On Applications of Some Special Functions in Statistics

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Abstract: ***In this paper we will introduce some probability distributions with help of some special functions like Gamma, k-Gamma functions, Beta, k-Beta functions, Bessel, modified Bessel functions and Laguerre polynomials and in mathematical analysis used Laplace transform. We will also obtain their cumulative density function, expected value, variance, Moment generating function and Characteristic function. Some characteristics and real life applications will be computed in tabulated for these distributions***

Keyword: Gamma function, Probability distribution, moment generating function, Laplace transforms,

I. INTRODUCTION

All special functions and their properties are field of mathematics, physics, statistics and engineering. Many probability distributions of special functions like Exponential, Weibull, Gamma, Beta, log-normal, Beta-normal and Beta Pareto distributions are used for analyze various problems related to human behavior and nature (Mathai et al [1], Rainville [5], Bapna et al [10], and Rehaman et al [9], Krishanamoorty [12] and Walac [4]) . Researchers (Bapna et al [10], Biondi et al [6], Madi et al [13], Subburaj et al [17],Cottne et al [8], Steffensen [11] and Yeates et al [14]) investigate many problems related to environments, queuing theory, climatology, animal behavior, ecology, actuarial science and machines reliability with the help computer software and mathematical modeling by characteristics of probability distributions

1.1 Probability distribution

A function $p(x)$ be discrete probability density function for countable values of random variable x then $p(x)$ has the following properties (Krishanamoorty [12], Walac [4])

$$\text{(i)} \quad p(x) \geq 0 \qquad \qquad \text{(ii)} \quad \sum_{i=0}^{n} p(x_i) = 1 \tag{1.1}$$

A function $f_x(x)$ be continuous probability density function for uncountable values of random variable x then $f_x(x)$ has the properties

$$\text{(i)} \quad f_x(x) \geq 0 \qquad \qquad \text{(ii)} \quad \int_{-\infty}^{\infty} f_x(x)dx = 1 \tag{1.2}$$

1.2 The k-Gamma and k-Beta functions

The Gamma function Γz is introduced by Euler (1707-1783). It is defined by

$$\Gamma z = \int_0^{\infty} x^{z-1} e^{-x} dx \qquad ; \qquad \operatorname{Re}(z) > 0 \tag{1.3}$$

$$\Gamma(z+1) = z\Gamma z \tag{1.4}$$

Many researchers like Euler, Carl Friendrich Gauss, Karl Weierstrass and Egan have extended the research work of Gamma function. They have introduced many applications of Gamma function which are useful in various branches (Mathai et al [1], Rainville [5])

The Beta function $B(\alpha, \beta)$ is introduced following by Euler and Legendre as

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dx \ ; \quad \operatorname{Re}(\alpha) > 0, R(\beta) > 0 \tag{1.5}$$

The Beta function in term of Gamma function is given by

$$B(\alpha, \beta) = \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)} \tag{1.6}$$

In 2007 R. Diaz and E. Pariguan have (see Daiz et al.[15], Daiz et al.[16]) introduced Pochhammer k-symbol and k-Gamma function that are defined as

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)(\alpha+3k)......(\alpha+(n-1)k) & ; n \geq 1, k > 0 \\ 1 & ; n = 0 \end{cases} \tag{1.7}$$

The k-Gamma function is defined as

$$\Gamma_k(z) = \int_0^{\infty} x^{z-1} e^{-\frac{x^k}{k}} dx \qquad ; \ \text{For} \quad k>0, \ \operatorname{Re}(z) > 0 \ , \tag{1.8}$$

Equation (9) we can write as

$$\Gamma_k(z) = k^{\frac{x}{k}} \Gamma\left(\frac{x}{k}\right) \tag{1.9}$$

Many authors have investigated some properties of k-Gamma function (see more detail Kokologiannak et al [2], Kokologiannak [3] and Merovci [7])

The k-Beta function $B_k(\alpha, \beta)$ for two variable α and β is defined by (see more detail Kokologiannak et al [2], Kokologiannak [3] and Merovci [7])

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$$B_k(\alpha,\beta)=\frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1} \quad ; \quad \mathrm{Re}(\alpha)>0, \ \mathrm{Re}(\beta)>0$$

(1.10)

Equation (1.10) in term (1.8) is given by

$$B_k(\alpha,\beta)=\frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \quad ; \quad \mathrm{Re}(\beta)>0$$

(1.11)

II. PROPOSED METHODOLOGY

The special functions have been widely used in science and technology with probability distribution theory for several kinds of problem in reliability analysis. Main results of this paper will be used following methodology

(1)In this paper we will construct new probability distributions with the help of some special functions.

