Commutative *l*-fuzzy languages

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***Abstract:** Here we study variety of monoid and semiring recognizable l-fuzzy languages. A l-fuzzy language λ over an alphabet A is called commutative l-fuzzy language if it satisfies the condition λ (puvq) = λ(pvuq) , for all u,v,p,q $\in A^*$. We prove that the set of all commutative l-fuzzy languages is $*$-variety and conjunctive variety of l-fuzzy languages.*

***Keywords :**Commutative l-fuzzy languages, Conjunctive variety of l-fuzzy languages, Generalized fuzzy languages, $*$-variety of l-fuzzy languages.*

I. INTRODUCTION

The theory of fuzzy language was developed as a generalization of the classical notion of (crisp)languages. The concept of fuzzy automaton was introduced by Wee in 1967. Zadeh and Lee generalized the classical notion of languages to the concept of fuzzy languages in 1969. More on recent development of algebraic theory of fuzzy automata and formal fuzzy languages can be found in the book by Mordeson and Malik [6]. The varieties of fuzzy languages were introduced by Petkovic [7]. Semiring recognizable languages was first studied by Polak [9]. In [10], he introduced the concept of syntactic semiring of a language and studied its properties. Also he established a one-one correspondence between the lattices of all conjunctive variety of languages and pseudovariety of finite idempotent semirings.

We introduce the notion of $*$-variety of monoid recognizable *l*-fuzzy languages in[4]. In [3] we introduce the notion of variety of semiring recognizable *l*-fuzzy languages. Also we obtain a one to one correspondence between varieties of semiring recognizable *l*-fuzzy languages and all pseudovarieties of finite idempotent semirings.

In this paper we describe commutative l-fuzzy languages. A family of recognizable *l*-fuzzy languages is a $*$-variety of *l*-fuzzy languages, if it is closed under joins, meets, complements, scalar products, quotients, inverse homomorphic images and cuts. We prove that the set of all commutative *l*-fuzzy languages is $*$-variety and conjunctive variety of *l*-fuzzy languages.

II. PRILIMINARIES

In this section we recall the basic definitions, results and notations that will be used in the sequel. All undefined terms are as in [5, 6, 8, 11]. A lattice is a partially ordered set in which every subset consisting of two element has a least upper bound and a greatest lower bound. A lattice *l* is called complemented if it is bounded and if every element in *l* has a complement. A lattice *l* is called a complete lattice if every nonempty subset of *l* has greatest lower bound and least upper bound in *l*. A lattice *l* is said to be distributive if for any element *a*,*b* and *c* of *l*, we have the following distributive properties.

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 2.1 (cf.[10])**.** An idempotent semiring is a nonempty set S together with two binary operations + and $\cdot$ and two constant elements 0 and 1 such that

i) (S,+,0) is a commutative idempotent monoid.

ii) (S,$\cdot$,1) is a monoid.

iii) the distributive laws $a\cdot(b+c) = a\cdot b + a\cdot c$ and $(a + b)\cdot c = a\cdot c + b\cdot c$ hold for every a,b,c $\in$ S.

iv) $0\cdot a = a\cdot 0 = 0$ for every a.

Let *A* be a finite set. When we deal with languages *A* is called an alphabet and elements of *A* are called letters. A finite sequence of letters in *A* is called a word. The length of the word *w* is the number of letters of *A* occurring in *w*. A word of length zero is called empty word and is denoted by ε. A^+ denotes the set of all nonempty words over an alphabet *A* and $A^* = A^+ \cup \{\varepsilon\}$ is a monoid under the operation concatenation, called free monoid over *A*. A subset of A^* is called the language *L* over an alphabet *A*.

Let $F(A^*)$ denote the set of all finite subsets of A^*. This set equipped with the operations usual union and multiplication $U \cdot V = \{uv \mid u \in U, v \in V\}$ form the free idempotent semiring over the alphabet *A*.

Let *l* be a complete complemented distributive lattice. Any function λ from A^* into *l* is called a *l*-fuzzy language over the alphabet *A*.

The complement $\bar{\lambda}$ of a l-fuzzy language λ is defined as $\bar{\lambda}(u) = \overline{\lambda(u)}$where $\overline{\lambda(u)}$ denotes the complement of λ(u) in *l*.[1]

For l-fuzzy languages λ_1, λ_2 over A , their join ($\vee$) and meet ($\wedge$) are defined by $(\lambda_1 \vee \lambda_2)(u) = \lambda_1(u) \vee \lambda_2(u)$ and $(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u)$.

