Abstract: In this disquisition, the notion $FFB$ structures is introduced. The intent of this article is to study the notions of homotopy, path homotopy and fundamental group via $FFB$ structure which is a triplet consisting of two fuzzy topological spaces and a continuous surjection between them. Many properties concerning these concepts are provided.

Keywords. $FFB$ structure, $FFB$ homotopy, $FFB$ path homotopy, $FFB$ fundamental group.

I. INTRODUCTION

Chang [2] introduced and developed the concept of fuzzy topological spaces based on the notion of fuzzy sets which was introduced by Zadeh in [11]. Later the theory of fuzzy topological spaces was developed by many researchers. In [4], I. M. James introduced the notion of fibrewise topology and promoted various notions of topology in fibrewise viewpoint. Also fibrewise versions of homotopy theory were studied in [4]. The notion of fuzzy homotopy theory was introduced by G. Culvacioglu and M. Citil in [3]. The fundamental group of fuzzy topological spaces was introduced by Abdul Razak Salleh and Mohammad Tap in [7] and [8]. Motivated by [7] and [8], fuzzy fundamental group in fuzzy topological spaces was extended to various fuzzy structure spaces in [9] and [6].

In this treatise, the notion of $FFB$ structures is introduced. The concepts of $FFB$ homotopy, $FFB$ path homotopy and $FFB$ fundamental group in $FFB$ structures are introduced and their properties are investigated. It is shown that the set of all $FFB$ path homotopy equivalence classes on the collection of $FFB$ loops forms a fundamental group and there exists a $FFB$ isomorphism between two $FFB$ fundamental groups.

II. PRELIMINARIES

Definition 2.1. [1] Let $(X, T)$ be a fuzzy topological space and $Y$ be an ordinary subset of $X$. Then $T_Y = \{\lambda/Y \mid \lambda \in T\}$ is a fuzzy topology on $Y$ and is called the induced or relative fuzzy topology. The pair $(Y, T_Y)$ is called a fuzzy subspace of $(X, T)$: $(Y, T_Y)$ is called a fuzzy open/fuzzy closed/fuzzy $\beta$-open fuzzy subspace if the characteristic function of $Y$ viz $\chi_Y$ is fuzzy open/fuzzy closed/fuzzy $\beta$-open respectively.

Definition 2.2. [10] Let $(X, T)$ be a (usual) topological space. The collection $\mathcal{F} = \{G: G$ is a fuzzy set in $X$ and $\text{Supp} G \in \tau\}$ is a fuzzy topology on $X$, called the fuzzy topology on $X$ introduced by T. $(X, \mathcal{F})$ is called the fuzzy topological space introduced by $(X, T)$. Let $i_i$ denote Euclidean subspace topology on $I$ and $(I, i_i)$ denote the fuzzy topological space introduced by the topological space $(I, i_i)$.

Definition 2.3. [4] If the base set is denoted by $B$, then a fibrewise set over $B$, consists of a set $X$ together with a function $p: X \rightarrow B$, called the projection. For each point $b$ of $B$ the fibre over $b$ is the subset $X_b = p^{-1}(b)$ of $X$.

Definition 2.4. [4] The definition of fibrewise homotopy involves the (fibrewise) cylinder $I \times X = (I \times B) \times_{\mathcal{B}} X$ on the domain, which comes equipped with a family of fibrewise embeddings $\sigma_t: X \rightarrow I \times X \ (0 \leq t \leq 1)$, where $\sigma_t(x) = (t, x) \ (x \in X)$. Let $\theta, \phi: X \rightarrow Y$ be fibrewise maps, where $X$ and $Y$ are fibrewise spaces over $B$. A fibrewise homotopy of $\theta$ into $\phi$ is a fibrewise map $f: I \times X \rightarrow Y$ such that $f \sigma_0 = \theta$ and $f \sigma_1 = \phi$.

Definition 2.5. [5] Given points $x$ and $y$ of the topological space $X$, a path in $X$ from $x$ to $y$ is a continuous map $f: [a, b] \rightarrow X$ of some closed interval in the real line into $X$ such that $f(a) = x$ and $f(b) = y$.

Definition 2.6. [5] A topological space $X$ is said to be path connected if every pair of points of $X$ can be joined by a path in $X$.

III. ATTRIBUTES OF FUZZY FIBREWISE STRUCTURES

In this section, the perception of $FFB$ structures is pioneered. In addition, the concepts of $FFB$ open (resp. closed) structures are instigated and some properties concerning these concepts are established.

Definition 3.1. The triplet $((X, \tau), p, (B, \sigma))$ which comprises two fuzzy topological spaces $(X, \tau)$, $(B, \sigma)$ and a fuzzy continuous surjection $p: (X, \tau) \rightarrow (B, \sigma)$ is called a fuzzy fibrewise (in short $FFB$) structure over $(B, \sigma)$ and is simply denoted by $(X, \tau, p, B, \sigma)$.

Then $(X, \tau)$ is termed the fuzzy total (or fuzzy fibred) space; $p$ is called the projection; $(B, \sigma)$ is said to be the fuzzy base space and for each $\lambda \in I^0$, the fuzzy set $p^{-1}(\lambda)$ is called the fuzzy fibre over $\lambda$. 

Fuzzy Fibrewise Homotopy

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If a fuzzy fibre $\mu \in \tau$, then $\mu$ is termed fuzzy open fibre. The complement of fuzzy open fibre is termed fuzzy closed fibre. If a fuzzy fibre is a fuzzy point $x_\mu$, then $x_\mu$ is said to be a fuzzy fibre point. The collection of fuzzy fibre points in $I^X$ is denoted by $\mathbb{F} FB P (X)$.  

**Definition 3.2.** Let $(X_1, p_1, B_2)$ be a $\mathbb{F} FB P$ structure over $(B_2, \tau_2)$ and $\lambda \in I^X$. Then the $\mathbb{F} FB P$ closure of $\lambda$ is defined and symbolised as $\mathbb{F} FB P c l (\lambda) = \{ \mu : \mu \geq \lambda \}$. 