(2)Some important properties these will be described.

(3)In mathematical analysis of main results will be used some special functions and Laplace transforms technique

III. MAIM RESULTS PROBABILITY DISTRIBUTION WITH GAMMA, K-GAMMA AND K-BETA FUNCTIONS

Theorem 3.1 Let X be a uncountable values of random variable then function

$$f_x(x,k)=\begin{cases}\dfrac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}} & ;0<x<\infty, k>0\\ 0 & ;otherwise\end{cases}$$

(3.1)

is follows equation (1.2). Where

$$\Gamma_k\left(\tfrac{1}{2}\right)=k^{\frac{1}{2k}-1}\Gamma\left(\tfrac{1}{2k}\right)$$

(3.2) **Proof.** Obviously

$f_x(x,k)\geq 0$; for $0<x<\infty, k>0$

Now

$$\int_0^\infty f_x(x,k)dx \;=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^\infty e^{-\frac{x^{2k}}{k}}dx$$

$\because$ put $x^2=u$

$$=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^\infty \frac{1}{2}u^{\frac{1}{2}-1}e^{-\frac{u^k}{k}}du=\frac{\Gamma_k\left(\frac{1}{2}\right)}{\Gamma_k\left(\frac{1}{2}\right)}=1$$

Thus theorem has been proved

3.2 The cumulative density function of theorem 3.1

The cumulative density function of equation (3.1) is given by

$$F_z(z)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^z e^{-\frac{x^{2k}}{k}}dx=\frac{\Gamma_k\left(\frac{1}{2},z^2\right)}{\Gamma_k\left(\frac{1}{2}\right)}$$

(3.3)

Equation (3.3) in series expansion as

$$F_z(z)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^z e^{-\frac{x^{2k}}{k}}dx=\frac{2}{k^{\frac{1}{2k}}\Gamma\left(\frac{1}{2k}\right)}\sum_{r=0}^{\infty}\left(-\frac{1}{k}\right)^r\frac{z^{2rk+1}}{r!(2kr+1)}$$

(3.4)

The survivor function $S_z(z)$ of equation (3.1) is given by

$$S_z(z)=P(X\geq z)=1-F_z(z)$$

$$=\int_z^\infty f_x(x)dx=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_z^\infty e^{-\frac{x^{2k}}{k}}dx=\frac{\gamma_{kc}\left(\frac{1}{2},z^2\right)}{\Gamma_k\left(\frac{1}{2}\right)}$$

(3.5)

Equation (3.3) can be expressed as

$$F_z(z)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^z e^{-\frac{x^{2k}}{k}}dx=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^u e^{-\frac{iu^{2k}}{k}}du \quad \because$$

Put $x=\sqrt[2k]{iu}$

$$F_u(u)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^z\left(\cos\left(\frac{u^{2k}}{k}\right)-i\sin\left(\frac{u^{2k}}{k}\right)\right)du$$

$$=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\left[\int_0^u\cos\left(\frac{u^{2k}}{k}\right)du-i\int_0^u\sin\left(\frac{u^{2k}}{k}\right)du\right]$$

$$=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\left[I_{ck}-iI_{sk}\right]$$

Where

$$I_{ck}=\int_0^u\cos\left(\frac{u^{2k}}{k}\right)du=\sum_{r=0}^{\infty}\left(\frac{-1}{k^{4k}}\right)^r\frac{u^{4rk+1}}{2r!(4rk+1)}$$

(3.6)

$$I_{sk}=\int_0^u\sin\left(\frac{u^{2k}}{k}\right)du=\sum_{r=0}^{\infty}\left(\frac{-1}{k^{4k+2}}\right)^r\frac{u^{4rk+3}}{(2r+1)!(4rk+3)}$$

(3.7)

