(m)}\, dz, \quad (12)$$

The norm in the function space $FS^m_2[a,b]$ for any functions $u(z),\ v(z) \in FS^m_2[a,\ b]$ is determined as;

$$\| u \|_{FS^m_2} = \sqrt{\langle u,\ \mathrm{u} \rangle_{FS^m_2}} \quad (13)$$

Definition 8. The inner product space of $FS^1_2[a,\ b]$ [28] is determined as:

$$\langle u,\ \mathrm{v} \rangle_{FS^1_2[a,\ b]} =$$
$$u(0)\, v(0) + \int_a^b u^{(1)}(z)\, v^{(1)}(z)\, dx,\ \forall\ u,\ v\ \in FS^m_2 \quad (14)$$

As the norm of $FS^1_2[a,\ b]$ and a special case from the general inner product is defined as;

$$\| u \|_{FS^1_2} = \sqrt{\langle u(z),\ \mathrm{u}(z) \rangle_{FS^1_2}}, \quad (15)$$

Theorem1. $FS^1_2[a,\ b]$ is RKHS with reproducing kernel given by [28] ;

$$U_Z(\mathrm{y}) = \begin{cases} 1+y, & y \le z \\ 1+z, & y > z \end{cases} \quad (16)$$

Definition 9. The inner product space of $FS^1_2[a,\ b]$ is determined as [29];

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$$\langle u, v\rangle_{FS_2^2[a,b]} = \sum_{i=0}^{1} u^{(i)}(a)\, v^{(i)}(a) + \int_0^1 u^{(2)}\, v^{(2)}\, dz, \quad \forall\, u, v \in FS_2^2[a, b] \tag{17}$$

And the norm of $FS_2^1[a, b]$ is defined as [29];

$$\| u \|_{FS_2^2} = \sqrt{\langle u(z), \mathrm{u(z)}\rangle_{FS_2^2}}, \tag{18}$$

The previous section contains the most important definitions that will be used to implement procedures for solving FTIVP depending on the set of steps that will be included in the next section.

III. PROCEDURE OF SOLUTION

In this part, the steps are given for making an analytical solution of FTIVP by using RKT. In light of this, the steps help us to derive a solution of FTIVP, where as the solution enable to computed iteratively during the two different fractional operator, where the solution contain the following steps:

Step 1: After identity the initial conditions in Equation (1) and using $r-$cut definition, we practice the RLIO in definition (1) for both parts of equations, which leads to new form for Equation (1) as follow:

$$I^{\beta}_{R,[a,z]} D^{\beta}_{c,[a,z]} y^{*}_{1,1r}(z) = I^{\beta}_{R,[a,z]} g^{*}_{1,1r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)),$$
$$y^{*}_{1,1r}(z_0) = 0,\ y^{*}_{1,2r}(z) = 0 \tag{19}$$

$$I^{\beta}_{RL,[a,z]} D^{\beta}_{c,[a,z]} y^{*}_{1,2r}(z) = I^{\beta}_{RL,[a,z]} g^{*}_{1,2r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)),$$
$$y^{*}_{1,1r}(z_0) = 0,\ y^{*}_{1,2r}(z) = 0 \tag{20}$$

By $I^{\alpha}_{RL} D^{\alpha}_{c}\, y(t) = y(t) - \sum_{k=0}^{n-1} y^{(k)}(0^{+}) \frac{z^{k}}{k!}$, $z > 0,\ n-1 < \alpha \le n.$ That's leads to next form:

$$y^{*}_{1.1r}(z) - y^{*}_{1,1r}(z_0) - y^{*}_{1,2r}(z_0) = I^{\beta}_{RL,[a,z]} g^{*}_{1,1r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)) \tag{21}$$

$$y^{*}_{1.2r}(z) - \mathrm{y}^{*}_{1,1r}(z_0) - \mathrm{y}^{*}_{1,2r}(z_0) = I^{\beta}_{RL,[a,z]} g^{*}_{1,2r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)) \tag{22}$$