**Definition 3.3.** Let $(X_1, p_1, B_2)$ be a $\mathbb{F} FB P$ structure over $(B_2, \tau_2)$ and $\lambda \in I^X$. Then the $\mathbb{F} FB P$ interior of $\lambda$ is defined and symbolised as $\mathbb{F} FB P i n t (\lambda) = \{ \mu : \mu \leq \lambda \}$. 

**Definition 3.4.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ structures over $(B_2, \tau_2)$. A function $\phi: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ is termed to be a $\mathbb{F} FB P$ continuous function if for each fuzzy fibre point $x_\lambda$ in $\mathbb{F} FB P (X)$ over the fuzzy point $x_\lambda \in B_2$ and for every fuzzy open fibre $\mu$ in $(Y_1, \tau_2)$ with $(x_\mu, x_\lambda) \in \mu$, there exists a fuzzy open fibre $\gamma$ in $(X_1, \tau_1)$ with $x_\lambda \in \gamma$ such that $\phi (x_\gamma) \leq \mu$. Equivalently, a $\mathbb{F} FB P$ function $\phi: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ is said to be a $\mathbb{F} FB P$ continuous function if for each fuzzy open fibre (resp. fuzzy closed fibre) $\lambda$ in $(Y_1, \tau_2)$, $\phi^{-1}(\lambda)$ is a fuzzy open fibre (resp. fuzzy closed fibre) in $(X_1, \tau_1)$. 

**Proposition 3.1.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ structures over the fuzzy topological space $(B_2, \tau_2)$. Let $\sigma_1: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ and $\sigma_2: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ continuous functions. If $\sigma_1 = \sigma_2$ then defined by $h(r) = f(r)$, $r \in I_1$; $g(r)$, $r \in I_2$ is a $\mathbb{F} FB P$ continuous function. 

**Remark 3.1.** Let $I$ be the unit interval. Let $\xi$ be an Euclidean subspace topology on $I$ and $(I, \xi)$ be a fuzzy topological space introduced by the (usual) topological space $(I, \xi)$. Then $(I, \xi, p, (B, \sigma))$ is a fuzzy fibrewise structure over $(B, \sigma)$. 

**Proposition 3.2.** Let $(X_1, p_1, B_2)$ be a $\mathbb{F} FB P$ structure over the fuzzy topological space $(B_2, \tau_2)$. Let $I_1$ and $I_2$ be (usual) closed in $(I, \xi)$ and $I_1 \cup I_2 = I$. Let $f: (I_1, \xi_1, p_1, B_2) \rightarrow (X_1, p_1, B_2)$ and $g: (I_2, \xi_2, p_2, B_3) \rightarrow (X_1, p_1, B_2)$ be any two $\mathbb{F} FB P$ continuous functions. Then $f|_{I_1 \cap I_2} = g|_{I_1 \cap I_2}$, then $h: (I_1, \xi_1, p_1, B_2) \rightarrow (X_1, p_1, B_2)$ defined by $h(r) = f(r)$, $r \in I_1$; $g(r)$, $r \in I_2$ is a $\mathbb{F} FB P$ continuous function. 

**Definition 3.5.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ structures over $(B_2, \tau_2)$. A function $\phi: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ is said to be a $\mathbb{F} FB P$ function if $\phi^{-1}(B_3)$ is a fuzzy open fibre in $(X_1, \tau_1)$. 

**Definition 3.6.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ structures over $(B_2, \tau_2)$. A function $\phi: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ is said to be a $\mathbb{F} FB P$ continuous function if for each fuzzy fibre point $x_\lambda$ in $(X_1, \tau_1)$ over the fuzzy point $x_\lambda \in F P (B)$ and for every fuzzy open fibre $\mu$ in $(Y_1, \tau_2)$ with $(x_\mu, x_\lambda) \in \mu$, there exists a fuzzy open fibre $\gamma$ in $(X_1, \tau_1)$ with $x_\lambda \in \gamma$ such that $\phi (x_\gamma) \leq \mu$. Equivalently, a $\mathbb{F} FB P$ function $\phi: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ is said to be a $\mathbb{F} FB P$ continuous function if for each fuzzy open fibre (resp. fuzzy closed fibre) $\lambda$ in $(Y_1, \tau_2)$, $\phi^{-1}(\lambda)$ is a fuzzy open fibre (resp. fuzzy closed fibre) in $(X_1, \tau_1)$. 

**Proposition 3.3.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ structures over the fuzzy topological space $(B_2, \tau_2)$. Let $\theta: (X_1, p_1, B_2) \rightarrow (Y_1, p_2, B_3)$ be any two $\mathbb{F} FB P$ continuous functions. $\mathbb{F} FB P$ homotopy of $\theta$ into $\phi$ is a $\mathbb{F} FB P$ continuous function $f: (I_1, \xi_1, p_1, B_2) \rightarrow (X_1, p_1, B_2)$ such that $f \circ \sigma_0 (x_\lambda) = \theta (x_\lambda)$ and $f \circ \sigma_1 (x_\lambda) = \phi (x_\lambda)$ for each fuzzy fibre point $x_\lambda \in F P P (X)$.

If there exists a $\mathbb{F} FB P$ homotopy of $\theta$ into $\phi$, then we say that $\theta$ is $\mathbb{F} FB P$ homotopic to $\phi$ denoted by $\theta \approx_{FB} \phi$. 

**Proposition 3.4.** Let $(X_1, p_1, B_2)$ and $(Y_1, p_2, B_3)$ be any three $\mathbb{F} FB P$ structures over the fuzzy topological space $(B_2, \tau_2)$. Then $\approx_{FB}$ is an equivalence relation. 

**Remark 3.1.** Let $I$ be the unit interval. Let $\xi$ be an Euclidean subspace topology on $I$ and $(I, \xi)$ be a fuzzy topological space introduced by the (usual) topological space $(I, \xi)$.
is a \( \mathcal{F} \) function and \( g \circ \sigma_0(x_3) = f \circ \sigma_1(x_1) = \theta_2(x_2) \). Hence \( \theta_2 \approx_{FB} \theta_1 \).

