

Methods for Optimal Separation of Income in Consumable and Accumulated Parts

Chernyaev A. P., Meerson A. Yu., Sukhorukova I. V., Fomin G. P.

Abstract: *The effectiveness of a particular type of business is determined by the terms of planning and management. The study is devoted to the urgent problem of finding the optimal method for dividing a product into consumed and accumulated parts. Various macro models of economic growth are considered. An optimality criterion is proposed in the form of utility per capita consumption. Growth rates of producers and economic characteristics are considered integrable functions arbitrarily time-dependent. At the same time, per capita consumption is optimized either in terms of utility or in its pure form. Income can be used either as consumer income to meet the needs of the population, or capital-forming to create new production capacities and compensate for old production capacities that are out of order. The results can be used to assess the economic growth rate of the national economy.*

Keywords: *economic growth, production function, per capita consumption, income, utility.*

I. INTRODUCTION

Theories of economic growth had two periods of rapid development (Zamulin O.A. and Sonin K.I. , 2019). At the first stage, the exogenous theories of Ramsey (Ramsey F.P.1928), Harrod – Domar (Harrod R.F. , 1939), (Domar E., 1946),(Meerson A.Yu. and Chernyaev A.P., 2014),Solow (Solow R.M. ,1956), (Solow R.M. 1957), (Kolemaev V.A. , 1998) and overlapping generations were created. At the second stage, endogenous theories appeared, the most famous of which are the models of Romer (Meerson A.Yu. and Chernyaev A.P. ,2019), (Romer P. , 1987) and Nordhaus (Nordhaus W., 2017). Currently, the advantages of endogenous theories over exogenous are noted (Sukhorukova I.V et al , 2019). Nevertheless, the Solow model is quite popular. The equation for capital-labor ratio following from the Solow model (Sukhorukova I.V. and Chistyakova N.A. , 2018) , (. Samarsky A.A. and Mikhailov A.P. , 1997) is integrated in the particular case of the Cobb – Douglas production function, and, therefore, the “Golden Solow rule” can be obtained.

The comparative analysis of macro-models of economic growth is investigated. The growth rate of manufacturers in the models under consideration is considered not constant, as it was previously traditionally considered, but is an integrable function that depends on time arbitrarily. Similarly, other economic characteristics are also assumed to be arbitrary integrable functions of time.

Revised Manuscript Received on February 8, 2020.

Chernyaev Alexander Petrovich, Professor, Department of Higher Mathematics, Moscow Institute of Physics and Technology (State University) - MIPT, Russia.

Meerson Alla Yurievna, Associate Professor, Department of Mathematical Methods in Economics, G.V. Russian University of Economics Plekhanov (REU named after G.V. Plekhanov), Russia.

Sukhorukova Irina Vladimirovna, Professor, Department of Higher Mathematics, Plekhanov Russian University of Economics, Russian Federation.

Fomin Gennady Petrovich, Department of Economic Cybernetics, Russian University of Trade and Economics, Russia.

The tasks are posed and solved to find methods of optimal in a certain sense division of income into consumed and accumulated parts. Moreover, in a different economic interpretation, consumption is optimized. Depending on the economic sense, it can be either aggregate or average per capita. The nature of consumption optimization can be different and is determined by economic sense. Aggregate consumption can be optimized in terms of utility and aggregate consumption in its pure form. The problem of finding a method for optimal separation of income into consumed and accumulated parts from the point of view of optimizing consumption is formulated and solved. The generality of the statement is ensured precisely by the fact that the growth rates of producers or consumers, as well as various economic factors are assumed to be integrable functions of time of an arbitrary nature.

II. RESULTS AND DISCUSSION

1. The main variables of the considered models of economic growth.

In the considered macro-models of economic growth, the number of independent producers or consumers $R = R(t)$ is considered to be continuously dependent on time t . The calculation is carried out for a period of time $[0, T]$. An increase in the number of manufacturers is supposed to satisfy the Cauchy problem

$$R'(t) = \alpha(t)R(t), \quad R(0) = R_0. \quad (1)$$

The solution to problem (1) has the form

$$R(t) = R_0 \exp\left(\int_0^t \alpha(\tau) d\tau\right)$$

Function $\alpha = \alpha(t)$ in (1) and (2) called the growth rate of producers or consumers and is considered known. Income can be used both as consumer income to meet the needs of the population, and as capital-forming for creating new production capacities, and to compensate for old production capacities that are out of order. National income; $Y = Y(t)$, thus divided into consumption $C = C(t)$ and accumulation $J = J(t)$.

$$Y = C + J.$$

We note that equality (3) is included, as an integral part, in both the Harrod – Domar macromodel, the Solow macromodel, and the macromodel with a power-dependent production function.