Sine $y^{*}_{1,1r}(z_0) = 0,\ y^{*}_{1,2r}(z) = 0$ that leads to the next form:

$$y^{*}_{1.1r}(z) = I^{\beta}_{RL,[a,z]} g^{*}_{1,1r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z))$$
$$y^{*}_{1.2r}(z) = I^{\beta}_{RL,[a,z]} g^{*}_{1,2r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z))$$
(23)

Therefore

$$y^{*}_{1.1r}(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z-t)^{\alpha-1} \left(g^{*}_{1,1r}(t, y^{*}_{1,1r}(t), y^{*}_{1,2r}(t)) \right) dt,\ z > a$$
$$y^{*}_{1.2r}(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z-t)^{\alpha-1} \left(g^{*}_{1,2r}(t, y^{*}_{1,1r}(t), y^{*}_{1,2r}(t)) \right) dt,\ z > a \tag{14}$$

Then,

$$\begin{aligned} y^{*}_{1.1r}(z) &= G^{*}_{1,1r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(\mathrm{z})) \\ y^{*}_{1.2r}(z) &= G^{*}_{1,2r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)) \\ y^{*}_{1,1r}(z_0) &= 0,\ y^{*}_{1,2r}(z) = 0 \end{aligned} \tag{15}$$

That's led to the below form for the solution of $y_{1,1r}(z)$ and $y_{1,2r}(z)$ as the following

$$\begin{aligned} y_{1,1r}(z) &= G^{*}_{1,1r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)) + z_{1,1r} \\ y_{1,2r}(z) &= G^{*}_{1,2r}(z, y^{*}_{1,1r}(z), y^{*}_{1,2r}(z)) + z_{1,2r} \end{aligned} \tag{16}$$

Immediately, for solving the last system by execution of the RKT, we add to the following steps;

Step 2: To use RKS tools for the first order want to estaplish the space $FS_2^{m+1}[a, b]$, where F mention for first and S for space.In the equation (1) $m = 1$, that's leads to estaplishing the space $FS_2^2[a, b]$.

Step 3: The inner product in $FS_2^2[a, b]$ is showing by:

$$\langle y, v\rangle_{FS_2^2[a,b],} = \sum_{i=0}^{1} y^{(i)}(a)\, v^{(i)}(a) + \int_0^1 y^{(2)}(z)\, v^{(2)}(z)\, dz,\ \forall\, u, v \in FS_2^2[a, b], \tag{27}$$

And the norm is given by:

$$\| y(z) \|_{FS_2^2[a,b]} = \sqrt{\langle y(z), \mathrm{y}(z)\rangle_{FS_2^2[a,b]}} \tag{28}$$

Step 4: space $FS_2^2[a, b]$ is a (RKHS), that performs to the next result: for each fixed $z \in [a, b]$, there exists $R_2K_z(y) \in FS_2^2[a, b]$ such that $\langle y(y), R_2K_z(y)\rangle_{FS_2^2[a,b]} = y(z)$ for any $y(y) \in FS_2^2[a, b]$ and $y \in [a, b]$.

The $R_2K_z(y)$ [29] given by:

$$R_2K_z(y) = \begin{cases} 1 + zy + \dfrac{yz^2}{2} - \dfrac{z^3}{6}, & y \le z \\ 1 + zy + \dfrac{zy^2}{2} - \dfrac{y^3}{6}, & y > z \end{cases} \tag{29}$$

For solving Equation (25) as a system by RKT need to define a differential operator

$$L_{m,jr}: FS_2^2[a,\ b] \to FS_2^1[a,\ b] \text{ where } m=1,\ j=1,\ 2. \quad (30)$$

In this step the equations (25) changed into the form below:

$$L_{1,mr}\, y^*_{1,1r}(z) = G^*_{1,1r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z)), \qquad (31)$$
$$y^*_{1,1r}(z_0)=0,\ y^*_{1,2r}(z)=0$$

Where
$z \in [a,\ b]$, $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z) \in FS_2^2[a,\ b]$,
$G^*_{1,1r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z))$
And $G^*_{1,2r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z)) \in FS_2^2[a,\ b]$.

