On Riemann-Liouville Fractional Calculus and F-Function

Cristina Gammeng, U. K. Saha, S. Maity

Abstract: ***The objective of the work is to explore certain correspondences that prevail amongst Riemann-Liouville fractional calculus and the F-function. The work is expected to find utilization in areas of fundamental linear order fractional differential and integral equations involving the F-function.***

Keywords: Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, F-function.

I. INTRODUCTION

The fractional calculus is one of the fastest expanding subject of the mathematical analysis. It is the area that deals with the integrals and derivatives of arbitrary order. It has been enforced in various field of science, engineering and technology. Numerous researchers have investigated their properties, applications, relationships with different special functions in detail. Today there are various form of fractional calculus operators but the Riemann-Liouville operator still remains he most widely and extensively used operator.

The operators of the left and right side of the Riemann-Liouville fractional calculus according to the section 2 of the book by Samko, Kilbas and Marichev [5] are defined as follows:

$$\left(I_{+}^{\alpha} g\right)(y)=\frac{1}{\Gamma(\alpha)}\int_{0}^{y}\frac{g(z)}{(y-z)^{1-\alpha}}dz, \alpha>0 \tag{1}$$

$$\left(I_{-}^{\alpha} g\right)(y)=\frac{1}{\Gamma(\alpha)}\int_{y}^{\infty}\frac{g(z)}{(z-y)^{1-\alpha}}dz, \alpha>0 \tag{2}$$

$$\left(D_{+}^{\alpha} g\right)(y)=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{+}^{1-\{\alpha\}} g\right)(y)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_{0}^{y}\frac{g(z)}{(y-z)^{\{\alpha\}}}dz, \alpha>0 \tag{3}$$

$$\left(D_{-}^{\alpha} g\right)(y)=\left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}} g\right)(y)=\frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_{y}^{\infty}\frac{g(z)}{(z-y)^{\{\alpha\}}}dz, \alpha>0$$

Where {α} is the fractional part of {α} and [α] is the (4)

Maximum integer not exceeding α.

Hartley and Lorenzo (1998) introduced the F-function as,

$$F_q(a,z)\equiv\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}, q>0 \tag{5}$$

II. MAIN RESULTS

Here, 4 theorems have been constructed regarding the Riemann-Liouville (R-L) fractional calculus and F-function.

Theorem 2.1 If I_{+}^{α} be the R-L fractional integral operator of the left side (1) and let μ>0, then

$$\left(I_{+}^{\alpha}[z^{\mu}F_q(a,z)]\right)(y)=y^{\mu+\alpha+q-1}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu+(m+1)q)}{\Gamma(\alpha+\mu+(m+1)q)} \tag{6}$$

Proof:

$$\Delta\equiv\left(I_{+}^{\alpha}[z^{\mu}F_q(a,z)]\right)(y)$$

$$=\frac{1}{\Gamma(\alpha)}\int_{0}^{y}(y-z)^{\alpha-1}z^{\mu}\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}dz$$

Swapping the order of integration and summation then examining the integral using beta function formula, we get

$$=y^{\mu+\alpha+q-1}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{1}{\Gamma(q)}\int_{0}^{1}(1-u)^{\alpha-1}u^{\mu+(m+1)q-1}du$$

$$=y^{\mu+\alpha+q-1}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu+(m+1)q)}{\Gamma(\alpha+\mu+(m+1)q)}$$

$$\left(I_{+}^{\alpha}[F_q(a,z)]\right)(y)=y^{\alpha+q}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma((m+1)q+1)}{\Gamma((m+1)q+\alpha+1)} \tag{7}$$

$$\left(I_{+}^{\alpha}[F_q(a,z)]\right)(y)=\frac{2aqy^{\alpha+2q}}{(\alpha+2q)!} \tag{8}$$

Theorem 2.4 If I_{-}^{α} be R-L fractional integral operator along the right side (2) and let μ>0, then

$$\left(I_{-}^{\alpha}[z^{-\alpha-\mu}F_q(a,z)]\right)(y)=y^{-\mu+mq}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu-(m+1)q)}{\Gamma(\alpha+\mu-(m+1)q)} \tag{9}$$

Proof: $\Delta\equiv\left(I_{-}^{\alpha}[z^{-\alpha-\mu}F_q(a,z)]\right)(y)$

Revised Manuscript Received on February 18, 2020.

***** Correspondence Author

Cristina Gammeng*, Department of Basic and Applied Science, National Institute of Technology, Arunachal Pradesh, India. E-mail: cgammeng@gmail.com

U. K. Saha, Department of Basic and Applied Science, National Institute of Technology, Arunachal Pradesh, India. E-mail: utpal@nitap.ac.in

S. Maity, Department of Basic and Applied Science, National Institute of Technology, Arunachal Pradesh, India. E-mail: susantamaiti@gmail.com

$$=\frac{1}{\Gamma(q)}\int_{y}^{\infty}(z-y)^{\alpha-1}z^{-\alpha-\mu}\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}$$

If we interchange the form of summation and integration,

$$=\frac{1}{\Gamma(q)}\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\int_{y}^{\infty}(z-y)^{\alpha-1}z^{-\alpha-\mu}(z)^{(m+1)q-1}dz$$

Now,using beta function in the integral, we get

$$=\frac{1}{\Gamma(q)}\sum_{m=0}^{\infty}\frac{a^m y^{-\mu+(m+1)q}}{\Gamma((m+1)q)}\int_{0}^{1}(1-u)^{\alpha-1}(u)^{\mu-(m+1)q+1}du$$

$$=y^{-\mu+mq}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q)}{\Gamma((m+1)q)\Gamma(\alpha+\mu-(m+1)q)}$$