(iii) Let \( \theta_1, \theta_2, \theta_3: (X_{t_1}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) be any three \( \mathcal{F} \) functions such that \( \theta_1 \approx_{FB} \theta_2 \) and \( \theta_2 \approx_{FB} \theta_3 \). Since \( \theta_1 \approx_{FB} \theta_2 \), there exists a \( \mathcal{F} \) function \( f: (I_{t_1}, p_3, B_{t_3}) \times (X_{t_1}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) such that \( f \circ \sigma_0(x_3) = \theta_1(x_1) \) and \( f \circ \sigma_1(x_1) = \theta_2(x_2) \) for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Similarly, since \( \theta_2 \approx_{FB} \theta_3 \), there exists a fuzzy fibrewise continuous function \( g: (I_{t_2}, p_3, B_{t_3}) \times (X_{t_2}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) such that \( g \circ \sigma_0(x_3) = \theta_2(x_2) \) and \( g \circ \sigma_1(x_1) = \theta_3(x_3) \) for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Let \( h: (I_{t_3}, p_3, B_{t_3}) \times (X_{t_3}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) be a fibrewise function such that

\[
\begin{align*}
  h \circ \sigma_1(x_3) &= \begin{cases} 
  f \circ \sigma_1(x_3), & 0 \leq t \leq 1/2; \\
  g \circ \sigma_2(x_3), & 1/2 < t \leq 1,
\end{cases} \\
\end{align*}
\]

for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Since \( f \) and \( g \) are fuzzy fibrewise continuous functions and by Proposition 3.2, \( h \) is a fuzzy fibrewise continuous function. Also \( h \circ \sigma_0(x_3) = f \circ \sigma_1(x_3) = \theta_1(x_1) \) and \( h \circ \sigma_1(x_1) = g \circ \sigma_2(x_3) = \theta_3(x_3) \).

Hence \( \theta_1 \approx_{FB} \theta_3 \). Thus \( \approx_{FB} \) is an equivalence relation.

### Proposition 3.4

Let \( (X_{t_1}, p_1, B_{t_3}) \) and \( (Y_{t_2}, p_2, B_{t_3}) \) and \( (Z_{t_1}, p_3, B_{t_3}) \) be any three \( \mathcal{F} \) structures over the fuzzy topological space \( (B, t_3) \). Let \( \theta, \phi: (X_{t_1}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) be any two \( \mathcal{F} \) continuous functions such that \( \theta \approx_{FB} \phi \). If \( \phi: (Y_{t_2}, p_2, B_{t_3}) \rightarrow (Z_{t_1}, p_3, B_{t_3}) \) is a \( \mathcal{F} \) continuous function, then \( \phi \circ \theta, \phi \circ \phi: (X_{t_1}, p_1, B_{t_3}) \rightarrow (Z_{t_1}, p_3, B_{t_3}) \) are \( \mathcal{F} \) continuous functions and \( \theta \approx_{FB} \phi \).

**Proof:** Let \( (I_{t_1}, p_3, B_{t_3}) \) be a \( \mathcal{F} \) structure over \( (B, t_3) \). Since \( \phi, \theta, \phi \circ \theta \) are \( \mathcal{F} \) continuous functions, \( \theta \circ \phi \) and \( \phi \circ \phi \) are \( \mathcal{F} \) continuous functions. Also \( \theta \approx_{FB} \phi \) implies that there exists a \( \mathcal{F} \) function \( f: (I_{t_1}, p_3, B_{t_3}) \times (X_{t_1}, p_1, B_{t_3}) \rightarrow (Y_{t_2}, p_2, B_{t_3}) \) such that \( f \circ \sigma_0(x_3) = \theta(x_1) \) and \( f \circ \sigma_1(x_1) = \phi(x_1) \) for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Let \( g: (I_{t_2}, p_3, B_{t_3}) \times (X_{t_1}, p_1, B_{t_3}) \rightarrow (Z_{t_1}, p_3, B_{t_3}) \) be a \( \mathcal{F} \) function such that \( g \circ \sigma_0(x_3) = (\phi \circ \sigma_1)(x_1) \) for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Since \( \phi \) and \( f \) are \( \mathcal{F} \) continuous functions, \( g \circ \sigma_0 = \phi \circ \sigma_1 \) is a fuzzy fibrewise continuous function. Further, \( g \) satisfies the following conditions:

\[
\begin{align*}
  g \circ \sigma_0(x_3) &= (\phi \circ \sigma_0)(x_1) = \phi(\theta(x_1)) = (\phi \circ \theta)(x_1), \\
  g \circ \sigma_1(x_1) &= (\phi \circ \sigma_1)(x_1) = \phi(\theta(x_1)) = \phi(\phi(x_1)), \\
\end{align*}
\]

for each fuzzy fibre point \( x_3 \in \mathcal{F} \). Hence \( \theta \approx_{FB} \phi \).

### Proposition 4.1

Let \( (X_{t_1}, p_1, B_{t_3}) \) be a \( \mathcal{F} \) structure over the fuzzy topological space \( (B, t_3) \). Let \( \alpha \) be a \( \mathcal{F} \) continuous function with \( \alpha(0) = x_\mu \) and \( \alpha(1) = x_\lambda \). Let \( \beta: (I_{t_1}, p_2, B_{t_3}) \rightarrow (X_{t_1}, p_1, B_{t_3}) \) be a \( \mathcal{F} \) continuous function such that \( \beta(0) = x_\mu \) and \( \beta(1) = x_\lambda' \). Then \( \beta \) is said to be a \( \mathcal{F} \) path in \( (X_{t_1}, p_1, B_{t_3}) \) with initial and end points \( x_\mu, x_\lambda \) respectively. Here \( x_\mu \) is called the initial point and \( x_\lambda \) is called the end point of the fuzzy fibrewise path \( \beta \).