Step 5: applying the differential operator:
$L_{m,jr}: FS_2^2[a,\ b] \to FS_2^1[a,\ b]$ where $m=1,\ j=1,\ 2.$
Then equation (25) changed into the form below:

$$L_{1,1r}\, y^*_{1,1r}(z) = G^*_{1,1r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z)),$$
$$L_{1,2r}\, y^*_{1,2r}(z) = G^*_{1,2r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z)), \qquad (32)$$
$$y^*_{1,1r}(z_0)=0,\ y^*_{1,2r}(z)=0$$

Where $z \in [a,\ b]$, $y^*_{1,1r}(z)$ and $y^*_{1,1r}(z) \in FS_2^2[a, b]$,
$G^*_{1,1r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z))$ and
$G^*_{1,12r}(z,\ y^*_{1,1r}(z),\ y^*_{1,2r}(z)) \in FS_2^1[a, b]$.

Step 6: An orthogonal function system of $FS_2^1[a, b]$, is requird to construct , for do that, let $\{z_k\}_{k=1}^{\infty}$ be a countable dense set in $[a,\ b]$.
Let $e_{k,m,jr}(z) = R_1K_{z_k}(z)$ where
$R_1K_z(z)$ is the RK of $FS_2^1[a,\ b]$, that is given by
$\forall\, y^*_{m,jr}(z) \in FS_2^1[a,\ b],\ j=1,\ 2.$

It follows:
$$\left\langle y^*_{1,jr}(z),\ e_{k,1,1r}(z)\right\rangle_{FS_2^1[a,\ b]} = \left\langle y^*_{1,jr}(z),\ RK_{z_k}(z)\right\rangle = y^*_{1,jr}(z_k).$$
Additionally, let $L^{ad}_{1,jr} e_{k,1,jr}(z) = \Psi_{k,1,jr}(z)$
where $L^{ad}_{1,jr}$ is an adjoint operator of $L_{1,jr}$, $j=1,\ 2.$
Thus, $\left\langle y^*_{k,1,jr}(z),\ \Psi_{k,1,jr}(z)\right\rangle_{FS_2^2[a,\ b]}$
$$= \left\langle y^*_{k,1,jr}(z),\ L^{ad}_{1,jr} e_{k,1,jr}(z)\right\rangle_{FS_2^2[a,\ b]}$$
$$= \left\langle L_{1,jr}\, y^*_{k,1,jr}(z),\ e_{k,1,jr}(z)\right\rangle_{FS_2^1[a,\ b]}$$
$$= L_{1,jr}\, y^*_{1,jr}(z_k),\quad j=1,\ 2,\ k=1,\ 2,\ \ldots$$
Hence, $\Psi_{k,1,jr}(z)$ expressed by the next form:
$$\Psi_{k,1,jr}(z) = L^{ad}_{1,jr} e_{k,1,jr}(z) =$$
$$= \left\langle L^{ad}_{1,jr} e_{k,1,jr}(z),\ R_2K_{z_k}(z)\right\rangle_{FS_2^2[a,\ b]}$$
$$= \left\langle e_{k,1,jr}(z),\ L_{1,jr} R_2K_{z_k}(z)\right\rangle_{FS_2^1[a,\ b]}$$
$$= L_{1,jr}\, R_2K_{z_k}(z)\Big|_{y=z_k},\quad j=1,\ 2,\ k=1,\ 2,\ \ldots$$

Theorem 2. For the Equations (32), let the inverse operator for $L_{1,jr},\ j=1,\ 2$ exists, there for, if $\{z_k\}_{k=1}^{\infty}$ is dense in $[a,\ b]$, then $\{\Psi_{k,m,jr}(z)\}_{(k,m,jr)=(1,1,1r)}^{(\infty,1,2r)}$ is the complete function of $FS_2^2[a,\ b]$.