Equations (3.6) and (3.7) like Fresnel integrals. Fresnel integrals have many applications in physics and engineering

3.3 Mean, Moments and Variance of theorem 3.1

Using equation (3.1)

$$E(x)=\int_0^\infty xf_x(x,k)dx$$

$$=\int_0^\infty\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}xe^{-\frac{x^{2k}}{k}}dx=\frac{\Gamma_k\left(\frac{3}{2}\right)}{\Gamma_k\left(\frac{1}{2}\right)}$$

$$= \frac{k^{\frac{3}{2k}-1}\Gamma\left(\frac{3}{2k}\right)}{k^{\frac{1}{2k}-1}\Gamma\left(\frac{1}{2k}\right)} = k^{\frac{1}{2k}}\frac{\Gamma\left(\frac{3}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)}$$

(3.8)

$$E\left(x^r\right) = \int_0^\infty x^r f_x(x,k)dx$$

$$= \frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^\infty x^r e^{-\frac{x^{2k}}{k}}dx = \frac{\Gamma_k\left(\frac{r+1}{2}\right)}{\Gamma_k\left(\frac{1}{2}\right)}$$

$$= k^{\frac{r}{2k}}\frac{\Gamma\left(\frac{r+1}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)}$$

(3.9) Equation (3.9) is represent r^th moment about origin of equation (3.1)

Using equation (3.9), the variance of equation (3.1) is given by

$$Var(x) = k^{\frac{1}{k}}\left[\frac{\Gamma\left(\frac{5}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)} - \left(\frac{\Gamma\left(\frac{3}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)}\right)^2\right]$$

(3.10) Note if k=1 then all results of theorem 3.1 like the standard normal distribution

Theorem 3.4 Let X be uncountable values of random variable then the function

$$f_x(x,\gamma,\alpha,\beta,k) = \begin{cases} \frac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1} e^{-\frac{\gamma\beta x^k}{k}} & ; x > 0 \\ 0 & ; otherwise \end{cases} \qquad (3.11)$$

follows equation (1.2). Where $\alpha > 0, \beta > 0, \gamma > 0$ and $k > 0$ are parameters

Proof. Obviously

$f_x(x,\gamma,\alpha,\beta,k) \geq 0$; $\forall x > 0, \alpha > 0, \beta >, k > 0$ and $\gamma > 0$

Now

$$\int_0^\infty f_x(x,\gamma,\alpha,\beta,k)dx$$

$$= \int_0^\infty \frac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1} e^{-\frac{\gamma\beta x^k}{k}}dx \qquad \because \text{ Put } e^{-\gamma x^k} = u$$

$$= \frac{1}{kB_k(\alpha,\beta)}\int_0^1 (1-u)^{\frac{\alpha}{k}-1} u^{\frac{\beta}{k}-1}du = 1$$

So that $f_x(x,\gamma,\alpha,\beta,k)$ follows equation (1.2)

3.5 The Moment Generating Function of Theorem 3.4

The moment generating function of equation (3.11) in tern of new parameter k>0, is given as

$$M_x(t) = \int_0^\infty e^{xt} f(x,\gamma,\alpha,\beta,k)dx$$

$$= \int_0^\infty e^{x^k t}\frac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1} e^{-\gamma\beta x^k}dx$$

$\because$ Put $e^{-\gamma x^k} = u$

$$= \frac{1}{kB_k(\alpha,\beta)}\int_0^1 u^{\frac{\beta}{k}-\frac{t}{\gamma}-1}(1-u)^{\frac{\alpha}{k}-1}du = \frac{B_k\left(\beta-\frac{tk}{\gamma},\alpha\right)}{B_k(\alpha,\beta)}$$

$$= \frac{\Gamma_k(\alpha+\beta)}{\Gamma_k\left(\alpha+\beta-\frac{kt}{\gamma}\right)}\frac{\Gamma_k\left(\beta-\frac{kt}{\gamma}\right)}{\Gamma_k(\beta)}$$

$$= \frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)\Gamma\left(\frac{\beta}{k}-\frac{t}{\gamma}\right)}{\Gamma\left(\frac{\alpha+\beta}{k}-\frac{t}{\gamma}\right)}$$

(3.12)

3.6 Probability distribution with Bessel functions

In this section we will discuss probability density function with the help of Bessel functions.