Let λ_1, λ_2be *l*-fuzzy languages over *A*. Then their left and right quotients are defined by

$(\lambda_1{}^{-1}\lambda_2)(u) = \vee_{v\in A^*}\left(\lambda_2(vu) \wedge \lambda_1(v)\right), u \in A^*$ and

$(\lambda_2\lambda_1{}^{-1})(u) = \vee_{v\in A^*}\left(\lambda_2(uv) \wedge \lambda_1(v)\right), u \in A^*$.

Let *A* and *B* be finite alphabets and $\varphi : A^* \to B^*$ be a homomorphism. Let λ be a *l*-fuzzy language over *B*. The inverse of λ under φ is a *l*-fuzzy language $\lambda\varphi^{-1}$ over *A* defined by $(\lambda\varphi^{-1})(u) = \lambda(\varphi(u))$, $u \in A^*$.

Let $c \in l$, then the scalar product $c\,.\lambda$ of the *l*-fuzzy language λ is defined as $(c \cdot \lambda)(u) = c \wedge \lambda(u)$.

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Let λ be a *l*-fuzzy language over *A*. The *c*-cut of λ is the crisp language λ_c defined by $\lambda_c = \{u \in A^* \mid \lambda(u) \geq c\}$.

A family of recognizable *l*-fuzzy languages is a ∗-variety of *l*-fuzzy languages, if it is closed under joins, meets, complements, scalar products, quotients, inverse homomorphic images and cuts.[4]

Let λ be a *l*-fuzzy language over *A*. The function λ_{min} : $F(A^*) \rightarrow l$ defined by

$$\lambda_{min}(U) = \wedge_{u \in U} \lambda(u), U \in F(A^*)$$

is called the generalized fuzzy language determined by λ . If $|U| = 1$ then $\lambda_{min}(u) = \lambda(u)$. So we can view λ_{min} as a generalization of λ .[2]

Let λ_{1min} and λ_{2min} be generalized fuzzy languages determined by λ_1 and λ_2 respectively. Then their meet, left quotient and right quotient are defined as follows

$$(\lambda_{1min} \wedge \lambda_{2min})(U) = \lambda_{1min}(U) \wedge \lambda_{2min}(U)$$

$$(\lambda_{1min}^{-1}\lambda_{2min})(U) = \vee_{v \in A^*} \left(\lambda_{2min}(vU) \wedge \lambda_{1min}(v)\right),$$

$$(\lambda_{2min}\lambda_{1min}^{-1})(U) = \vee_{v \in A^*} \left(\lambda_{2min}(Uv) \wedge \lambda_{1min}(v)\right),$$

for $U \in F(A^*)$

Let *A* and *B* be finite alphabets and φ from $F(A^*)$ to $F(B^*)$ be a semiring homomorphism and λ_{min} be a generalized fuzzy language determined by a *l*-fuzzy language λ over *B*. Then the inverse homomorphic image of λ_{min} is a *l*-fuzzy subset $\lambda_{min}\varphi^{-1}$ of $F(A^*)$ defined by

$$(\lambda_{min}\varphi^{-1})(U) = \lambda_{min}(\varphi(U)), U \in F(A^*)$$

Theorem 2.2. [3] Let λ, λ_1, λ_2 be *l*-fuzzy languages over A. Then

(i) $(\lambda_1 \wedge \lambda_2)_{min} = \lambda_{1min} \wedge \lambda_{2min}$

(ii) $(\lambda_1^{-1}\lambda_2)_{min} = (\lambda_{1min}^{-1}\lambda_{2min})$

(iii) $(\lambda_2\lambda_1^{-1})_{min} = \lambda_{2min}\lambda_{1min}^{-1}$

Theorem 2.3. [3] Let A and B be finite alphabets and φ from $F(A^*)$ to $F(B^*)$ be a semiring homomorphism. If λ is a *l*-fuzzy language over B, then $(\lambda\varphi^{-1})_{min} = \lambda_{min}\varphi^{-1}$.

Definition 2.4. [3] Let lC be a family of *l*-fuzzy languages and lC_{min} be the family of associated generalized *l*-fuzzy languages. We say that lC is a conjunctive variety if lC_{min} is closed under finite meet, quotients and inverse homomorphic images.