Proof:
$\forall$ fixed $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z) \in FS_2^2[a\ b]$.
Let
$$\left\langle y^*_{1,jr}(z), L^{ad}_{1,jr}\ e_{k,1,jr}(z)\right\rangle_{FS_2^2}$$
$$= \left\langle L_{1,jr}\, y^*_{1,jr}(z),\ e_{k,1,jr}\right\rangle_{FS_2^1} = 0,\ j=1,\ 2.$$
Note that $\{z_k\}_{k=1}^{\infty}$ is dense in $[a,\ b]$.
Thence
$L_{1,1r}\ y^*_{1,1r}(z)=0,\ L_{1,2r}\ y^*_{1,2r}(z)=0.$
$y^*_{1,1r}(z)=0$ and $y^*_{1,2r}(z)=0$, from the $L^{ad}_{1,1r}$ and $L^{ad}_{1,2r}$, and the continuity of $y^*_{1,1r}(z)$ and $y^*_{1,2r}(z)$. proof is complete. ■

Now, for use Gram-Schmidt orthonormalization construct an orthonormal system $\{\overline{\Psi}_{k,q,sr}\}_{(k,q,sr)=(1,1,1r)}^{(\infty,1,jr)}$ of $FS_2^2[a,\ b]$, via the Gram-Schmidt orthogonalization process where
$$\overline{\psi}_{k,m}(z) = \sum_{l=1}^{k}\sum_{m=1}^{2} \beta_{l,m}\psi_{l,m}(z) \qquad (33)$$
Where $\beta_{k,m}$ is orthogonalization coefficient.
We rearrange all equation by mention $1{,}1r=1$ and $1{,}2r=2$.
Theorem 3. Let $\{z_k\}_{k=1}^{\infty}$ be dense in $[a,\ \mathrm{b}]$, and the unique solution (32) is on $FS_2^2[a,\ b]$. Then the exact solution of Equations (33) is given by
$$\sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{i}\sum_{q=1}^{m} \beta_{l,q} G^*_m\left(z_l,\ y^*_m(z_l)\right)\overline{\psi_{k,m}}(z) \qquad (34)$$
The approximate solution $y^*_{m.jr}(z)$ is obtained by taking finitely many terms for (34) in the representation form of
$$y^{*N}_m = \sum_{k=1}^{N}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m} \beta_{l,q} G^*_l\left(z_l,\ y^*_m(z_l)\right)\overline{\psi_{k,m}}(z) \qquad (35)$$
Proof: By applying Theorem 2, we can see that. $\{\Psi_{k,m}(z)\}_{(k,m)=(1,1)}^{(\infty,2)}$ is the complete orthonormal basis,
$$\left\langle y^*_m(z),\ e_m(z)\right\rangle_{FS_2^1[a,\ b]} = \left\langle y^*_m(z),\ R_1K_{z_k}(z)\right\rangle = y^*_m(z_k)$$
For each

$$y^*_m(z_k)y^*_m(z) \in FS_2^2[a,\ b],\ \sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y^*_m(z),\ \overline{\Psi}_{k,mr}(z)\right\rangle \overline{\Psi}_{k,m}(z)$$

is the Fourier series expansion of $\{\Psi_{k,m}(z)\}_{(k,m)=(1,1)}^{(\infty,2)}$, the series