Modified Bessel differential equation of order ν is defined as (Mathai et al [1], Rainville [5])

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 - \nu^2\right)y = 0$$

(3.13)

Equation (3.13) has the solution

$Y_n(x) = AI_\nu(x) + BK_\nu(x)$; A and B being arbitrary constant

Where $I_\nu(x)$ denotes modified Bessel function of the order ν of the first kind and

$$I_\nu(x) = \sum_{r=0}^\infty \frac{(x/2)^{\nu+2r}}{\Gamma(r+1)\Gamma(r+\nu+1)}$$

(3.14)

Equation (3.14) is like growing exponential (Mathai et al [1], Rainville [5])

Where $K_\nu(x)$ denotes modified Bessel function of order ν of the second kind and

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi} \qquad \nu \neq 0, \pm 1, \pm 2, \pm 3. \;.\;.$$

(3.15)

Equation (3.15) is like decaying exponential(Mathai et al [1], Rainville [5]).

If $\nu = \frac{1}{2}$ and $-\frac{1}{2}$ then equations (3.14) and (3.15) will be(Mathai et al [1], Rainville [5])

(i) $I_{1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{\sinh x}{\sqrt{x}}$ and $I_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{\cosh x}{\sqrt{x}}$

(3.16)

(ii)
$$K_{1/2}(x)=\sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}$$
(3.17)

Theorem 3.7 Let X be continuous random variable then the function

$$f_x(x)=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x) \quad ; \quad x>0,\lambda>2$$
(3.18)

follows equation (1.2)

Proof. Clearly

$$f_x(x)\geq 0; x>0$$

Now

$$\int_0^{\infty} f_x(x)dx$$

$$=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{-\lambda x}I_{-1/2}(x)I_{1/2}(x)dx$$

$$=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{-\lambda x}\frac{2\sinh x\cosh x}{\pi x}dx$$

$$=\frac{1}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{-\lambda x}\left(\frac{e^{2x}-e^{-2x}}{x}\right)$$

Solved by the help of Laplace transform technique

$$=\frac{\log\left(\frac{\lambda+2}{\lambda-2}\right)}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}=1$$

Hence $f_x(x)$ has satisfied equation (1.2)

3.8 The moment generating function of theorem 3.7

The moment generating function of equation (3.18) is given as

$$M_x(t)=\int_0^{\infty} e^{xt}f_x(x)dx$$

$$=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{xt}e^{-\lambda x}I_{-1/2}(x)I_{1/2}(x)dx$$

$$=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{-x(\lambda-t)}\frac{2\sinh x\cosh x}{\pi x}dx$$

$$=\frac{1}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^{\infty} e^{-x(\lambda-t)}\frac{e^{2x}-e^{-2x}}{x}dx$$

Solve by the help of Laplace transform

$$\Rightarrow M_x(t)=\frac{\log\left(\frac{\lambda-t+2}{\lambda-t-2}\right)}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}$$
(3.19)

Thus equation (3.19) is moment generating function of theorem 3.7

Theorem 3.9 Let X be uncountable values of random variable then the function

$$f_x(x)=\frac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x) \;; x>0$$
(3.20)

follows equation (1.2)

Proof. Clearly

$$f_x(x)\geq 0 \;;\forall x>0$$

Now

$$\int_0^{\infty} f_x(x)dx$$

$$=\frac{1}{\log 3}\int_0^{\infty} e^{-x}I_{1/2}(x)K_{1/2}(x)dx$$

$$=\frac{1}{\log 3}\int_0^{\infty} e^{-x}\frac{e^{-x}\sinh x}{x}dx$$

$$=\frac{1}{\log 3}\int_0^{\infty} e^{-x}\frac{1-e^{-2x}}{x}dx$$

Solved by the help of Laplace transform technique

$$=\frac{\log 3}{\log 3}=1$$

Hence $f_x(x)$ has satisfied equation (1.2)

