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Optimal Linear Arrangement of Turán Graphs

Arul Jeya Shalini, J. Nancy Delaila, J. Maria Julie

***Abstract:** Graph embedding in parallel processing techniques has acquired considerable attention and hence raised as an efficient approach for reducing overhead data into low-dimensional space. Optimal layout and congestion are powerful parameters to examine the capability of embedding. In this study, Modified Congestion and λ -Partition lemmas are utilized to obtain the optimal layout of Turán graph into path and windmill graphs.*

***Keywords:** Embedding, Congestion, Turán graph, Layout.*

I. INTRODUCTION

Graph theory, a discipline of mathematics, has enormous applications that exist in the fields of operation research, chemistry, social sciences, linguistics, electrical engineering, computer networks, etc. Embedding problems are dealt in graph theory as it represents the simulation of systems for parallel processors. Simulation of a network from one to the other can be developed mathematically as a problem of graph embedding [1] which is an ordered pair (g, R_g) of injective mapping between G_1 (guest graph) and G_2 (host graph) such that

(i) $g: V(G_1) \rightarrow V(G_2)$;

(ii) $R_g : E(G_1) \rightarrow R_g(x,y)$ such that $R_g(x,y)$ is a route in G_2 between $g(x)$ and $g(y)$ for $(x,y) \in E(G_1)$.

We represent the ordered pair (g, R_g) as g, for brevity. The congestion [2] of an edge e of G_2, is the number of edges (x, y) of G_1 such that e is in the path $R_g(x,y)$ between $g(x)$ and $g(y)$ in G_2. In other words,

$$EC_g(G_1, G_2(e)) = |\{(x,y) \in E(G_1) : e \in R_g(x,y)\}|.$$

The layout [3] of an embedding g of G_1 into G_2 is given by

$$L_g(G_1,G_2) = \sum_{(x,y)\in E(G_1)} |R_g(x,y)| = \sum_{e \in E(G_2)} EC_g(G_1, G_2(e)).$$

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Fig. 1 depicts the embedding of K_5 into P_5. The layout problem of G_1 into G_2 is $L(G_1, G_2) = min\, L_g(G_1, G_2)$.

A mathematical framework of embedding emerges from VLSI circuit designs, network reliability, numerical analysis, graph drawing, parallel processing networks, etc [3]. Several results exist for different architectures on the optimal layout problem (See [4, 5, 6, 7]).

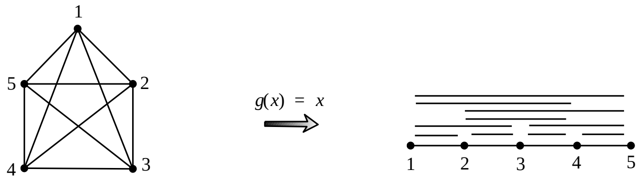

Fig. 1. Embedding g from K_5 into P_5 with $L_g(K_5, P_5) = 20$

The Maximum Subgraph Problem (MSP) [8] is a NP-complete problem wherein find $X \subseteq V(G)$, $I_G(X) = \{(x,y) \in E(G) : x, y \in X\}$ and for every $1 \le k \le |V(G)|$, let $I_G(k) = \max_{|X| \subseteq V(G), |X| = k} |I_G(X)|$. Then the objective of MSP is to identify $X \subseteq V(G)$, such that $|X| = k$ and $I_G(k) = |I_G(X)|$. Such a set X is the optimal set and the relation between optimality constrain and edge congestion is indicated in the lemma below.

Lemma 1. *(Modified Congestion Lemma) [9] Let g be an embedding of any arbitrary graph G_1 into G_2. Let S be an edge cut of G_2 such that the removal of S disengages G_2 into two components G_2^1 and G_2^2 and let $G_1^1 = G_1[g^{-1}(G_2^1)]$ and $G_1^2 = G_1[g^{-1}(G_2^2)]$. Furthermore, S satisfies the following conditions: For (x, y) in G_1,*

(i) $|E(R_g(x,y)) \cap S| = 0$, if $(x,y) \in G_1^i$ $i = 1, 2$.

(ii) $|E(R_g(x,y)) \cap S| = 1$, if $x \in G_1^1$ and $y \in G_1^2$.

(iii) G_1^1 and G_1^2 are optimal sets.