Theorem 2.5 If D_{+}^{α} be the R-L fractional derivative operator along the left side (3) and let μ>0, then

$$\left(D_{+}^{\alpha}[z^{\mu-1}F_q(a,z)]\right)(y)=y^{\mu+q-\alpha-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)}$$

(10)

Proof:

$$\Delta=\left(D_{+}^{\alpha}[z^{\mu-1}F_q(a,z)]\right)(y)$$

$$=\left(\frac{d}{dy}\right)^{[\alpha]+1}\left(I_{+}^{1-\{\alpha\}}[z^{\mu-1}F_q(a,z)]\right)(y)$$

$$=\left(\frac{d}{dy}\right)^{[\alpha]+1}\left[\frac{1}{\Gamma(1-\{\alpha\})}\int_{0}^{y}(y-z)^{-\{\alpha\}}z^{\mu-1}\sum_{m=0}^{\infty}\frac{a^m z^{(m+1)q-1}}{\Gamma((m+1)q)}dz\right]$$

$$=\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\left(\frac{d}{dy}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_{0}^{y}(y-z)^{-\{\alpha\}}z^{\mu-1}z^{(m+1)q-1}dz$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu+1-\{\alpha\})}\left(\frac{d}{dy}\right)^{[\alpha]+1}y^{\mu+(m+1)q-\{\alpha\}}$$

$$=\sum_{m=0}^{\infty}\frac{a^m y^{\mu+(m+1)q-\{\alpha\}}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)}$$

$$=y^{\mu+q-\alpha-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)}$$

Corollary 2.6 For μ=1, there holds the formula

$$\left(D_{+}^{\alpha}[F_q(a,z)]\right)(y)=y^{q-\alpha}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+1)}{\Gamma((m+1)q)\Gamma((m+1)q+1-\alpha)}$$

(11)

Corollary 2.7 For μ=1, m=1, (10) reduces to

$$\left(D_{+}^{\alpha}[F_q(a,z)]\right)(y)=\frac{2aqy^{2q-\alpha}}{\Gamma(q+1-\alpha)}$$

(12)

Theorem 2.8 If D_{-}^{α} be the R-L fractional derivative operator of the right side (4) and let μ>0, then there holds the formula

$$\left(D_{-}^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)=y^{-\mu+q-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}$$

(13)

Proof:

$$\Delta\equiv\left(D_{-}^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)$$

$$=\left(-\frac{d}{dy}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}[z^{\alpha-\mu}F_q(a,z)]\right)(y)$$

$$=\left(-\frac{d}{dy}\right)^{[\alpha]+1}\left[\frac{1}{\Gamma(1-\{\alpha\})}\int_{y}^{\infty}(z-y)^{-\{\alpha\}}z^{\alpha-\mu}\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}dz\right]$$

$$=\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\left(-\frac{d}{dy}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_{y}^{\infty}(z-y)^{-\{\alpha\}}(z)^{\alpha-\mu+(m+1)q-1}dz$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}\left(-\frac{d}{dy}\right)^{[\alpha]+1}(y)^{-\{\alpha\}+\alpha-\mu+(m+1)q}$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma(\mu-(m+1)q+1)(y)^{-\mu+(m+1)q-1}}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}$$

$$=y^{-\mu+q-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}$$

Corollary 2.9 For m=1, (13) reduces to

$$\left(D_{-}^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)=\frac{ay^{2q-\mu-1}\Gamma(\mu-2q+1)}{\Gamma(2q)\Gamma(\mu-\alpha-2q+1)}$$

(14)

III. CONCLUSION

The outcomes derived in this study is envisaged to have application in the findings of cases emerging in fundamental linear order fractional differential and integral equations and problems which are of fractional order in areas of physical sciences and engineering where the F-function plays a very pivotal role.

REFERENCES

1. T. T. Hartley and C. F. Lorenzo, " A solution to the fundamental linear fractional differential equation," NASA/TP-1998-208963.
2. V. Kiryakova, "Generalized fractional calculus and application," Research notes in Math. Series Gordon and Breach science publishers, Reading, PA, vol. 301, 1993.
3. J. L. Lavoie, T. J. Osler, and R. Tremblay, "Fractional derivatives and special function," SIAM review, vol. 18, pp. 240-268.
4. K. S. Miller, and B. Ross, "An introduction to fractional calculus and fractional differential equation," A wiley-interscience publication. John wiley & sons/ inc. New york,1993.
5. S. G. Samko, A. A. Kilbas, and O. I. Marichev, "Fractional integrals and derivatives: Theory and applications," Gordon and Breach science publishers, Reading, PA, 1993.

AUTHORS PROFILE

Ms. Cristina Gammeng, is pursuing her Ph.D degree in the Department of Basic and Applied Science at National Institute of Technology, Arunachal Pradesh, India. She is presently serving as a Lecturer in Govt. Polytechnic College, Arunachal Pradesh. She has communicated more than 4 papers in various scopus journals.

Dr. U. K. Saha, is currently working as an assistant professor in the Department of Basic & Applied Science at National Institute of Technology, Arunachal Pradesh, India. Before joining the institution he has worked as assistant professor in a Govt. College of Assam, India. He has more than 24 years teaching experience and 8 years of research experience. He has published more than 20 papers in different national and international journals of repute, He is a life member of various professional bodies.

Dr. S. Maity, is an Assistant Professor of Mathematics and Head of Department of Basic and Applied Science at National Institute of Technology, Arunachal Pradesh. He received his M.sc and Ph.D degrees in Applied Mathematics from the University of Calcutta in 2004 and 2012 respectively. His research work includes the modeling of thin film, heat and mass transfer, nanofluids, thermocapillary flows. He has published several research articles in many world leading journals.