3.10 The moment generating function of theorem 3.9

The moment generating function of equation (3.20) is given as

$$M_x(t)=\int_0^{\infty} e^{xt}f_x(x)dx$$

$$=\int_0^{\infty} e^{xt}\left[\frac{1}{\log 3}I_{1/2}(x)K_{1/2}(x)\right]dx$$

$$=\frac{1}{\log 3}\int_0^{\infty} e^{-x(1-t)}\frac{1-e^{-2x}}{x}dx$$

Solved by the help of Laplace transform

$$=\frac{\log\left(1+\frac{2}{1-t}\right)}{\log 3}$$
(3.21)

Equation (3.21) represents the M.G.F. of theorem 3.9
Above discussed theorems 3.1, 3.4, 3.7 and 3.1 of moment generating functions (M.G.F's) and characteristic functions in following table-1

Table-1

S. N	Probability distribution	$M_x(t)$	$\phi_x(t)$
1	$f_x(k,x)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}}$;$0<x<\infty, k>0$;	$\sum_{r0}^{\infty}\frac{t^r}{r!}k^{\frac{r}{2k}}\frac{\Gamma\left(\frac{r+1}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)}$	$\sum_{r0}^{\infty}\frac{(it)^r}{r!}k^{\frac{r}{2k}}\frac{\Gamma\left(\frac{r+1}{2k}\right)}{\Gamma\left(\frac{1}{2k}\right)}$
2	$f_x(x,\alpha,m,n,k)=\frac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}$;$x>0$	$\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)\Gamma\left(\frac{\beta}{k}-\frac{t}{\gamma}\right)}{\Gamma\left(\frac{\alpha+\beta}{k}-\frac{t}{\alpha}\right)}$	$\frac{\Gamma\left(\frac{\alpha+\beta}{k}\right)\Gamma\left(\frac{\beta}{k}-\frac{ti}{\gamma}\right)}{\Gamma\left(\frac{\alpha+\beta}{k}-\frac{ti}{\gamma}\right)}$
3	$f_x(x)=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)$;$x>0,\lambda>2$	$\frac{\log\left(\frac{\lambda-t+2}{\lambda-t-2}\right)}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}$	$\frac{\log\left(\frac{\lambda-ti+2}{\lambda-ti-2}\right)}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}$
4	$f_x(x)=\frac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x)$;$x>0$	$\frac{\log\left(1+\frac{2}{1-t}\right)}{\log 3}$	$\frac{\log\left(1+\frac{2}{1-ti}\right)}{\log 3}$

3.11 Life Time of Component $F_x(x)$, Survivor function $S_x(x)$, Hazard Rate Function $h(x)$ and Mean residue Life Function $\kappa(x)$

Let $f_x(x)$ is follows equation (1.2) then the probability of failure till time x is given by (Krishanamoorty [12], Walac [4])

$$F_x(x)=P(X\le x)=\int_{-\infty}^{x}f_x(x)dx \quad , \quad \text{for} \quad x>0$$

(3.22) The probability that the component survivor until time x is denoted $S_x(x)$ and can be introduce as

$$S_x(x)=F(X\ge x)=\int_x^{\infty}f_x(x)dx$$

(3.23) Let $f_x(x)$ follows equation (1.2) then hazard rate function $h(x)$ and mean residue life function $\kappa(x)$ are defined by respectively (Krishanamoorty [12], Walac [4])

$$h(x)=\frac{f_x(x)}{S(x)}$$

(3.24) $\kappa(x) =$

$$E(X-x|X\ge x)=\frac{\int_x^{\infty}(t-x)f_t(t)dt}{S(x)}=\frac{\int_x^{\infty}tf_t(t)dt}{S(x)}-x$$

(3.25) Above discussed theorems 3.1, 3.4, 3.7 and 3.9 of life time components, survivor functions, hazard rate functions and mean residue life functions are given following in tabulated respectively

Table-2

S. N	Probability distribution	$F_x(x)$
1	$f_x(k,x)=\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}}$;$0<x<\infty, k>0$;	$\frac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_0^x e^{-\frac{x^{2k}}{k}}dx$
2	$f_x(x,\gamma,\alpha,\beta,k)=\frac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}$;$x>0$	$\frac{\gamma}{B_k(\alpha,\beta)}\int_0^x x^{k-1}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}dx$
3	$f_x(x)=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)$;$x>0,\lambda>2$	$\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_0^x e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)dx$
4	$f_x(x)=\frac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x)$;$x>0$	$\frac{1}{\log 3}\int_0^x e^{-x}I_{1/2}(x)K_{1/2}(x)dx$