Then $EC_g(S)$ is minimum

$$EC_g(S) = \sum_{v \in V(G_1^1)} deg_{G_1}(v) - 2|E(G_1^1)|.$$

Lemma 2.*(λ -Partition Lemma) [4] Let g be an embedding from G_1 into G_2. Let $E^{\lambda}(G_2)$ denote a collection of edges of G_2 with each edge in G_2 repeated exactly λ times. Let $\{S_1, S_2, \ldots, S_p\}$ be a partition of $E^k(G_2)$. Then*

$$L_g(G_1, G_2) = \frac{1}{\lambda} \sum_{i=1}^{p} EC_g(S_i).$$

II. PRELIMINARIES

In this rapid pace of technological development, the efficiency of many digital systems is delineated by its interconnection architecture. Therefore building interconnection network with high-performance will maximize the ability of parallel computers. A complete interconnected network or well-connected topology has direct links with each processor in a parallel processing system.

The routing of messages within the processing elements becomes a straightforward process as each processor is associated with one another and there remains a single link to be traversed. Therefore, it ensures minimum time delay in transmitting a message to any destination within the network system and neglects inefficient communication. A complete graph represents the complete interconnected network and is defined as follows:

An inverse edgeless (null) graph with *r* nodes is a complete graph K_r. A complete *t*-partite graph $K_{n_1,n_2,\ldots,n_t}$ is a simple graph with *t* independent sets (called partites) in that two nodes are adjacent if and only if the nodes belong to different partites. In general, such graphs are also called as complete multipartite graphs. If order of all the partites is equal in $K_{n_1,n_2,\ldots,n_t}$ (i.e, $n_1 = n_2 = \ldots = n_t = r$), then it is termed as balanced complete multipartite graph K_r^t.

In 1941, Pál Turán introduced generalisation of K_{n_1,n_2} (bipartite graph), namely the Turán graph to assert his renowned Turán's Theorem in the field of extremal graph theory. For $n \geq t \geq 2$, the Turán graph $T(n,t)$ is the complete *t*-partite graph on *n* nodes with partites of order nearly equal. Intuitively, the Turán graph can be considered as a nearly balanced complete multipartite graph in which $n = tr + j$, $1 \leq j \leq t-1$ such that *j* partites have order $r+1$ while the other partites have order *r*. Hence degree of each node in $T(n,t)$ is either $n-r-1$ or $n-r$. The Turán's Theorem provides an upper bound on the size of Turán graph that is $|E(T(n,t))| \leq \frac{t-1}{t}\frac{n^2}{2}$. In particular, the exact number of edges in Turán graph [10] is $\frac{(t-1)(n^2-j^2)}{2t} + \binom{j}{2}$. In this paper, we stress our attention on Turán graph as it posses high fault tolerance ability for a network to prevent disruptions. Since Turán graph $T(n,t)$ is developed from K_r^t, the conditions of K_r^t can be further extended to $T(n,t)$ if $T(n,t)$ has the form $K_{\underbrace{r+1,r+1,\ldots,}_{j\ times}\underbrace{r,r,r}_{(t-j)\ times}}$.

Consider the labeling of nodes in $K_{n_1,n_2,\ldots,n_t}$ explained in [11] for Turán graph and it can be redefined as the nodes of each partite V_i, $1 \leq i \leq t$, of $T(n,t)$ is labeled as $(k-1)t+i$, when $i \leq j$, $1 \leq k \leq r+1$ and $i > j$, $1 \leq k \leq r$. We denote the ordering as $\mathscr{K}$ and propose a closed formulation as per MSP to count the cardinality of edges in $T(n,t)$. For $1 \leq x \leq n$, let $x = qt + k$, $0 \leq k \leq t$, $0 \leq q \leq r$,

$$I_{T(n,t)}(x) = \frac{1}{2}[q(qt+2k)(t-1) + k(k-1)].$$

Theorem 1. *[11] For $1 \leq x \leq n$, the set of first x nodes of $T(n,t)$ taken in the ordering $\mathscr{K}$ yields an optimal set.*

Corollary 1. *For $1 \leq x \leq n$, any set of x consecutive labels of ordering $\mathscr{K}$ provides an optimal set in $T(n,t)$.*

III. LINEAR ARRANGEMENT OF TURÁN GRAPHS

In this section, we present the embedding of Turán graph into the path graph and its layout computation. If path P_n is the host graph of an embedding, then the optimal layout problem is considered as bandwidth or optimal linear arrangement.

Theorem 2. *The optimal linear arrangement of Turán graph $T(n,t)$ with $n = tr + j$, $1 \leq j \leq t-1$ nodes is*

$$L(T(n,t),P_n) = \frac{1}{6}[j^3 + 3j^2(r(t-1)+1) + j(3r^2(t-1)t + 3r(t-1) - 6t + 2) + (t-1)t(r^3t - 2r + 3)].$$

Proof. Label the nodes of Turán graph by the ordering $\mathscr{K}$ and the nodes of P_n from 1 to *n* initiating from the left most node to right. Let *g* be an embedding of $T(n,t)$ into P_n given by $g(x) = x$.