2. Macro model of Harrod – Domar. It is clear that the Harrod – Domar macromodel is a consumer model and, in the classical setting, does not explicitly have a production function. The main conditions of the macro model are as follows: all savings are invested in the economy, that is, savings are identified with investments, and the investments themselves are proportional to the growth rate of income

$$J = BY' \tag{4}$$

where B – is the capital-intensive ratio of income growth, or incremental capital-intensiveness, and

$\frac{1}{B}$ – is the marginal product of capital, or marginal return

on assets at the macroeconomic level. Substituting (4) in (3) we obtain

$$Y(t) = C(t) + BY'(t).$$

Earlier, the coefficient of incremental capital intensity was considered positive and constant [5]:

$$B = \text{const} > 0.$$

For case (6), the solution of differential equation (5) is known [5] and is given by the formula:

$$Y(t) = Y_0 e^{\frac{t-t_0}{B}} - \frac{1}{B} \int_{t_0}^t C(\tau) e^{\frac{t-\tau}{B}} d\tau.$$

It is also assumed that the initial condition is satisfied:

$$Y(t_0) = Y_0 > 0, \tag{8}$$

which together with equation (5) forms the Cauchy problem. In doing so, we assume:

$$B = B(t).$$

As is known [5], under conditions (8) and (9), the solution of differential equation (5) will be given by the formula

$$Y(t) = Y_0 e^{\int_{t_0}^t \frac{ds}{B(s)}} - e^{\int_{t_0}^t \frac{ds}{B(s)}} \int_{t_0}^t \frac{C(\tau)}{B(\tau)} e^{-\int_{t_0}^{\tau} \frac{ds}{B(s)}} d\tau.$$

Formula (7) is obviously a special case of (10) if $B = \text{const} > 0$ (6). To solve the main problem, you need to add the condition of any optimization of consumption. You can, for example, consider the problem of maximizing the integral discounted utility of consumption [5].

3. Solow macromodel with the Cobb – Douglas production function. The Solow model [6–9] is also based on equality (3). In the Solow model, the average per capita

capital-to-labor ratio; $k = K/R$, where K – funds satisfy the first-order differential equation:

$$\frac{dk}{dt} = -\lambda k + \rho A k^\alpha$$

and the initial condition

$$k(0) = k_0 > 0.$$

Equation (12) and condition (13) are the Cauchy problem.

Here $\rho = J/Y$ – accumulation rate $\rho = \text{const}$, $0 < \rho < 1$; [8]. Wherein the $\lambda = \mu + \nu$, where μ – is the share of fixed assets disposed of during the year $\mu = \text{const}$, and $\nu = R'/R$ – annual growth rate of labor resources. The following restrictions apply μ and ν : $0 < \mu < 1$, $-1 < \nu < 1$.

The Cobb – Douglas production function has the form $F(K, R) = AK^\alpha R^{1-\alpha}$, $A = \text{const} > 0$, $0 < \alpha < 1$ [6–8].

Equation (12) is the Bernoulli equation, having solved it taking into account the initial condition (13) under the additional condition $\lambda = \text{const}$, we obtain:

$$k(t) = e^{-\lambda t} \left[\frac{\rho A}{\lambda} e^{\lambda(1-\alpha)t} - \frac{\rho A}{\lambda} + k_0^{1-\alpha} \right]^{\frac{1}{1-\alpha}}. \tag{14}$$

Find the limit of expression (14) for $t \rightarrow +\infty$:

$$k_\infty = \lim_{t \rightarrow +\infty} k(t) = \left(\frac{\rho A}{\lambda} \right)^{\frac{1}{1-\alpha}}.$$

Therefore, the specific consumption, that is, consumption per worker, also converges to a stationary value:

$$c_\infty = \lim_{t \rightarrow +\infty} \frac{C(t)}{R(t)} = \lim_{t \rightarrow +\infty} \frac{(1-\rho)Y(t)}{R(t)} = (1-\rho)A \left(\frac{\rho A}{\lambda} \right)^{\frac{\alpha}{1-\alpha}}.$$

Based on (16), we write: (9)

$$\begin{aligned} c_\infty &= (1-\rho)A \left(\frac{\rho A}{\lambda} \right)^{\frac{\alpha}{1-\alpha}} = A^{1+\frac{\alpha}{1-\alpha}} \lambda^{-\frac{\alpha}{1-\alpha}} \left(\rho^{1-\alpha} - \rho^{1+\frac{\alpha}{1-\alpha}} \right) = \\ &= A^{1-\alpha} \lambda^{-\frac{\alpha}{1-\alpha}} \left(\rho^{\frac{\alpha}{1-\alpha}} - \rho^{\frac{1}{1-\alpha}} \right). \end{aligned}$$