$\sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \overline{\psi}_{k,m}(z)\right\rangle\overline{\psi}_{k,m}(z)$ is convergent.
(36)

$$y_m^*(z)=\sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \overline{\psi}_{k,m}(z)\right\rangle\overline{\psi}_{k,m}(z) \quad (37)$$

$$y_{m,jr}^*(z)=\sum_{k=1}^{\infty}\sum_{m=1}^{2}\left\langle y_m^*(z),\ \sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\psi_{l,q}(z)\right\rangle\overline{\psi}_{k,m}(z) \quad (38)$$

$$y_{m,jr}^*(z)=\sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left\langle y_m^*(z),\ L_m^{ad}e_{l,q}(z)\right\rangle\overline{\psi}_{k,m}(z) \quad (39)$$

$$y_{m,jr}^*(z)=\sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left\langle L_m y_m^*(z),\ e_{l,q}(z)\right\rangle\overline{\psi}_{k,m}(z) \quad \text{in}$$

$FS_2^2[a,\ b]$. (40)

$$y_{m,jr}^*(z)=\sum_{k=1}^{\infty}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z,\ y_m^*(z_l)\right)\overline{\psi}_{k,m}(z) \quad (41)$$

$$y_{m,jr}^{*N}(z)=\sum_{k=1}^{N}\sum_{m=1}^{2}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z,\ y_q^*(z_l)\right)\overline{\psi}_{k,m}(z) \quad (42)$$

Proof is complete. ■

As results:

1:- The approximant solution $y_m(z),\ m=1,\ 2.$ by using RLIO can be presented by:

$$\begin{bmatrix}\left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\frac{1}{\Gamma(\beta)}\int_a^{z_l}(z_l-t_l)^{\alpha-1}\left(g_{1,2r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)dt_l*\overline{\psi}_{km}(z)\right]\\ +z_{1.1r},\\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\frac{1}{\Gamma(\beta)}\int_a^{z_l}(z_l-t_l)^{\alpha-1}\left(g_{1,1r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)dt_l*\overline{\psi}_{km}(z)\right]\\ +z_{1,1r}\end{bmatrix}$$

(43)

$$=\begin{bmatrix}\left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z_l,\ y_{F1}^*(z_l)\right)\overline{\psi}_{km}(z)\right]+z_{1.1r},\\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}G_q^*\left(z_l,\ y_{F1}^*(z_l)\right)\overline{\psi}_{km}(z)\right]+z_{1,1r}\end{bmatrix}=\left[y_{1r}^N(z),\ y_{2r}^N(z)\right]$$

(44)

$$y_{mr}^{NR}(z)=\begin{pmatrix}N-\text{terms by RLIO},\\ y_{odd(m)r_q}^*(z)+z_{1,1r},\ y_{even(m)r_q}^*(z)+z_{1,2r}\end{pmatrix}=\left[y_{1,1r}(z),\ \mathrm{y}_{1,2r}(z)\right]$$

(45)

2:- The approximate solution $y_m(z),\ m=1,\ 2.$ by using CFDO can be presented by:

$$\begin{bmatrix}\left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left(g_{1,1r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)*\overline{\psi}_{km}(z)\right]\\ +z_{1.1r},\\ \left[\sum_{k=1}^{n}\sum_{m=1}^{2n}\sum_{l=1}^{k}\sum_{q=1}^{m}\beta_{l,q}\left(g_{1,2r}^*(t,\ y_{1,1r}^*(t_l),\ y_{1,2r}^*(t_l))\right)*\overline{\psi}_{km}(z)\right]\\ +z_{1,1r}\end{bmatrix}$$

$$=\left[y_{1r}^N(z),\ y_{2r}^N(z)\right]$$

(46)

$$y_{mr}^{NC}(z)=\begin{pmatrix}N-\text{terms by CFDO},\\ y_{odd(m)r_q}^*(z)+z_{1,1r},\ y_{even(m)r_q}^*(z)+z_{1,2r}\end{pmatrix}$$

$$=\left[y_{1,1r}(z),\ \mathrm{y}_{1,2r}(z)\right]$$

(47)

Step 7: Applying the steps as algorithm on the Eq. (1) by using the software "Wolfram Mathematical version 10" to demonstrate accuracy and efficiency of the approximate solution.