### Proposition 4.1

Let \( (X_{t_1}, p_1, B_{t_3}) \) be a \( \mathcal{F} \) structure over the fuzzy topological space \( (B, t_3) \). Let \( \alpha \) be a \( \mathcal{F} \) continuous function with \( \alpha(0) = x_\mu \) and \( \alpha(1) = x_\lambda \). Let \( \beta: (I_{t_1}, p_2, B_{t_3}) \rightarrow (X_{t_1}, p_1, B_{t_3}) \) be a \( \mathcal{F} \) continuous function such that \( \beta(0) = x_\mu \) and \( \beta(1) = x_\lambda \). Then \( \beta \) is a fuzzy fibrewise continuous function. Then \( \beta(t) \) is defined for all \( t \in \mathcal{T} \). Hence \( \beta \) is a fuzzy fibrewise continuous function. Then \( \beta(t) = \alpha(1 - t) = \alpha(1) = x_\lambda \). Hence \( \beta \) is a fuzzy fibrewise path in \( (X_{t_1}, p_1, B_{t_3}) \) with initial and end points \( x_\mu, x_\lambda \) respectively.

### Proposition 4.2

Let \( (X_{t_1}, p_1, B_{t_3}) \) and \( (I_{t_2}, p_2, B_{t_3}) \) be any three \( \mathcal{F} \) structures over the fuzzy topological space \( (B, t_3) \). Let \( \alpha \) and \( \beta \) be two \( \mathcal{F} \) paths in \( (X_{t_1}, p_1, B_{t_3}) \). Then \( \alpha \) is said to be \( \mathcal{F} \) path homotopic to \( \beta \), denoted by \( \alpha \approx_{FB} \beta \), if they have the same initial point \( x_\mu \) and the same end point \( x_\lambda \) and if there exists a \( \mathcal{F} \) continuous function \( F: (I_{t_2}, p_2, B_{t_3}) \times (I_{t_2}, p_2, B_{t_3}) \rightarrow (X_{t_1}, p_1, B_{t_3}) \) such that \( F(r, 0) = \alpha(r) \) and \( F(r, 1) = \beta(r) \).
The function is called the path homotopy between \( \alpha \) and \( \beta \).

**Definition 4.3.** Let \((X_1, p_1, B_1)\) and \((I, p_2, B_2)\) be any two fuzzy fibrewise structures over the fuzzy topological space \((B, \tau_3)\). Let \(x_{\mu}, x_{\mu}' \in FFP(X)\). If \( \alpha \) is a fuzzy fibrewise path in \((X_1, p_1, B_1)\) from \(x_{\mu}\) to \(x_{\mu}'\) and if \( \beta \) is a fuzzy fibrewise path in \((X_2, p_2, B_2)\) from \(x_{\mu}'\) to \(x_{\mu}''\), then the fuzzy fibrewise path product \( \alpha \ast \beta \) is the fuzzy fibrewise path in \((X_1, p_1, B_1)\) from \(x_{\mu}\) to \(x_{\mu}''\) defined by

\[
(\alpha \ast \beta)(r) = \begin{cases} 
\alpha(2r), & \text{if } 0 \leq r \leq 1/2 \\
\beta(2r - 1), & \text{if } 1/2 \leq r \leq 1.
\end{cases}
\]

**Definition 4.4.** Let \((X_1, p_1, B_1)\) and \((I, p_2, B_2)\) be any two \( FFB \) structures over the fuzzy topological space \((B, \tau_3)\). A \( FFB \) path \( \alpha: (I, p_2, B_2) \rightarrow (X_1, p_1, B_1) \) in \((X_1, p_1, B_1)\) that begins and ends at \(x_{\mu}\) (that is \( \alpha(0) = \alpha(1) = x_{\mu}\)) is called a \( FFB \) loop in \((X_1, p_1, B_1)\) at \(x_{\mu}\). The collection of all \( FFB \) loops in \((X_1, p_1, B_1)\) is denoted by \( \Gamma((X_1, p_1, B_1), x_{\mu}) \).

Then the fuzzy fibre point \( x_{\mu} \) is said to be fuzzy fibre base point of \((X_1, p_1, B_1)\).

**Proposition 4.2.** Let \((X_1, p_1, B_1)\) be a \( FFB \) structure over the fuzzy topological space \((B, \tau_3)\) and let \(x_{\mu} \in FFBP(X)\). Then the relation \( \approx_{fp} \) is an equivalence relation on \( \Gamma((X_1, p_1, B_1), x_{\mu}) \).

**Proof:** The proof is vivid.

**Notation 4.1.** Let \((X_1, p_1, B_1)\) be a \( FFB \) structure over the fuzzy topological space \((B, \tau_3)\) and let \(x_{\mu} \in FFBP(X)\). Let \( \alpha \in \Gamma((X_1, p_1, B_1), x_{\mu}) \) and let \([\alpha]\) denotes the \( FFB \) path homotopy equivalence class that contains \( \alpha \). Then the set of all \( FFB \) path homotopy equivalence classes on \( \Gamma((X_1, p_1, B_1), x_{\mu}) \) is denoted by \( Y((X_1, p_1, B_1), x_{\mu}) \).

Now an operation \( \circ \) on \( Y((X_1, p_1, B_1), x_{\mu}) \) is defined by \( [\alpha] \circ [\beta] = [\alpha \ast \beta] \).

**Proposition 4.3.** Let \((X_1, p_1, B_1)\) be a \( FFB \) structure over the fuzzy topological space \((B, \tau_3)\) and let \(x_{\mu} \in FFBP(X)\). Also let \( y_1, y_2, \eta_1, \eta_2 \in \Gamma((X_1, p_1, B_1), x_{\mu}) \).

If \( y_1 \approx_{fp} y_2 \) and \( \eta_1 \approx_{fp} \eta_2 \), then \( y_1 \ast \eta_1 \approx_{fp} y_2 \ast \eta_2 \).