We form a new value

$$\hat{c} = A^{-\frac{1}{1-\alpha}} \lambda^{\frac{\alpha}{1-\alpha}} c_\infty = \rho^{\frac{\alpha}{1-\alpha}} - \rho^{\frac{1}{1-\alpha}},$$

which is different from c_∞ by a constant. We differentiate now (17) by ρ :

$$\hat{c}'_{\rho} = \frac{\alpha}{1-\alpha} \rho^{\frac{\alpha}{1-\alpha}-1} - \frac{1}{1-\alpha} \rho^{\frac{1}{1-\alpha}-1} = \frac{\alpha}{1-\alpha} \rho^{\frac{2\alpha}{1-\alpha}} - \frac{1}{1-\alpha} \rho^{\frac{\alpha}{1-\alpha}} = \frac{\rho^{\frac{2\alpha}{1-\alpha}}}{1-\alpha} (\alpha - \rho).$$

The right-hand side of (18) vanishes only at $\rho = \alpha$. But this is only a necessary condition for the extremum. However, when the variable ρ passes through α the value (18), it changes sign from plus to minus. And this is a sufficient condition for the maximum.

The optimal rate of accumulation in stationary mode is equal to the coefficient of elasticity for funds. This is the “golden rule” of economic growth for the Cobb – Douglas production function. For other production functions, this rule, quite possibly, will be different [8].

4. Macromodel with power-dependent production function. Capacity $M = M(t)$ is the maximum product release [12]. Then

$$Y = F(M, R)$$

The known properties (19), which are postulated [12] and (19) are conveniently represented in the form:

$$F = M(t)f(x(t))$$

Note that in the written formula (20), the quantity

$$x = x(t) = R(t)/M(t)$$

equal to the number of manufacturers per unit of capacity. Function $f(x)$ with $x \in [0, x_M]$, where $x_M = R_M/M$, and $R_M = R_M(t)$ the number of jobs at power $M(t)$, and $f(x_M) = 1$ [12], possesses properties $f(0) = 0$, $f' > 0$ – as output increases with the number of manufacturers, $f'' < 0$ – since there is saturation [12]. The problem of finding the optimal method for dividing the manufactured product into C and J (3) is studied. We will optimize the average per capita consumption, that is, the amount of product consumed by one worker $c = c(t) = C(t)/R(t)$.

Let $I = I(t)$ – be the number of units of new power, then

$$J(t) = a(t)I(t),$$

where $a = a(t) > 0$ – is the coefficient of incremental capital intensity - the amount of capital-forming product to create a unit of new capacity. The degradation of fixed assets leads to a decrease in capacity by bM , where $b = b(t)$ – is the retirement rate.

So, we have the rate of change of power over time

$$M'(t) = I(t) - b(t)M(t)$$

Suppose [10] that proportionality holds

$$I(t) = \beta(t)M(t),$$

Where $\beta = \beta(t)$ is the reciprocal of the characteristic time of power buildup. Then from (23) and (24)

$$M'(t) = [\beta(t) - b(t)]M(t) \tag{18}$$

Integrating equation (25), we obtain

$$M(t) = M_0 \exp\left(\int_0^t [\beta(\tau) - b(\tau)]d\tau\right)$$

Here, the initial condition $M(0) = M_0 > 0$ is added to equation (26). From (3), (19), (20), (22) and (24) for per capita consumption we have:

$$c(t) = \frac{C(t)}{R(t)} = \frac{Y(t) - J(t)}{R(t)} = \frac{M(t)f(x(t)) - a(t)I(t)}{R(t)} = \frac{M(t)f(x(t)) - a(t)\beta(t)M(t)}{R(t)}$$

Transforming the latter taking into account (21), we obtain

$$c(t) = [f(x(t)) - a(t)\beta(t)]/[R(t)/M(t)] = [f(x(t)) - a(t)\beta(t)]/x(t) \tag{27} \tag{19}$$

Next, we consider the case

$$a(t)\beta(t) = \gamma = const > 0 \tag{20}$$

It follows from (27) and (28) that

$$c = \frac{f(x) - \gamma}{x} \tag{21}$$

The maximum of expression (29) is achieved provided

$$\frac{dc}{dx} = \frac{d}{dx} \left[\frac{f(x) - \gamma}{x} \right] = \frac{xf'(x) - f(x) + \gamma}{x^2} = 0$$

Equality (30) gives an equation for determining the number of producers per unit of power for x_m the desired mode of maximizing per capita consumption

$$x_m f'(x_m) - f(x_m) + \gamma = 0$$

Additionally, we assume agreement R_0 from (1) and (2) with M_0 for $t = 0$: $x_m = R_0/M_0$.