Table-3

S. N	Probability distribution	$S_x(x)$
1	$f_x(k,x)=\dfrac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}}\qquad ;0<x<\infty, k>0;$	$\dfrac{2}{\Gamma_k\left(\frac{1}{2}\right)}\int_x^{\infty} e^{-\frac{x^{2k}}{k}}dx$
2	$f_x(x,\gamma,\alpha,\beta,k)=\dfrac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}$ $;x>0$	$\dfrac{\gamma}{B_k(\alpha,\beta)}\int_x^{\infty} x^{k-1}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}dx$
3	$f_x(x)=\dfrac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)\quad ;x>0,\lambda>2$	$\dfrac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}\int_x^{\infty} e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)dx$
4	$f_x(x)=\dfrac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x)\ ;x>0$	$\dfrac{1}{\log 3}\int_x^{\infty} e^{-x}I_{1/2}(x)K_{1/2}(x)dx$

Table-4

S. N	Probability distribution	$h(x)$
1	$f_x(k,x)=\dfrac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}}\qquad ;0<x<\infty, k>0;$	$\dfrac{e^{-\frac{x^{2k}}{k}}}{\int_x^{\infty} e^{-\frac{x^{2k}}{k}}dx}$
2	$f_x(x,\gamma,\alpha,\beta,k)=\dfrac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}$ $;x>0$	$\dfrac{x^{k-1}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\gamma\beta x^k}}{\int_x^{\infty} x^{k-1}\left(1-e^{\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\gamma\beta x^k}dx}$
3	$f_x(x)=\dfrac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)\quad ;x>0,\lambda>2$	$\dfrac{e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)}{\int_x^{\infty} e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)dx}$
4	$f_x(x)=\dfrac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x)\ ;x>0$	$\dfrac{e^{-x}I_{1/2}(x)I_{-1/2}(x)}{\int_x^{\infty} e^{-x}I_{1/2}(x)I_{-1/2}(x)dx}$

Table-5

S. N	Probability distribution	$\kappa(x)$
1	$f_x(k,x)=\dfrac{2}{\Gamma_k\left(\frac{1}{2}\right)}e^{-\frac{x^{2k}}{k}}\qquad ;0<x<\infty, k>0;$	$\dfrac{\int_x^{\infty} xe^{-\frac{x^{2k}}{k}}dx}{\int_x^{\infty} e^{-\frac{x^{2k}}{k}}dx}-x$
2	$f_x(x,\gamma,\alpha,\beta,k)=\dfrac{\gamma x^{k-1}}{B_k(\alpha,\beta)}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\frac{\gamma\beta x^k}{k}}$ $;x>0$	$\dfrac{\int_x^{\infty} xx^{k-1}\left(1-e^{-\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\gamma\beta x^k}dx}{\int_x^{\infty} x^{k-1}\left(1-e^{\gamma x^k}\right)^{\frac{\alpha}{k}-1}e^{-\gamma\beta x^k}dx}-x$

3	$f_x(x)=\frac{2\pi}{\log\left(\frac{\lambda+2}{\lambda-2}\right)}e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x) \quad ;x>0,\lambda>2$	$\frac{\int_x^{\infty} xe^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)dx}{\int_x^{\infty} e^{-\lambda x}I_{1/2}(x)I_{-1/2}(x)dx}-x$
4	$f_x(x)=\frac{e^{-x}}{\log 3}I_{1/2}(x)K_{1/2}(x) \;;x>0$	$\frac{\int_x^{\infty} xe^{-x}I_{1/2}(x)I_{-1/2}(x)dx}{\int_x^{\infty} e^{-x}I_{1/2}(x)I_{-1/2}(x)dx}-x$

IV. CONCLUSIONS

For this paper the authors have concluded that above discussed theorems 3.1, 3.4, 3.7 and 3.9 (probability distributions) have area under the curve is unity. Also authors discussed like cumulative density function, mean, variance, moments and M.G.F.'s for these theorems. By the help of these theorems the authors have obtained in tabulated some applications of real life.

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