IV. NUMERICAL EXAMPLES

During the general formula of the Equation (1) [12], the below example can be prepared.

Example 4.1. let the following FTIVP of $\alpha=1$ order.

$$\begin{aligned}&D_{c,a,z}^{\alpha}y(z)=2z^2+3z-y(z),\\ &y(z_0)=\ \text{FTN}=(1,\ 2,\ 3),\ 0\le z\le 1.\end{aligned} \quad (48)$$

Solution: By the last algorithm;

$$\begin{aligned}&D_{c,a,z}^{\alpha}y_{1,1r}(z)=2z^2+3z-y_{1,1r}(z),\\ &y_{1,1r}(z_0)=1+r,\ r\in[0,\ 1].\\ &D_{c,a,z}^{\alpha}y_{1,2r}(z)=2z^2+3z-y_{1,2r}(z),\\ &y_{1,2r}(z_0)=3-r,\ r\in[0,\ 1].\end{aligned} \quad (49)$$

In this example at $N=50$ approximant solution has been finding, the solution was acquired twice by the method with different process, in the first process by applied CFDO and second process by RLIO, the results were as follows:

During comparison of results on Tables 1, 2, 3 and 4 and figures 1, 2, 3 and 4. The reserch gave a convergence in the results and effective in both process, but there is a difference in the stability and behavior of the error based in various of operator, and the effectiveness in the application by considering the time it takes to get results was for the benefit the process that was used Caputo fractional derivative operator more than Riemann fractional integral.

Table 1. Result of $y_{1,1r}(z)$ using CFDO where $r=0$

z	Exact Solution for $y_{1,1r}(z)$, $r=0$	Approximant Solution for $y_{1,1r}(z)$, $r=0$	Error
0.0	1.0	1.0	0.0
0.1	0.92	0.9200051991	$5.199092309 \times 10^{-8}$
0.2	0.88	0.8800104063	$1.040629265 \times 10^{-5}$
0.3	0.88	0.8800156734	1.56733578510^{-5}
0.4	0.92	0.9200210565	2.10564901810^{-5}
0.5	1.0	1.000026618	2.66181051810^{-5}
0.6	1.12	1.120032429	3.24289174410^{-5}
0.7	1.28	1.28003857	3.8570410510^{-5}
0.8	1.48	1.480045138	4.51377687810^{-5}
0.9	1.72	1.720052243	5.22433618310^{-5}
1.0	2.0	2.000060021	6.00208868710^{-5}