Divide each term on the left side of equation (31) by $f(x_m)$

$$\gamma/f(x_m) = 1 - x_m f'(x_m)/f(x_m)$$

The accumulation rate $n = n(t)$ is defined as $n = n(t) = J(t)/Y(t)$.

We now transform this rate of accumulation using (22), (19), (20) and (28):

$$n(t) = a(t)I(t)/F(M(t), R(t)) = a(t)\beta(t)M(t)/M(t)f(x(t)) = \gamma/f(x(t)) \tag{24}$$

In (33) we substitute x_m instead x . Given an expression (32)

$$n_m = \gamma / f(x(t)) = 1 - x_m f'(x_m) / f(x_m) \quad (34)$$

Here, the rate of accumulation (33) is called the rate of the “golden rule of growth” Solow [12].

III. CONCLUSION

The last considered economic growth model is more general than in [12] due to the fact that the growth rate of employed workers, the amount of capital-forming product to create a unit of new capacity, the rate of retirement of new capacity and the characteristic time of capacity increase are integrable functions of arbitrary time. The considered problem of maximizing per capita consumption is solved in the presence of condition (28). Only under this condition is expression (29) valid. Economically, expression (29) cannot describe consumption in a small right neighborhood of zero. Indeed, in this neighborhood, the right-hand side of (29) becomes negative. So the economic sense of condition (28) is that this condition is constraining. We also note that vanishing of the numerator of the right-hand side of (30) leads to the same right-hand side as that of (29). And if we call the second factor of the right-hand side of (20) the specific production function, then we obtain the following economic interpretation of the result. If constraint condition (28) allows, then the optimal average per capita consumption may be equal to the derivative of the specific production function.

Note that from the point of view of the problem considered in the article, the third model is more convenient than the second, because the exact solution of the differential equation for capital-labor ratio is not required. However, the assumption is not made of the identity of the growth rate of capacity and increase in the number of employees. The results can be used to assess the economic growth rate of the national economy.

REFERENCES

1. Zamulin O.A. and Sonin K.I. “The economic growth. 2018 and lessons for Russia” *Issues of Economics* No. 1, pp. 11–36, 2019. (in Russ.)
2. Ramsey F.P. “A mathematical theory of saving” *Economic Journal*. Vol. 38, № 152, pp. 543-559, 1928.
3. Harrod R.F. “An Essay in Dynamic Theory” *Economic Journal*, No. 49, pp. 14–33, 1939.
4. Domar E. “Capital Expansion, Rate of Growth and Employment” *Econometrica*, Vol. 14, No. 2, pp. 137–147, 1946.
5. Meerson A.Yu., and Chernyaev A.P. “The variational problem of optimizing consumption of the Harrod – Domar model of economic dynamics with a variable coefficient of capital intensity of revenue growth” *Transactions of the Free Economic Society of Russia*. Moscow, Vol. 186, pp. 502–506, 2014. (in Russ.)
6. Solow R.M. “Contribution to the Theory of Economic Growth” *The Quarterly Journal of Economics*. Vol. 70, No 1, pp. 65–94, 1956
7. Solow R.M. “Technical Change and the Aggregate Production Function” *The Review of Economics and Statistics*. Vol. 39, No. 3, pp. 312–320, 1957
8. Kolemaev V.A. “Mathematical Economics”. - M.: UNITY, 240 p., 1998 (in Russ.)
9. Meerson A.Yu. and Chernyaev A.P. “Tasks of optimal per capita consumption management with the equation of coupling for capital-labor ratio” *Transactions of MIPT*. Vol. 11. No. 2, pp. 27–37, 2019

10. Romer P. “Growth Based on Increasing Returns Due to Specialization” *American Economic Review*. Vol. 77, pp. 56-62, 1987
11. Nordhaus W. “Integrated assessment models of climate change” *NBER Reporter.*, No.3, pp. 16-12, 2017
12. Sukhorukova I.V., Fomin G.P. et al “Hybrid Method for Multi-Criteria Risk Minimization”. *Espacios*, Vol. 40, p 14, 2019
13. Sukhorukova I.V. and Chistyakova N.A. “Optimization of the Formation of the Capital Structure of the Insurance Company, Taking into Account the National Specifics of Insurance” *Journal of Reviews on Global Economics*, No. 7, pp. 146-151, 2018.
14. Samarsky A.A. and Mikhailov A.P. “Mathematical Modeling: Ideas. Methods Examples”. Moscow: Science. Fizmatlit, 320 p., 1997 (in Russ.)