**Proof:** Let \((I, p_2, B_2)\) and \((I, p_3, B_3)\) be any two \( FFB \) structures over the fuzzy topological space \((B, \tau_3)\). As \( y_1 \approx_{fp} y_2 \) and \( \eta_1 \approx_{fp} \eta_2 \), there exist \( FFB \) continuous functions \( F, G: (I, p_2, B_2) \times (I, p_3, B_3) \rightarrow (X_1, p_1, B_1) \) such that

\[
F(r, 0) = y_1(r), \quad F(r, 1) = y_2(r), \quad G(r, 0) = \eta_1(r), \quad G(r, 1) = \eta_2(r), \quad \text{where } (0 \leq r \leq 1),
\]

\[
F(0, s) = F(1, s) = G(0, s) = G(1, s) = x_{\mu}, \quad \text{where } (0 \leq s \leq 1).
\]

Let \( H: (I, p_2, B_2) \times (I, p_3, B_3) \rightarrow (X_1, p_1, B_1) \) be defined by

\[
H(r, s) = \begin{cases} 
F(2r, s), & \text{if } 0 \leq r \leq 1/2 \text{ and } 0 \leq s \leq 1 \\
G(2r - 1, s), & \text{if } 1/2 \leq r \leq 1, 0 \leq s \leq 1.
\end{cases}
\]

Hence \( H \) is well defined and it is a \( FFB \) continuous function. Moreover,

\[
H(r, 0) = \begin{cases} 
F(2r, 0), & \text{if } 0 \leq r \leq 1/2 \\
G((2r - 1), 0), & \text{if } 1/2 \leq r \leq 1.
\end{cases}
\]

and

\[
H(r, 1) = \begin{cases} 
\eta_1(2r - 1), & \text{if } 0 \leq r \leq 1/2 \\
\eta_2(2r - 1), & \text{if } 1/2 \leq r \leq 1.
\end{cases}
\]

Now an operation \( \circ \) is defined by \( (y_1 \ast \eta_1)(r) = (y_2 \ast \eta_2)(r) \) for all \( r \in [0, 1] \) and \( H(0, s) = F(0, s) = x_{\mu}, H(1, s) = G(1, s) = x_{\mu} \) for all \( 0 \leq s \leq 1 \). Hence \( y_1 \ast \eta_1 \approx_{fp} y_2 \ast \eta_2 \).

By the above proposition, it is apparent that the operation \( \ast \) is well defined.

**Proposition 4.4.** Let \((X_1, p_1, B_1)\), \((I, p_2, B_2)\) and \((I, p_3, B_3)\) be any three \( FFB \) structures over the fuzzy topological space \((B, \tau_3)\) and let \(x_{\mu} \in FFBP(X)\). Also let \( [y_1], [y_2] \) and \([y_3] \in \Gamma((X_1, p_1, B_1), x_{\mu})\). Then \( ([y_1] \circ [y_2] \circ [y_3]) = ([y_1] \ast ([y_2] \ast [y_3])) \).

**Proof:** It is enough to prove that \( (y_1 \ast y_2) \ast y_3 = y_1 \ast (y_2 \ast y_3) \). For all \( r \in [0, 1] \), it is clear that,

\[
([y_1] \ast y_2)(2r) = (y_1 \ast y_2)(2r), \quad \text{if } 0 \leq r \leq 1/2
\]

\[
(y_3)(2r - 1) = (y_3)(2r - 1), \quad \text{if } 1/2 \leq r \leq 1.
\]

This implies

\[
([y_1] \ast y_2) \ast y_3) = (y_1 \ast (y_2 \ast y_3)) = ([y_1] \ast ([y_2] \ast [y_3])).
\]
Also
\[
\begin{align*}
[y_1 * (y_2 * y_3)](r) &= y_1(2r), \\
(y_2 * y_3)(2r - 1) &= y_2(4r - 2),
\end{align*}
\]
This implies
\[
[y_1 * (y_2 * y_3)](r) = \begin{cases} y_1(2r), & \text{if } 0 \leq r \leq 1/2 \\
y_2(4r - 2), & \text{if } 1/2 \leq r \leq 1 \\
y_3(2r - 1), & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
Hence \(H\) is a \(F\)F\(B\) continuous function and
\[
H(r, 0) = \begin{cases} x_3 = e(r), & \text{if } 0 \leq r \leq 1/2 \\
y(2r - 1), & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
\[
(r, 0) = \begin{cases} (\gamma * \gamma)(r), & \text{if } 0 \leq r \leq 1/2 \\
(\gamma(2r - 1) - \gamma)(r), & \text{if } 1/2 \leq r \leq 1 \\
\gamma(2r - 1), & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
and \(H(r, 1) = \gamma(r)\) for all \(r \in [0, 1]\). Also \(H(0, s) = H(1, s) = x_3\) for all \(s \in [0, 1]\). Hence \((e * \gamma) \approx \gamma\). That is \([e] * [\gamma] = [\gamma]\). Therefore \([\gamma] * [e] = [e] * [\gamma] = \gamma\) for each \([\gamma] \in Y((x_1, p_1, B_{3r}), x_3)\).

Hence \([e] = 1\) is the identity element of \(Y((x_1, p_1, B_{3r}), x_3)\).

Definition 4.5. Let \(Y\) be the \(F\)F\(B\) path in \((X_{1r}, p_1, B_{3r})\) from \(x_3\) to \(x'_{1r}\), where \(x_3, x'_{1r} \in FFBP(X)\). Let \(\tilde{\gamma}\) be the \(F\)F\(B\) path from \(x'_{1r}\) to \(x_3\) defined by \(\tilde{\gamma}(r) = \gamma(1 - r)\), for all \(r \in I\). Then \(\tilde{\gamma}\) is called the inverse of \(\gamma\).

Proposition 4.6. Let \((X_{1r}, p_1, B_{3r})\), \((I_{1r}, p_2, B_{3r})\) and \((I_{1r}, p_3, B_{3r})\) be any three \(F\)F\(B\) structures over the fuzzy topological space \((B, r_s)\) and let \(x_3 \in FFBP(X)\). If \(\delta \in \Gamma((X_{1r}, p_1, B_{3r}), x_3)\), then there exists a \(F\)F\(B\) continuous function \(\delta: (I_{1r}, p_2, B_{3r}) \rightarrow (X_{1r}, p_1, B_{3r})\) such that \(\delta \in \Gamma((x_1, p_1, B_{3r}), x_3)\).