III. COMMUTATIVE *L*-FUZZY LANUAGES

Let λ be a *l*-fuzzy language over an alphabet *A*. Then λ is called commutative *l*-fuzzy language if it satisfies the condition

$$\lambda(puvq) = \lambda(pvuq)$$

for all *u,v,p,q* ∈ A^*. The class of commutative *l*-fuzzy languages on A^* is denoted by ClFL(A^*).

Example 3.1. Let A = {a,b} and *l* = ({1,2,3,6}, LCM, GCD) be a complete distributive lattice. Let λ: $A^* \rightarrow l$ be given by

$$\lambda(u) = \begin{cases} 2 \text{ if } |u| \text{ is even} \\ 3 \text{ if } |u| \text{ is odd} \end{cases}$$

Then λ is a commutative *l*-fuzzy language.

Lemma 3.2. Let $\lambda \in$ ClFL(A^*), then $\bar{\lambda} \in$ ClFL(A^*).

Proof. Since $\lambda \in$ ClFL(A^*), we have $\lambda(puvq) = \lambda(pvuq)$ for all *p,q,u,v* ∈ A^*. So

$$\bar{\lambda}(puvq) = \overline{\lambda(puvq)} = \overline{\lambda(pvuq)} = \bar{\lambda}(pvuq)$$

for all *p,q,u,v* ∈ A^*. Thus $\bar{\lambda} \in$ ClFL(A^*). Hence ClFL(A^*) is closed under complements.

Lemma 3.3. ClFL(A^*) is closed under scalar multiplication.

Proof. Let $\lambda \in ClFL(A^*)$ and $c \in l$, then

$$(c \cdot \lambda)(puvq) = c \wedge (\lambda(puvq)) = c \wedge \lambda(pvuq) = (c \cdot \lambda)(pvuq)$$

for all *u,v,p,q* ∈ A^* . Thus $c \cdot \lambda \in$ ClFL(A^*). Hence ClFL(A^*) is closed under scalar multiplication.

The following result shows that ClFL(A^*) is closed under join and meet.

Lemma 3.4. Let $\lambda_1, \lambda_2 \in$ ClFL(A^*). Then $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are in ClFL(A^*).

Proof. Since $\lambda_1, \lambda_2 \in$ ClFL(A^*), we have λ_1(puvq) = λ_1(pvuq) and λ_2 (puvq) = λ_2 (pvuq) for all p,q,u,v ∈ A^*. So

$$(\lambda_1 \vee \lambda_2)(puvq) = \lambda_1(puvq) \vee \lambda_2(puvq) = \lambda_1(pvuq) \vee \lambda_2(pvuq) = (\lambda_1 \vee \lambda_2)(pvuq)$$

for all p,q,u,v ∈ A^*. Thus $(\lambda_1 \vee \lambda_2) \in$ ClFL(A^*).

Since $\lambda_1 \wedge \lambda_2 = \overline{(\overline{\lambda_1} \vee \overline{\lambda_2})}$ we have $(\lambda_1 \wedge \lambda_2) \in$ ClFL(A^*).

Lemma 3.5. Let λ be a commutative *l*-fuzzy language on A^*, B be a finite alphabet and $\phi : B^* \rightarrow A^*$ be a homomorphism. Then $\lambda\phi^{-1}$ is a commutative *l*-fuzzy language over B where $\lambda\phi^{-1}(u) = \lambda(\phi(u))$ for all u ∈ B^*.

Proof. Since $\lambda \in$ ClFL(A^*), we have $\lambda(puvq) = \lambda(pvuq)$ for all *p,q,u,v* ∈ A^*.So

$$\lambda\phi^{-1}(puvq) = \lambda(\phi(puvq)) = \lambda(\phi(p)\phi(u)\phi(v)\phi(q)) = \lambda(\phi(p)\phi(v)\phi(u)\phi(q)) = \lambda(\phi(pvuq)) = \lambda\phi^{-1}(pvuq)$$

for all *p,q,u,v* ∈ A^*. Thus $\lambda\phi^{-1}$ is a commutative *l*-fuzzy language over *B*.

From the above lemma it follows that ClFL(A^*) is closed under the inverse homomorphic images .

Lemma 3.6. Let $\lambda_1, \lambda_2 \in$ ClFL(A^*). Then

(*i*) $\lambda_1^{-1}\lambda_2 \in$ ClFL(A^*).