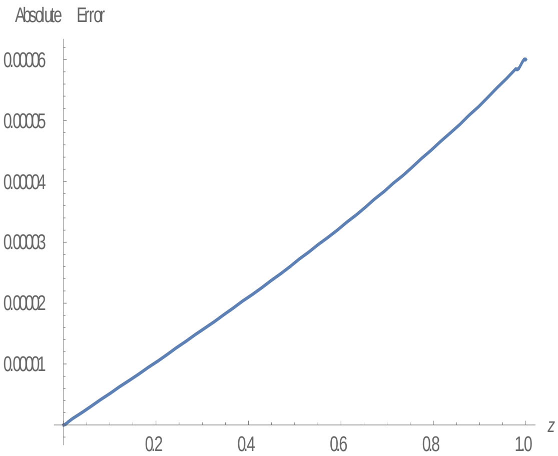

Figure 1. The error of $y_{1,1r}(z)$ using CFDO where $r=0$

Table 2. Result of $y_{1,1r}(z)$ using RLIO where $r=0$

z	Exact Solution for $y_{1,2r}(z)$, $r=0$	Approximate Solution for $y_{1,2r}(z)$, $r=0$	Error
0.0	3.0	3.0	0.0
0.1	2.73	2.729697225	$2.23894273 \times 10^{-5}$
0.2	2.52	2.517482178	$2.067192369 \times 10^{-5}$
0.3	2.36	2.361654798	$1.835633338 \times 10^{-5}$
0.4	2.26	2.260656389	$1.629646799 \times 10^{-5}$
0.5	2.21	2.213075782	$1.446240576 \times 10^{-5}$
0.6	2.22	2.217636102	$1.282983124 \times 10^{-5}$
0.7	2.27	2.273181985	$1.137700472 \times 10^{-5}$
0.8	2.38	2.378668013	$1.008450251 \times 10^{-5}$
0.9	2.53	2.533148254	$8.93496632 \times 10^{-6}$
1.0	2.74	2.735766795	$7.912908885 \times 10^{-6}$

z	Exact Solution for $y_{1,1r}(z)$, $r=0$	Approximant Solution for $y_{1,1r}(z)$, $r=0$	Error
0.0	1.0	1.0	0.0
0.1	0.92	0.9200114071	$1.140710144 \times 10^{-5}$
0.2	0.88	0.8800106101	$1.06100609 \times 10^{-5}$
0.3	0.88	0.8800095995	$9.599492645 \times 10^{-6}$
0.4	0.92	0.920008686	$8.68597954 \times 10^{-6}$
0.5	1.0	1.000007859	$7.859395585 \times 10^{-6}$
0.6	1.12	1.120007111	$7.111470887 \times 10^{-6}$
0.7	1.28	1.280006435	$6.434719647 \times 10^{-6}$
0.8	1.48	1.480005822	$5.822370829 \times 10^{-6}$
0.9	1.72	1.720005268	$5.268294685 \times 10^{-6}$
1.0	2.0	2.000004767	$4.766961052 \times 10^{-6}$

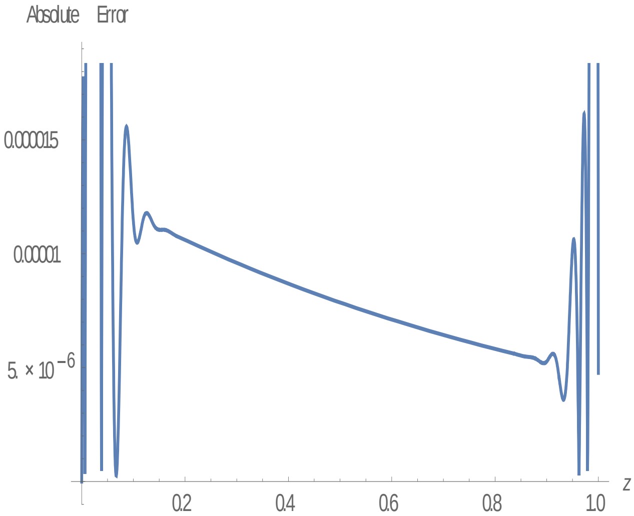

Figure 2. The error $y_{1,1r}(z)$ using RLIO where $r=0$

Table 3. Result of $y_{1,2r}(z)$ using CFDO where $r=0$

z	Exact Solution for $y_{1,2r}(z)$, $r=0$	Approximant Solution for $y_{1,2r}(z)$, $r=0$	Error
0.0	3.0	3.0	0.0
0.1	2.73	2.729677327	$2.490565271 \times 10^{-6}$
0.2	2.52	2.517467131	$5.624442558 \times 10^{-6}$
0.3	2.36	2.361645799	$9.357205942 \times 10^{-6}$
0.4	2.26	2.260653756	$1.366424237 \times 10^{-5}$
0.5	2.21	2.21307986	1.85405448810^{-5}
0.6	2.22	2.217647273	2.40011498110^{-5}
0.7	2.27	2.27320069	3.00822503810^{-5}
0.8	2.38	2.378694771	3.68430362610^{-5}
0.9	2.53	2.533183688	4.43683282310^{-5}
1.0	2.74	2.735811654	5.27720966710^{-5}

