International Journal of Recent Technology and Engineering (IJRTE)
ISSN: 2277-3878 (Online), Volume-8 Issue-6, March 2020

On Riemann-Liouville Fractional Calculus and F-Function

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***Abstract:** The objective of the work is to explore certain correspondences that prevail amongst Riemann-Liouville fractional calculus and the F-function. The work is expected to find utilization in areas of fundamental linear order fractional differential and integral equations involving the F-function.*

***Keywords:** Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, F-function.*

I. INTRODUCTION

The fractional calculus is one of the fastest expanding subject of the mathematical analysis. It is the area that deals with the integrals and derivatives of arbitrary order. It has been enforced in various field of science, engineering and technology. Numerous researchers have investigated their properties, applications, relationships with different special functions in detail. Today there are various form of fractional calculus operators but the Riemann-Liouville operator still remains he most widely and extensively used operator.

The operators of the left and right side of the Riemann-Liouville fractional calculus according to the section 2 of the book by Samko, Kilbas and Marichev [5] are defined as follows:

$$\left(I_+^\alpha g\right)(y)=\frac{1}{\Gamma(\alpha)}\int_0^y \frac{g(z)}{(y-z)^{1-\alpha}}dz, \alpha>0 \tag{1}$$

$$\left(I_-^\alpha g\right)(y)=\frac{1}{\Gamma(\alpha)}\int_y^\infty \frac{g(z)}{(z-y)^{1-\alpha}}dz, \alpha>0 \tag{2}$$

$$\left(D_+^\alpha g\right)(y)=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_+^{1-\{\alpha\}} g\right)(y)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^y \frac{g(z)}{(y-z)^{\{\alpha\}}}dz, \alpha>0 \tag{3}$$

Manuscript received on February 10, 2020.
Revised Manuscript received on February 20, 2020.
Manuscript published on March 30, 2020.
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$$\left(D_-^\alpha g\right)(y)=\left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I_-^{1-\{\alpha\}} g\right)(y)=\frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_y^\infty \frac{g(z)}{(z-y)^{\{\alpha\}}}dz, \alpha>0 \tag{4}$$

(5)

Where {α} is the fractional part of {α} and [α] is the maximum integer not exceeding α.
Hartley and Lorenzo (1998) introduced the F-function as,

$$F_q(a,z)\equiv\sum_{m=0}^\infty \frac{a^m (z)^{(m+1)q-1}}{\Gamma((m+1)q)}, q>0$$

II. MAIN RESULTS

Here, 4 theorems have been constructed regarding the Riemann-Liouville (R-L) fractional calculus and F-function.

Theorem 2.1 If I_+^α be the R-L fractional integral operator of the left side (1) and let μ>0, then

$$\left(I_+^\alpha[z^\mu F_q(a,z)]\right)(y)=y^{\mu+\alpha+q-1}\sum_{m=0}^\infty \frac{a^m (y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu+(m+1)q)}{\Gamma(\alpha+\mu+(m+1)q)} \tag{6}$$

Proof:

$$\Delta\equiv\left(I_+^\alpha[z^\mu F_q(a,z)]\right)(y)$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^y (y-z)^{\alpha-1}z^\mu\sum_{m=0}^\infty \frac{a^m (z)^{(m+1)q-1}}{\Gamma((m+1)q)}dz$$

$$=y^{\mu+\alpha+q-1}\sum_{m=0}^\infty \frac{a^m (y)^{mq}}{\Gamma((m+1)q)}\frac{1}{\Gamma(q)}\int_0^1 (1-u)^{\alpha-1}u^{\mu+(m+1)q-1}du$$

$$=y^{\mu+\alpha+q-1}\sum_{m=0}^\infty \frac{a^m (y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu+(m+1)q)}{\Gamma(\alpha+\mu+(m+1)q)}$$

$$=\left(\frac{d}{dy}\right)^{[\alpha]+1}\left[\frac{1}{\Gamma(1-\{\alpha\})}\int_0^y (y-z)^{-\{\alpha\}}z^{\mu-1}\sum_{m=0}^\infty \frac{a^m z^{(m+1)q-1}}{\Gamma((m+1)q)}dz\right]$$

$$\left(I_+^\alpha[F_q(a,z)]\right)(y)=y^{\alpha+q}\sum_{m=0}^\infty \frac{a^m (y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma((m+1)q+1)}{\Gamma((m+1)q+\alpha+1)} \tag{7}$$

$$\left(I_+^\alpha[F_q(a,z)]\right)(y)=\frac{2aqy^{\alpha+2q}}{(\alpha+2q)!} \tag{8}$$

Theorem 2.4 If I_-^α be R-L fractional integral operator along the right side (2) and let μ>0, then

$$\left(I_-^{\alpha}[z^{-\alpha-\mu}F_q(a,z)]\right)(y)=y^{-\mu+mq}\sum_{m=0}^{\infty}\frac{a^m(y)^{mq}}{\Gamma((m+1)q)}\frac{\Gamma(\mu-(m+1)q)}{\Gamma(\alpha+\mu-(m+1)q)}$$