Table 1 shows the partition of edges for the path graph P_n as shown in Fig. 2. From the partition of edge sets,

$$E(P_n) = \{S_i, 1 \leq i \leq t-1\} \cup \{SS_i, 1 \leq i \leq r\} \cup \{ST_i^k, 1 \leq i \leq t-1, 1 \leq i \leq r-1\} \cup \{ST_r^{k_1}, 1 \leq k_1 \leq j-1\}.$$

By Theorem 1, we conclude that the preimage of the components defined in column (iv) of Table 1 is optimal and satisfies all the conditions of Lemma 1. Therefore, edge congestion of all the cut edges is minimum and

$$EC_g(S_i) = \begin{cases} i[(t-1)r+j] - i - 2I_{T(n,t)}(i) & : i < j \\ i[(t-1)r+j] - j - 2I_{T(n,t)}(i) & : i \geq j \end{cases}$$
$$= \begin{cases} i[(t-1)r+j] - i - i(i-1) & : i < j \\ i[(t-1)r+j] - j - i(i-1) & : i \geq j. \end{cases}$$

$$EC_g(SS_i) = it((t-1)r+j) - ij - 2I_{T(n,t)}(it)$$
$$= it((t-1)r+j) - ij - i^2t(t-1).$$

$$EC_g(ST_i^k) = (it+k)((t-1)r+j-1) + i - 2I_{T(n,t)}(it+k)$$
$$= (it+k)((t-1)r+j-1) + i - i(it+2k)(t-1) - k(k-1).$$

$$EC_g(ST_r^{k_1}) = (rt+k_1)((t-1)r+j-1) + r - 2I_{T(n,t)}(rt+k_1)$$
$$= (rt+k_1)((t-1)r+j-1) + r - r(rt+2k_1)(t-1) - k_1(k_1-1).$$

By 1-Partition Lemma, we conclude that

$$L(T(n,t),P_n) = \sum_{i=1}^{t-1} EC_g(S_i) + \sum_{i=1}^{r} EC_g(SS_i) + \sum_{i=1}^{r-1}\sum_{k=1}^{t-1} EC_g(ST_i^k) + \sum_{k_1=1}^{j-1} EC_g(ST_r^{k_1})$$
$$= \frac{1}{6}[j^3 + 3j^2(r(t-1)+1) + j(3r^2(t-1)t + 3r(t-1) - 6t + 2) + (t-1)t(r^3t - 2r + 3)].$$

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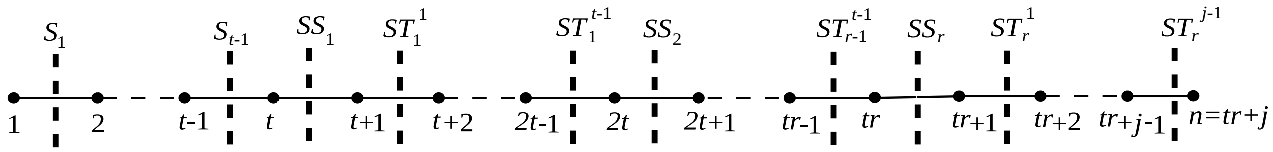

Fig. 2. Different cuts of P_n

Table- I: Description of different cut edges in P_n

Cut edge description	Range	Components	node set of the component
$S_i=\{(i,i+1)\}$	$1\le i\le t-1$	$X_i,\bar{X}_i$	$V(X_i)=\{1,2,\ldots,i\}$
$SS_i=\{(it,it+1)\}$	$1\le i\le r$	$Y_i,\bar{Y}_i$	$V(Y_i)=\{1,2,\ldots,it\}$
$ST_i^k=\{(it+k,it+k+1)\}$	$1\le k\le t-1$, $1\le i\le r-1$	$Z_i^k,\bar{Z}_i^k$	$V(Z_i^k)=\{1,2,\ldots,it+k\}$
$ST_r^{k_1}=\{(rt+k_1,rt+k_1+1)\}$	$1\le k_1\le j-1$	$Z_r^{k_1},\bar{Z}_r^{k_1}$	$V(Z_r^{k_1})=\{1,2,\ldots,rt+k_1\}$

IV. EMBEDDING TURÁN GRAPHS INTO WINDMILL GRAPHS

We begin this section with the definition of windmill graphs.