(*ii*) $\lambda_2\lambda_1^{-1} \in$ClFL($A^*$).

Proof. (i) Since $\lambda_1, \lambda_2 \in$ ClFL(A^*), we have λ_1 (*puvq*) = λ_1 (*pvuq*) and λ_2 (*puvq*) =λ_2 (*pvuq*) for all *p,q,u,v* ∈ A^*. So

$$(\lambda_1^{-1}\lambda_2)(puvq) = \vee_{w \in A^*} \{\lambda_2(wpuvq) \wedge \lambda_1(w)\} = \vee_{w \in A^*} \{\lambda_2((wp)uvq) \wedge \lambda_1(w)\} = \vee_{w \in A^*} \{\lambda_2((wp)vuq) \wedge \lambda_1(w)\} = \vee_{w \in A^*} \{\lambda_2(wpvuq) \wedge \lambda_1(w)\} = (\lambda_1^{-1}\lambda_2)(pvuq)$$

for all p,q,u,v ∈ A^*. Thus $\lambda_1^{-1}\lambda_2 \in$ ClFL(A^*).

ii) Similarly if $\lambda_1, \lambda_2 \in$ ClFL(A^*), then $\lambda_2\lambda_1^{-1} \in$ ClFL(A^*). Thus ClFL(A^*) is closed under left and right quotients.

Lemma 3.7. Let $\lambda \in$ ClFL(A^*) and $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$ for all c ∈ *l* .Then ClFL(A^*) is closed under the c-cut. (ie, $\chi_{\lambda_c} \in$ ClFL(A^*) for all c ∈ *l*).

Proof. Since $\lambda \in$ ClFL(A^*), we have $\lambda(puvq) = \lambda(pvuq)$ for all *p,q,u,v* ∈ A^*.

So $puvq \in \lambda_c \Leftrightarrow c \leq \lambda(puvq) = \lambda(pvuq) \Leftrightarrow pvuq \in \lambda_c$ for all *p,q,u,v* ∈ A^*. Hence ClFL(A^*) is closed under *c*-cut.

Theorem 3.8. ClFL(A^*) is a ∗-variety of *l*-fuzzy languages.

Proof. Follows from Lemmas 3.2, 3.3, 3.4, 3.5,3.6 and 3.7.

Let $\lambda \in$ ClFL(A^*) and λ_{min} be the generalized fuzzy language determined by λ . Then from the definition of λ_{min}, we have

$$\lambda_{min}(pUVq) = \wedge_{uv\in UV} \lambda(puvq)$$
$$= \wedge_{uv\in UV} \lambda(pvuq)$$
$$= \lambda_{min}(pVUq)$$

for all $p,q \in A^*$ and $U,V \in F(A^*)$. Thus λ is commutative if and only if the generalized fuzzy language determined by λ satisfies the condition

$$\lambda_{min}(pUVq) = \lambda_{min}(pVUq),$$

for all $p,q \in A^*$ and $U,V \in F(A^*)$.

The following result shows that $ClFL_{min}(A^*)$ is closed under the operation meet $\wedge$.

Lemma3.9. If λ_1 and λ_2 are in $ClFL(A^*)$, then $(\lambda_1 \wedge \lambda_2)_{min}$ belongs to $ClFL_{min}(A^*)$.

Proof. Let $\lambda_1, \lambda_2 \in ClFL(A^*)$ then $\lambda_{1min}(pUVq) = \lambda_{1min}(pVUq)$ and $\lambda_{2min}(pUVq) = \lambda_{2min}(pVUq)$, for all $p,q \in A^*$ and $U,V \in F(A^*)$. By the definition of $\wedge$, We have

$$(\lambda_{1min} \wedge \lambda_{2min})(pUVq) = \lambda_{1min}(pUVq) \wedge \lambda_{2min}(pUVq)$$
$$= \lambda_{1min}(pVUq) \wedge \lambda_{2min}(pVUq)$$
$$= (\lambda_{1min} \wedge \lambda_{2min})(pVUq)$$

or all $p,q \in A^*$ and $U,V \in F(A^*)$. Thus $(\lambda_{1min} \wedge \lambda_{2min}) \in ClFL_{min}(A^*)$.Since $(\lambda_1 \wedge \lambda_2)_{min} = \lambda_{1min} \wedge \lambda_{2min}$, $(\lambda_1 \wedge \lambda 2)min$ belongs to $ClFL_{min}(A^*)$.