Proof: $\Delta\equiv\left(I_-^{\alpha}[z^{-\alpha-\mu}F_q(a,z)]\right)(y)$

$$=\frac{1}{\Gamma(q)}\int_y^{\infty}(z-y)^{\alpha-1}z^{-\alpha-\mu}\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}$$

If we interchange the form of summation and integration,

$$=\frac{1}{\Gamma(q)}\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\int_y^{\infty}(z-y)^{\alpha-1}z^{-\alpha-\mu}(z)^{(m+1)q-1}dz$$

$$=\frac{1}{\Gamma(q)}\sum_{m=0}^{\infty}\frac{a^m y^{-\mu+(m+1)q}}{\Gamma((m+1)q)}\int_0^1(1-u)^{\alpha-1}(u)^{\mu-(m+1)q+1}du$$

$$=y^{-\mu+mq}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q)}{\Gamma((m+1)q)\Gamma(\alpha+\mu-(m+1)q)}$$

Theorem 2.5 If D_+^{α} be the R-L fractional derivative operator along the left side (3) and let μ>0, then

$$\left(D_+^{\alpha}[z^{\mu-1}F_q(a,z)]\right)(y)=y^{\mu+q-\alpha-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)} \tag{10}$$

Proof:

$$\Delta=\left(D_+^{\alpha}[z^{\mu-1}F_q(a,z)]\right)(y)$$

$$=\left(\frac{d}{dy}\right)^{[\alpha]+1}\left(I_+^{1-\{\alpha\}}[z^{\mu-1}F_q(a,z)]\right)(y)$$

$$=\left(\frac{d}{dy}\right)^{[\alpha]+1}\left[\frac{1}{\Gamma(1-\{\alpha\})}\int_0^y(y-z)^{-\{\alpha\}}z^{\mu-1}\sum_{m=0}^{\infty}\frac{a^m z^{(m+1)q-1}}{\Gamma((m+1)q)}dz\right]$$

$$=\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\left(\frac{d}{dy}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_0^y(y-z)^{-\{\alpha\}}z^{\mu-1}z^{(m+1)q-1}dz$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu+1-\{\alpha\})}\left(\frac{d}{dy}\right)^{[\alpha]+1}y^{\mu+(m+1)q-\{\alpha\}}$$

$$=\sum_{m=0}^{\infty}\frac{a^m y^{\mu+(m+1)q-\{\alpha\}}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)}$$

$$=y^{\mu+q-\alpha-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+\mu)}{\Gamma((m+1)q)\Gamma((m+1)q+\mu-\alpha)}$$

Corollary 2.6 For μ=1, there holds the formula

$$\left(D_+^{\alpha}[F_q(a,z)]\right)(y)=y^{q-\alpha}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma((m+1)q+1)}{\Gamma((m+1)q)\Gamma((m+1)q+1-\alpha)} \tag{11}$$

Corollary 2.7 For μ=1, m=1, (10) reduces to

$$\left(D_+^{\alpha}[F_q(a,z)]\right)(y)=\frac{2aqy^{2q-\alpha}}{\Gamma(q+1-\alpha)} \tag{12}$$

Theorem 2.8 If D_-^{α} be the R-L fractional derivative operator of the right side (4) and let μ>0, then there holds the formula

$$\left(D_-^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)=y^{-\mu+q-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)} \tag{13}$$

Proof:

$$\Delta\equiv\left(D_-^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)$$

$$=\left(-\frac{d}{dy}\right)^{[\alpha]+1}\left(I_-^{1-\{\alpha\}}[z^{\alpha-\mu}F_q(a,z)]\right)(y)$$

$$=\left(-\frac{d}{dy}\right)^{[\alpha]+1}\left[\frac{1}{\Gamma(1-\{\alpha\})}\int_y^{\infty}(z-y)^{-\{\alpha\}}z^{\alpha-\mu}\sum_{m=0}^{\infty}\frac{a^m(z)^{(m+1)q-1}}{\Gamma((m+1)q)}dz\right]$$

$$=\sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)q)}\left(-\frac{d}{dy}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_y^{\infty}(z-y)^{-\{\alpha\}}(z)^{\alpha-\mu+(m+1)q-1}dz$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}\left(-\frac{d}{dy}\right)^{[\alpha]+1}(y)^{-\{\alpha\}+\alpha-\mu+(m+1)q}$$

$$=\sum_{m=0}^{\infty}\frac{a^m\Gamma(\mu-(m+1)q+1)(y)^{-\mu+(m+1)q-1}}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}$$

$$=y^{-\mu+q-1}\sum_{m=0}^{\infty}\frac{a^m y^{mq}\Gamma(\mu-(m+1)q+1)}{\Gamma((m+1)q)\Gamma(\mu-\alpha-(m+1)q+1)}$$

Corollary 2.9 For m=1, (13) reduces to

$$\left(D_-^{\alpha}[z^{\alpha-\mu}F_q(a,z)]\right)(y)=\frac{ay^{2q-\mu-1}\Gamma(\mu-2q+1)}{\Gamma(2q)\Gamma(\mu-\alpha-2q+1)} \tag{14}$$

III. CONCLUSION

The outcomes derived in this study is envisaged to have application in the findings of cases emerging in fundamental linear order fractional differential and integral equations and problems which are of fractional order in areas of physical sciences and engineering where the F-function plays a very pivotal role.

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