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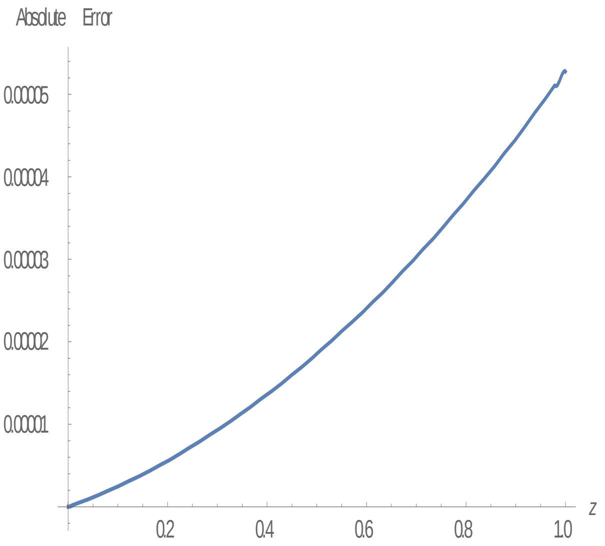

Figure 3. The error of $y_{1,2r}(z)$ using CFDO where $r = 0$

Table 4. Result of $y_{1,2r}(z)$ using RLIO where $r = 0$

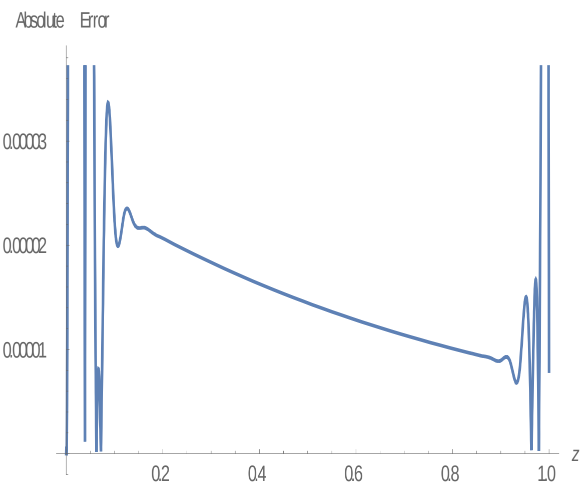

Figure 4. The error $y_{1,2r}(z)$ using RLIO where $r = 0$

It is noticeable that while applying the proposed solution using the same software, the solution using the Riemann definition takes longer to get results.

V. CONCLUSION

Approximate solution of fuzzy fractional differential equation is presented where the initial value given as a triangular fuzzy number with different fractional operator, The results in both examples show the effectiveness of the method used in both cases, the results give a various in the behaviour of error, also gave a variation in the time it takes to solve in favour of the CFDO, however it is shown more stable in the behavior of error in favour of RLIO.

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AUTHORS PROFILE

Moath Ali Alshorman : received the Master Degree from Unimap, Malaysia in 2017. He is a PhD student in Faculty of Informatics and Computing, University Sultan Zainal Abidin, Terengganu, Malaysia. He has published some journal papers.

Nurnadiah Zamri received the PhD Degree from Universiti Malaysia Terengganu, Malaysia in 2015. She is a senior lecturer in the Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, Malaysia. Her research of interest include fuzzy decision making, fuzzy expert system and cloud computing. She has published more than 40 journal papers, proceeding papers and book of chapter.

Asia Khalaf Albzeirat received the PhD Degree from Unimap, Malaysia in 2018. She is a researcher in the Department of Mathematics, Mutah University, Mutah, Jordan.. Her research of interest include fuzzy, differential equations, fractional differential equations, set theory. She has published 8 journal papers,.

Mohammed Ali received the PhD in Mathematics from Memorial University Of Newfoundland, 2009. He is a Associate Professor in the Department of Mathematics, Jordan University of Science and Technology, Irbid, Jordan. His research of interest include differential Equations and fractional differential equations, fuzzy differential equation. He has published more than 20 journal papers, proceeding papers