Proof: By definitio of \(\delta\), it is apparent that \(\delta\) is a \(F\)F\(B\) continuous function. Also \(\delta(0) = \delta(1) = \delta(0)\). Hence \(\delta \in \Gamma((x_1, p_1, B_{3r}), x_3)\).

Proposition 4.7. Let \((X_{1r}, p_1, B_{3r})\), \((I_{1r}, p_2, B_{3r})\) and \((I_{1r}, p_3, B_{3r})\) be any three \(F\)F\(B\) structures over the fuzzy topological space \((B, r_s)\) and let \(x_3 \in FFBP(X)\). If \([\gamma] \in Y((X_{1r}, p_1, B_{3r}), x_3)\), then \([\gamma] \in Y((X_{1r}, p_1, B_{3r}), x_3)\) such that \([\gamma] * [\tilde{\gamma}] = [\tilde{\gamma}] * [\gamma] = [e]\).

Proof: It is enough to prove that \(\gamma * \tilde{\gamma} \approx \gamma\) and \(\gamma * \tilde{\gamma} \approx \gamma\). Now let \(H: (I_{1r}, p_2, B_{3r}) \times (I_{1r}, p_3, B_{3r}) \rightarrow (X_{1r}, p_1, B_{3r})\) be defined by
\[
H(r, s) = \begin{cases} x_3, & \text{if } 0 \leq s \leq 1 \text{ and } 0 \leq r \leq 1/2 \\
(\gamma(2r - s), & \text{if } 0 \leq s \leq 1, s/2 \leq r \leq 1/2, \\
(\gamma(2r - 2s), & \text{if } 0 \leq s \leq 1, 1/2 \leq r \leq 1 - (s/2), \\
x_3, & \text{if } 0 \leq s \leq 1, 1 - (s/2) \leq r \leq 1. \end{cases}
\]
Hence \(H\) is a \(F\)F\(B\) continuous function and
\[
H(r, 0) = \begin{cases} x_3, & \text{if } 0 \leq r \leq 1/2 \\
y(2r), & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
\[
(x_3 = e(r), \text{if } 0 \leq r \leq 1/2)
\]
and \(H(r, 1) = \gamma(r)\) for all \(r \in [0, 1]\). Also \(H(0, s) = H(1, s) = x_3\) for all \(s \in [0, 1]\). Hence \((\gamma * e) \approx \gamma\). That is \([\gamma] * [e] = [\gamma]\).

Similarly let \(H\) be defined by
\[
H(r, s) = \begin{cases} x_3, & \text{if } 0 \leq s \leq 1 \text{ and } 0 \leq r \leq 1/2 \\
(\gamma(2r - s), & \text{if } 0 \leq s \leq 1, s/2 \leq r \leq 1/2, \\
(\gamma(2r - 2s), & \text{if } 0 \leq s \leq 1, 1/2 \leq r \leq 1 - (s/2), \\
x_3, & \text{if } 0 \leq s \leq 1, 1 - (s/2) \leq r \leq 1. \end{cases}
\]
Hence \(H\) is a \(F\)F\(B\) continuous function and
\[
H(r, 0) = \begin{cases} x_3, & \text{if } 0 \leq r \leq 1/2 \\
y(2r), & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
\[
(\gamma(2r), \text{if } 0 \leq r \leq 1/2, \\
(\gamma(2r), \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
and \(H(r, 1) = e(r)\) for all \(r \in [0, 1]\). Also \(H(0, s) = H(1, s) = x_3\) for all \(s \in [0, 1]\).
Hence \((\gamma \ast \gamma) \approx_{FP} \varepsilon\). That is \([\gamma] \ast [\gamma] = [\varepsilon]\).

Similarly let \(H\) be defined by
\[
H(r,s) =
\begin{cases}
  x_r, & \text{if } 0 \leq s \leq 1 \text{ and } 0 \leq r \leq s/2, \\
  y(1-2r+s), & \text{if } 0 \leq s \leq 1 \text{ and } s/2 \leq r \leq 1/2, \\
  y(2r+s-1), & \text{if } 0 \leq s \leq 1, 1/2 \leq r \leq 1 - (s/2), \\
  x_r, & \text{if } 0 \leq s \leq 1, 1 - (s/2) \leq r \leq 1.
\end{cases}
\]

Hence \(H\) is a \(FFB\) continuous function and
\[
H(r,0) = \begin{cases} (1-2r), & \text{if } 0 \leq r \leq 1/2 \\
2r-1, & \text{if } 1/2 \leq r \leq 1 \end{cases}
\]
and \(H(r,1) = e(r)\) for all \(r \in [0,1]\). Also \(H(0,s) = H(1,s) = x_2\) for all \(s \in [0,1]\). Hence \((\gamma \ast \gamma) \approx_{FP} \varepsilon\). That is \([\gamma] \ast [\gamma] = [\varepsilon]\). Therefore \([\gamma] \ast [\gamma] = [\varepsilon] = [\varepsilon].

From the Propositions 4.4, 4.5 and 4.7, it is vivid that \(Y((X_t, p_1, B_t), x_2)\) is a group under the operation \(*\). Also \(Y((X_t, p_1, B_t), x_2)\) is said to be the \(FFB\) fundamental group of \((X_t, p_1, B_t)\) at \(x_2\).

V. FUZZY FIBREWISE FUNDAMENTAL GROUP

In this section, some interesting properties of \(FFB\) fundamental group are established. It is shown that there exists a \(FFB\) isomorphism between two \(FFB\) fundamental groups provided they are \(FFB\) path connected structures.

Definition 5.1. Let \((X_t, p_1, B_t)\) be a \(FFB\) structure over the fuzzy topological space \((B_t, \tau)\). Then \((X_t, p_1, B_t)\) is said to be an \(FFB\) path connected if for every pair of fuzzy fibre points \(x_1, x_2 \in FFBP(X)\), there is a \(FFB\) path \(\delta\) in \((X_t, p_1, B_t)\) such that \(\delta(0) = x_1\) and \(\delta(1) = x_2\).