AUTHORS PROFILE



Chernyaev Alexander Petrovich - Doctor of Physical and Mathematical Sciences, professor by rank, Professor - Department of Higher Mathematics. Place of work - Moscow Institute of Physics and Technology (State University) - MIPT, Department of Higher Mathematics. He graduated from the Faculty of Mechanics and Mathematics of Moscow State University named after M.V. Lomonosov Moscow State University in 1971. In 1974 he graduated from the Moscow Institute of Physics and Technology. In 1974, he defended his thesis for the degree of candidate of physical and mathematical sciences. In 1975, Chernyaev A.P. was awarded the degree of candidate of physical and mathematical sciences. Since 1974, he joined the Department of Higher Mathematics at the Moscow Institute of Physics and Technology. In 1981, Chernyaev A.P. was awarded the title of associate professor of higher mathematics. In 1993, he defended his thesis for the degree of Doctor of Physics and Mathematics. In the same 1993, Chernyaev A.P. was awarded the degree of Doctor of Physics and Mathematics. In 1998, he was awarded the academic title of professor at the Department of Higher Mathematics. In the position of professor Chernyaev A.P. works at the department of higher mathematics MIPT works to the present. Sphere of scientific interests A. Chernyaev quite wide. He published scientific articles on the theory of functions of a real variable, differential equations, fluid and gas mechanics, mathematical modeling, set theory, mathematical economics, and questions of the philosophy of science related to these scientific disciplines. He is also interested in the issues of teaching the listed scientific disciplines.



Meerson Alla Yurievna - candidate of physical and mathematical sciences, associate professor by rank, associate professor of the department of mathematical methods in economics. Place of employment - G.V. Russian University of Economics Plekhanov (REU named after G.V. Plekhanov), Department of Mathematical Methods in Economics. She graduated from the Moscow Institute of Transport Engineers (MIIT) with a degree in automated control systems in 1985. From 1993 to 2007, Meerson A.Yu. She worked at the Department of Economics and Organization of Production of the Moscow Automotive Institute (MAMI). In 2004, she defended her thesis for the degree of candidate of physical and mathematical sciences. In the same 2004, Meerson A.Yu. The degree of candidate of physical and mathematical sciences was awarded. In 2007, she joined the Department of Mathematical Methods in Economics at the G.V. Plekhanov. In 2010, Meerson A.Yu. was awarded the title of assistant professor of mathematical methods in economics. In the position of assistant professor Meerson A.Yu. works at the department of mathematical methods in economics Plekhanov to the present. Field of scientific interests Meerson A.Yu. is in mathematical economics and the issues of teaching mathematical economics to students of economic specialties. She is equally interested in both microeconomic issues and macroeconomic models. She is also interested in simulation issues.



Sukhorukova Irina Vladimirovna, Academic status: Professor of the Department of Higher and Applied Mathematics Academic degree: Doctor of Economic Sciences, Candidate of Physical and Mathematical Sciences
Position: Professor of the Department of Higher Mathematics, Plekhanov Russian University of Economics, Russian Federation .

In 1991 she graduated with honors from Moscow State University M.V. Lomonosov and entered into full-time graduate school. The scientific degree of the candidate of physical and mathematical sciences after the defense of the thesis was awarded by the dissertational council of the Moscow State University M.V. Lomonosov in 1994. The degree of Doctor of Economic Sciences after the defense of the thesis was awarded in 2002. In 2007, he was awarded the title of professor in the department of higher and applied mathematics. The area of scientific interests of Sukhorukova I.V. is in mathematical economics



Fomin Gennady Petrovich, was born on May 9, 1941, graduated from the Moscow Aviation Institute in 1965. Then in the same university in 1973 he defended his dissertation in the field of mathematical modeling of physics of failures of electronics elements, according to which he published several monographs: "Physics of failures of electronics elements", "Modeling of failures of computer components", etc. In 1979 he moved to work at the Russian University of Trade and Economics at the Department of Economic Cybernetics, where he taught classes in the Economic and Mathematical Modeling of Trade and Economic Processes course, in which he issued several textbooks and textbooks: Mathematical Methods and Models in Commercial Activities , (2005,2009) "Economic-mathematical methods and models in commercial activity" (Yurait, 2015). "Mathematical methods in economics: 777 problems with comments and answers" (Knoruss, 2017). "Honorary worker of higher vocational education of the Russian Federation "(2007). Scientific research ; Mathematical modeling of trade risk development mechanisms.