Definition 1. [6] *For $t\ge 2$, the windmill graph $W_{K_{r+1}}^t$ on $tr+1$ nodes, has t copies of the complete graph K_{r+1} with a common node.*

Remark. *We denote the t copies of $W_{K_{r+1}}^t$ by K_{r+1}^i, $1\le i\le t$. For convenience, we write $w_i=ir+1$, $1\le i\le t$ and $w_0=0$.*

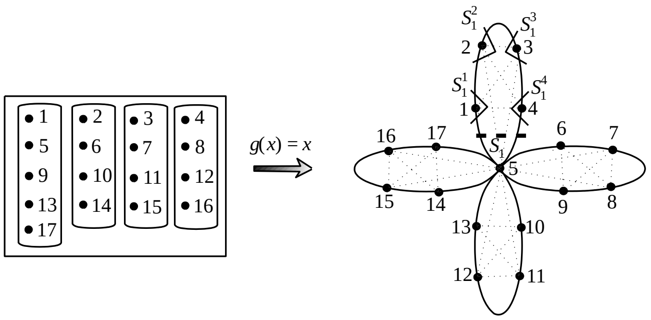

Fig. 3. Embedding of $T(17,4)$ into $W_{K_5}^4$ with edge cuts

Theorem 3. *For $l\ge 1$ $r=lt$, the layout of embedding Turán graphs $T(n,t)$ with $n=tr+1$ nodes and into Windmill graphs $W_{K_{r+1}}^t$ is given by*

$$L(T(n,t),W_{K_{r+1}}^t)=lt(t-1)(2lt^2-lt+2).$$

Proof. Label the nodes of K_{r+1}^1 of $W_{K_{r+1}}^t$, $1,2,\ldots,r+1$ and label the nodes of K_{r+1}^i, $2\le i\le t$ in $W_{K_{r+1}}^t$ as $w_{i-1}+j$, $1\le j\le r$ as shown in Fig. 3. The embedding g of $T(n,t)$ into $W_{K_{r+1}}^t$ defined by $g(x)=x$ gives the optimal layout.

For $1\le i\le t$, $1\le j\le r$, let $S_i^j=\{(w_{i-1}+j,v):v\in(V(K_{r+1}^i)-\{w_{i-1}+j\})\}$ be the edge cut of windmill graph. $E(W_{K_{r+1}}^t)-S_i^j$ disengaged the graph into X_1 and X_2 where $V(X_1)=\{w_{i-1}+j\}$. The preimages of X_1 and X_2 are optimal. Then S_i^j satisfies the conditions of Lemma 1 and $EC_g(S_i^j)$, $1\le i\le t$, $1\le j\le r$ is minimum and

$$EC_g(S_i^j)=\begin{cases}(t-1)r & :\text{if } v=kt+1, 0\le k\le l-1 \,\&\, v\in V(W_{K_{r+1}}^t)\\(t-1)r+1 & :\text{else.}\end{cases}$$

For $1\le i\le t$, let $S_i=\{(r+1,v_1)\cup(r+1,v_2):v_1,v_2\in(V(K_{r+1}^i)-(r+1))\}$ be the edge cut of windmill graph. Then, $E(W_{K_{r+1}}^t)-S_i$, splits the graph into X_1 and X_2 where $V(X_1)=\{w_{i-1}+j:1\le j\le r\}$. By Corollary 1, the preimages of X_1 and X_2 are optimal. Then S_i satisfies all the conditions of Lemma 1. Hence

$$EC_g(S_i)=l(t-1)r+(r-l)((t-1)r+1)-2I_{T(n,t)}(r)$$
$$=l(t-1)[lt^2-lt+1]$$

is minimum for $1\le i\le t$. From the above edge cuts, we see that $E^2(W_{K_{r+1}}^t)=\{S_i^j:1\le i\le t,1\le j\le r\}\cup\{S_i:1\le i\le t\}$.

Then by 2-Partition Lemma,

$$L(T(n,t),W_{K_{r+1}}^t)=\frac{1}{2}\Big[\sum_{i=1}^{t}\sum_{j=1}^{r}EC_g(S_i^j)+\sum_{i=1}^{t}EC_g(S_i)\Big]$$
$$=lt(t-1)(2lt^2-lt+2).$$

V. CONCLUSION

As Turán structure has high fault tolerance and connection within all the processors, it is accustomed to broadcasting and switching. In this paper, we have obtained the optimal linear arrangement of Turán graphs using MSP. Further applying the λ-partition lemma, we computed the optimal layout of Turán graph into windmill graph.

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