Definition 5.2. Let \(Y((X_t, p_1, B_t), x_1)\) and \(Y((X_t, p_1, B_t), x_2)\) be any two \(FFB\) fundamental groups. A \(FFB\) function \(f: Y((X_t, p_1, B_t), x_1) \rightarrow Y((X_t, p_1, B_t), x_2)\) is said to be a \(FFB\) homomorphism if
\[
f([\gamma_1] \circ [\gamma_2]) = f([\gamma_1]) \circ f([\gamma_2])
\]
for all \([\gamma_1], [\gamma_2] \in Y((X_t, p_1, B_t), x_1)\). Further, the \(FFB\) homomorphism is said to be \(FFB\) isomorphism if it is bijective.

Proposition 5.1. Let \((X_t, p_1, B_t)\) be a \(FFB\) path connected structure over the fuzzy topological space \((B_t, \tau)\) and \(y_{1,2} \in FFBP(Y)\). Let \(Y((X_t, p_1, B_t), x_1)\) and \(Y((X_t, p_1, B_t), x_2)\) be any two \(FFB\) fundamental groups where \(x_1, x_2 \in FFBP(X)\). Then \(Y((X_t, p_1, B_t), x_1)\) and \(Y((X_t, p_1, B_t), x_2)\) are \(FFB\) isomorphic.

Proof : Let \(\delta\) be a \(FFB\) path in \((X_t, p_1, B_t)\) from \(x_1\) to \(x_2\) and hence \(\delta\) is a \(FFB\) path in \((X_t, p_1, B_t)\) from \(x_\mu\) to \(x_\lambda\) defined by \(\delta(r) = (1-r)\) for all \(r \in I\). If \([\gamma] \in Y((X_t, p_1, B_t), x_\lambda)\), then
\[
[\delta] \ast [\gamma] \ast [\delta] \in Y((X_t, p_1, B_t), x_\lambda).
\]
Define a \(FFB\) function \(\gamma: Y((X_t, p_1, B_t), x_\lambda) \rightarrow Y((X_t, p_1, B_t), x_\mu)\) by
\[
\delta([\gamma]) = [\delta] \ast [\gamma] \ast [\delta].
\]
It is apparent that \(\delta\) is well defined since the operation \(*\) is well defined.

First to prove \(\delta\) is a \(FFB\) homomorphism, let \(\gamma_1, \gamma_2 \in Y((X_t, p_1, B_t), x_\lambda)\). Then
\[
\delta([\gamma_1]) \ast \delta([\gamma_2]) = [\delta] \ast [\gamma_1] \ast [\delta] \ast [\gamma_2] \ast [\delta].
\]
Next to prove \(\delta\) is bijective, it is enough to show that if \(\alpha\) denotes the fuzzy fibrewise path \(\delta\), then \(\alpha\) is an inverse of \(\delta\). Now,
\[
\delta([\alpha]) = [\delta] \ast [\beta] \ast [\delta] \ast [\alpha],
\]
where \([\beta] \in Y((X_t, p_1, B_t), x_\lambda)\) implies \(\delta([\beta]) = [\delta] \ast [\beta] \ast [\delta],\) since \(\alpha = \delta\). Then
\[
\delta([\delta([\alpha])]) = [\delta] \ast [\delta] \ast [\delta] \ast [\delta] = [\delta].
\]
Similarly, \(\delta([\delta([\beta])]) = [\beta]\), for each \([\beta] \in Y((X_t, p_1, B_t), x_\lambda)\). Therefore \(\delta\) is bijective and hence \(FFB\) isomorphism.

Notation 5.1. Let \(h: (X_t, p_1, B_t) \rightarrow (Y_{t', p_2, B_{t'}})\) be a \(FFB\) continuous function that carries the fuzzy fibre point \(x_2\) to the fuzzy fibre point \(y_{1,2}\), where \(X_t \in FFBP(X)\) and \(Y_{t'} \in FFBP(Y)\). In this case, let \(h\) be denoted by \(h: (X_t, p_1, B_t) \rightarrow (Y_{t'}, p_2, B_{t'})\).

Proposition 5.2. Let \((X_t, p_1, B_t)\) and \((Y_{t'}, p_2, B_{t'})\) be any two \(FFB\) path connected structures. Let \((l_1, p_3, B_{t'})\) and \((l_2, p_4, B_{t'})\) be any two \(FFB\) structures over the fuzzy topological space \((B_t, \tau)\). Then \(h: (X_t, p_1, B_t) \rightarrow (Y_{t'}, p_2, B_{t'})\) is a \(FFB\) continuous function. Then \(h\) induces a \(FFB\) homomorphism \(h: Y((X_t, p_1, B_t), x_2) \rightarrow Y((Y_{t'}, p_2, B_{t'}), y_{1,2})\).

Proof : Let \([\gamma_1] \in Y((X_t, p_1, B_t), x_2)\). Then \(\gamma_1: (l_2, p_3, B_{t'}) \rightarrow (X_t, p_1, B_t)\) is a \(FFB\) continuous function such that \(\gamma_1(0) = x_2\). Hence \(h \circ \gamma_1: (l_2, p_3, B_{t'}) \rightarrow (Y_{t'}, p_2, B_{t'})\) is a \(FFB\) continuous function such that \(h \circ \gamma_1(0) = (h \circ \gamma_1)(1) = y_{1,2}\). Hence
\[
h \circ \gamma_1 \in Y((Y_{t'}, p_2, B_{t'}), y_{1,2})\).
\]
Define \(h: Y((X_t, p_1, B_t), x_2) \rightarrow Y((Y_{t'}, p_2, B_{t'}), y_{1,2})\) by \(h([\gamma_1]) = h \circ \gamma_1\).