The following resultshows that $ClFL_{min}(A^*)$ is closed under quotients.

Lemma 3.10. *If* $\lambda_1, \lambda_2 \in ClFL(A^*)$, then $(\lambda_1{}^{-1}\lambda_2)_{min}$ and $(\lambda_2\lambda_1{}^{-1})_{min}$ are in $ClFL_{min}(A^*)$.

Proof. Let $\lambda_1, \lambda_2 \in ClFL(A^*)$ then $\lambda_{1min}(pUVq) = \lambda_{1min}(pVUq)$ and $\lambda_{2min}(pUVq) = \lambda_{2min}(pVUq)$, for all $p, q \in A^*$ and $U,V \in F(A^*)$. By the definition of left quotient, we have

$$(\lambda_{1min}^{-1}\lambda_{2min})(pUVq)$$
$$= \vee_{w\in A^*} (\lambda_{2min}(w(pUVq)) \wedge \lambda_{1min}(w)),$$
$$= \vee_{w\in A^*} (\lambda_{2min}(wp(UV)q) \wedge \lambda_{1min}(w)),$$
$$= \vee_{w\in A^*} (\lambda_{2min}(wp(VU)q) \wedge \lambda_{1min}(w)),$$
$$= \vee_{w\in A^*} (\lambda_{2min}(w(pVUq)) \wedge \lambda_{1min}(w)),$$
$$= (\lambda_{1min}^{-1}\lambda_{2min})(pVUq)$$

for all $p,q \in A^*$ and $U,V \in F(A^*)$. Thus $(\lambda_{1min}^{-1}\lambda_{2min}) \in ClFL_{min}(A^*)$. Similarly we can prove that $\lambda_{2min}\lambda_{1min}^{-1} \in ClFL_{min}(A^*)$. Since $(\lambda_1{}^{-1}\lambda_2)_{min} = (\lambda_{1min}^{-1}\lambda_{2min})$ and $(\lambda_2\lambda_1{}^{-1})_{min} = \lambda_{2min}\lambda_{1min}^{-1}$, we have $(\lambda_1{}^{-1}\lambda_2)_{min}$ and $(\lambda_2\lambda_1{}^{-1})_{min}$ belongs to $ClFL_{min}(A^*)$.

Lemma 3.11. Let A,B be finite alphabets, $\varphi : F(A^*) \to F(B^*)$ be a homomorphism and $\lambda \in ClFL_{min}(A^*)$. Then $(\lambda\varphi^{-1})_{min} \in ClFL_{min}(A^*)$.

Proof. We have

$$(\lambda_{min}\varphi^{-1})(pUVq) = \lambda_{min}(\varphi(pUVq)),$$
$$= \lambda_{min}(\varphi(p)\varphi(U)\varphi(V)\varphi(q))$$
$$= \lambda_{min}(\varphi(p)\varphi(V)\varphi(U)\varphi(q))$$
$$= \lambda_{min}(\varphi(pVUq))$$
$$= (\lambda_{min}\varphi^{-1})(pVUq)$$

for all p,q $p,q \in A^*$ *and* $U,V \in F(A^*)$. Thus $(\lambda\varphi^{-1})_{min} \in ClFL_{min}(A^*)$. Since $(\lambda\varphi^{-1})_{min} = \lambda_{min}\varphi^{-1}$, $(\lambda\varphi^{-1})_{min} \in ClFL_{min}(A^*)$.

from the above lemma it follows that $ClFL_{min}(A^*)$ is closed under inverse homomorphic images.

Theorem 3.12 $ClFL(A^*)$ is a conjunctive variety of *l*-fuzzy languages.

Proof. By Lemmas 3.9 and 3.10, $ClFL_{min}(A^*)$ is closed under meet and quotients. By Lemma 3.11, $ClFL_{min}(A^*)$ is closed under inverse homomorphic images. Hence $ClFL(A^*)$ is a conjunctive variety of *l*-fuzzy languages.

IV. CONCLUSION

We have studied varieties of class of *l*-fuzzy languages.
The varieties of different class of fuzzy languages have been studied by many authors. In this paper we have studied varieties of commutative *l*-fuzzy languages. We have proved that the class of commutative *l*-fuzzy languages is $*$-variety and conjunctive variety of *l*-fuzzy languages.

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