To show \(h\) is well defined it is enough to show that \(h \circ \gamma_1 \in Y((X_t, p_1, B_t), x_2)\) and \(\gamma_1 \approx_{FP} \gamma_2\). Then \(h \circ \gamma_1 \approx_{FP} h \circ \gamma_2\). Since
\( \gamma_1 \equiv_{FP} \gamma_2 \), there exists a \( FFB \) continuous function

\[ F : (I_1, p_3, B_{t_3}) \times (I_2, p_4, B_{t_4}) \rightarrow (Y_{t_1}, p_1, B_{t_1}) \]

such that

\[ F(r, 0) = \gamma_1(r) \text{ and } F(r, 1) = \gamma_2(r), \]

where \( 0 \leq r \leq 1 \).

Then

\[ F(0, s) = F(1, s) = x_{s}, \text{ where } (0 \leq s \leq 1). \]

Let \( G : (I_1, p_2, B_{t_3}) \times (I_2, p_4, B_{t_4}) \rightarrow (Y_{t_2}, p_2, B_{t_2}) \) be defined by

\[ G(r, s) = (h \circ F)(r, s). \]

Then

\[ G(r, 0) = (h \circ \gamma_1)(r), \text{ and } G(r, 1) = (h \circ \gamma_2)(r) = x_{s}, \]

where \( 0 \leq r \leq 1 \), and

\[ G(0, s) = G(1, s) = y_{\mu}, \text{ where } (0 \leq s \leq 1). \]

Hence \( (h \circ \gamma_1) \equiv_{FP} (h \circ \gamma_2) \) and so \( h_* \) is well defined.

Next to show \( h_* \) is a \( FFB \) homomorphism, let \( \gamma_1, \gamma_2 \in Y((X_{t_1}, p_1, B_{t_1}), x_{s_1}). \) It is clear that

\[ (\gamma_1 \ast \gamma_2)(r) = \begin{cases} \gamma_1(2r), & \text{if } 0 \leq r \leq 1/2, \\ \gamma_2(2r - 1), & \text{if } 1/2 \leq r \leq 1. \end{cases} \]

Then

\[ (h \circ (\gamma_1 \ast \gamma_2))(r) = \begin{cases} (h \circ \gamma_1)(2r), & \text{if } 0 \leq r \leq 1/2 \\ (h \circ \gamma_2)(2r - 1), & \text{if } 1/2 \leq r \leq 1 \end{cases} = (h \circ \gamma_1) \ast (h \circ \gamma_2)(r). \]

Therefore,

\[ h_*([\gamma_1] \ast [\gamma_2]) = h_*([\gamma_1] \ast [\gamma_2]) = [h \circ (\gamma_1 \ast \gamma_2)] = [(h \circ \gamma_1) \ast (h \circ \gamma_2)] = [h \circ \gamma_1] \ast [h \circ \gamma_2] = (h_*[\gamma_1]) \ast (h_*[\gamma_2]). \]

Hence \( h_* \) is a fuzzy fibrewise homomorphism.

**Proposition 5.3.** If

\[ h : ((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \rightarrow (Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \]

\[ k : ((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \rightarrow ((Z_{t_3}, p_3, B_{t_3}), x_{s_3}) \]

are the \( FFB \) continuous functions, then \( (k \circ h)_* = k_* \circ h_* \).

**Proof:** Let \([y] \in Y((X_{t_1}, p_1, B_{t_1}), x_{s_1}). \) Now

\[ (k \circ h)_*[y] = [(k \circ h) \circ y] = [k \circ (h \circ y)] = k_*[h_*[y]] = [k \circ (h \circ y)]. \]

Hence \( (k \circ h)_* = k_* \circ h_* \).

**Proposition 5.4.** Let \((X_{t_1}, p_1, B_{t_1}) \) be a \( FFB \) path connected structure and let \( x_{s_1} \in FFBP(X) \). If

\[ i_* : ((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \rightarrow (Y_{t_2}, p_2, B_{t_2}), x_{s_2} \]

is the \( FFB \) identity function, then \( i_* \) is the \( FFB \) identity homomorphism.

**Proof:** The proof is similar to the above proposition, since

\[ i_*[f] = [i \circ f] = [f]. \]

**Definition 5.3.** Let \((X_{t_1}, p_1, B_{t_1}) \) and \((Y_{t_2}, p_2, B_{t_2}) \) be \( FFB \) structures over the fuzzy topological space \((B_{t_3}, t_3) \). A \( FFB \) function \( f : (X_{t_1}, p_1, B_{t_1}) \rightarrow (Y_{t_2}, p_2, B_{t_2}) \) is said to be \( FFB \) homeomorphism if \( f \) is bijective, \( f \) and \( f^{-1} \) are \( FFB \) continuous.

**Proposition 5.5.** Let \( h : ((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \rightarrow (Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \) be a \( FFB \) homeomorphism between two \( FFB \) path connected structures. Then \( h_* : Y((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \rightarrow Y((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \) is a \( FFB \) isomorphism.

**Proof:** Let \( i_* : Y((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \rightarrow Y((X_{t_1}, p_1, B_{t_1}), y_{\mu}) \), and \( j_* : Y((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \rightarrow Y((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \) Let

\[ k : Y((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \rightarrow Y((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \]

be the inverse of \( h \). Since \( h \) is a \( FFB \) homeomorphism, \( k \) is \( FFB \) continuous. Then \( (k \circ h)_* = (k \circ h)_* = i_* \), where \( i_* \) is the \( FFB \) identity function of \((X_{t_1}, p_1, B_{t_1}), x_{s_1}) \) and \( h \circ k_* = (h \circ k)_* = j_* \), where \( j_* \) is the \( FFB \) identity function of \((Y_{t_2}, p_2, B_{t_2}), y_{\mu}) \). Hence \( k_* \) is the inverse of \( h_* \). Thus \( h_* \) is a \( FFB \) isomorphism.

**VI. CONCLUSION**

In this paper, it is investigated that the group properties are satisfied by the set of all \( FFB \) path homotopy equivalence classes on the collection of \( FFB \) loops. Further, the results in this treatise motivate to study the applications of \( FFB \) fundamental groups.

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