International Journal of Recent Technology and Engineering (IJRTE)
ISSN: 2277-3878 (Online), Volume-8 Issue-6, March 2020

Fuzzy Fibrewise Homotopy

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***Abstract :** In this disquisition, the notion $\mathcal{FFB}$ structures is introduced. The intend of this article is to study the notions of homotopy, path homotopy and fundamental group via $\mathcal{FFB}$ structure which is a triplet consisting of two fuzzy topological spaces and a continuous surjection between them. Many properties concerning these concepts are provided.*

***Keywords.** $\mathcal{FFB}$ structure, $\mathcal{FFB}$ homotopy, $\mathcal{FFB}$ path homotopy, $\mathcal{FFB}$ fundamental group.*

I. INTRODUCTION

Chang [2] introduced and developed the concept of fuzzy topological spaces based on the notion of fuzzy sets which was introduced by Zadeh in [11]. Later the theory of fuzzy topological spaces was developed by many researchers. In [4], I. M. James introduced the notion of fibrewise topology and promoted various notions of topology in fibrewise viewpoint. Also fibrewise versions of homotopy theory were studied in [4]. The notion of fuzzy homotopy theory was introduced by G. Culvacioglu and M. Citil in [3]. The fundamental group of fuzzy topological spaces was introduced by Abdul Razak Salleh and Mohammad Tap in [7] and [8]. Motivated by [7] and [8], fuzzy fundamental group in fuzzy topological spaces was extended to various fuzzy structure spaces in [9] and [6].

In this treatise, the notion of $\mathcal{FFB}$ structures is introduced. The concepts of $\mathcal{FFB}$ homotopy, $\mathcal{FFB}$ path homotopy and $\mathcal{FFB}$ fundamental group in $\mathcal{FFB}$ structures are introduced and their properties are investigated. It is shown that the set of all $\mathcal{FFB}$ path homotopy equivalence classes on the collection of $\mathcal{FFB}$ loops forms a fundamental group and there exists a $\mathcal{FFB}$ isomorphism between two $\mathcal{FFB}$ fundamental groups.

II. PRELIMINARIES

Definition 2.1. [1] Let (X,T) be a fuzzy topological space and Y be an ordinary subset of X. Then $T_Y = \{\lambda/Y \mid \lambda \in T\}$ is a fuzzy topology on Y and is called the induced or relative fuzzy topology. The pair (Y, T_Y) is called a fuzzy subspace of (X,T) : (Y,T_Y) is called a fuzzy open/ fuzzy closed / fuzzy β-open fuzzy subspace if the characteristic function of Y viz χ_Y is fuzzy open/ fuzzy closed /fuzzy β-open respectively.

Definition 2.2. [10] Let (X,τ) be a (usual) topological space. The collection $\tilde{T} = \{G : G$ is a fuzzy set in X and $Supp\ G \in \tau\}$ is a fuzzy topology on X, called the fuzzy topology on X introduced by T. $(X,\tilde{T})$ is called the fuzzy topological space introduced by (X,T). Let $\tilde{\varepsilon}_I$ denote Euclidean subspace topology on I and $(I,\tilde{\varepsilon}_I)$ denote the fuzzy topological space introduced by the topological space (I,ε_I).

Definition 2.3. [4] If the base set is denoted by B, then a fibrewise set over B, consists of a set X together with a function $p: X \to B$, called the projection. For each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X.

Definition 2.4. [4] The definition of fibrewise homotopy involves the (fibrewise) cylinder $I \times X = (I \times B) \times_B X$ on the domain, which comes equipped with a family of fibrewise embeddings $\sigma_t : X \to I \times X$ $(0 \le t \le 1)$, where $\sigma_t(x) = (t,x)$ $(x \in X)$. Let $\theta, \phi : X \to Y$ be fibrewise maps, where X and Y are fibrewise spaces over B. A fibrewise homotopy of θ into ϕ is a fibrewise map $f : I \times X \to Y$ such that $f\sigma_0 = \theta$ and $f\sigma_1 = \phi$.

Definition 2.5. [5] Given points x and y of the topological space X, a path in X from x to y is a continuous map $f : [a,b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$.

Definition 2.6. [5] A topological space X is said to be path connected if every pair of points of X can be joined by a path in X.

III. ATTRIBUTES OF FUZZY FIBREWISE STRUCTURES

In this section, the perception of $\mathcal{FFB}$ structures is pioneered. In addition, the concepts of $\mathcal{FFB}$ open (resp. closed) structures are instigated and some properties concerning these concepts are established.

Definition 3.1. The triplet $((X,\tau), p, (B,\sigma))$ which comprises two fuzzy topological spaces (X,τ), (B,σ) and a fuzzy continuous surjection $p : (X,\tau) \to (B,\sigma)$ is called a fuzzy fibrewise (in short $\mathcal{FFB}$) structure over (B,σ) and is simply denoted by (X_τ, p, B_σ).

Then (X,τ) is termed the fuzzy total (or fuzzy fibred) space; p is called the projection; (B,σ) is said to be the fuzzy base space and for each $\lambda \in I^B$, the fuzzy set $p^{-1}(\lambda)$ is called the fuzzy fibre over λ.

Manuscript received on February 10, 2020.
Revised Manuscript received on February 20, 2020.
Manuscript published on March 30, 2020.
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If a fuzzy fibre $\mu \in \tau$, then μ is termed fuzzy open fibre. The complement of fuzzy open fibre is termed fuzzy closed fibre. If a fuzzy fibre is a fuzzy point x_μ, then x_μ is said to be a fuzzy fibre point. The collection of fuzzy fibre points in I^X is denoted by $\mathcal{FFBP}(X)$.

Definition 3.2. Let $(X_{\tau_1}, p_1, B_{\tau_2})$ be a $\mathcal{FFB}$ structure over (B, τ_2) and $\lambda \in I^X$. Then the $\mathcal{FFB}$ interior of λ is defined and symbolised as $\mathcal{FFB}int(\lambda) = \vee \{\mu : \mu \le \lambda$ and μ is a fuzzy open fibre in $(X, \tau_1)\}$.

Definition 3.3. Let $(X_{\tau_1}, p_1, B_{\tau_2})$ be a $\mathcal{FFB}$ structure over (B, τ_2) and $\lambda \in I^X$. Then the $\mathcal{FFB}$ closure of λ is defined and symbolised as $\mathcal{FFB}cl(\lambda) = \wedge \{\mu : \mu \ge \lambda$ and μ is a fuzzy closed fibre in $(X, \tau_1)\}$.

Definition 3.4. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and $(Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ structures over (B, τ_3). A function $\phi : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ is termed to be a $\mathcal{FFB}$ function if $p_2 \circ \phi = p_1$.

Definition 3.5. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and $(Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ structures over (B, τ_3). A $\mathcal{FFB}$ function $\phi : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ is said to be a $\mathcal{FFB}$ continuous function if for each fuzzy fibre point $x_\lambda \in \mathcal{FFBP}(X)$ over the fuzzy point $z_\eta \in FP(B)$ and for every fuzzy open fibre μ in (Y, τ_2) with $\phi(x_\lambda) \le \mu$, there exists a fuzzy open fibre γ in (X, τ_1) with $x_\lambda \le \gamma$ such that $\phi(\gamma) \le \mu$.

Equivalently, a $\mathcal{FFB}$ function $\phi : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ is said to be a $\mathcal{FFB}$ continuous function if for each fuzzy open fibre (resp. fuzzy closed fibre) λ in (Y, τ_2), $\phi^{-1}(\lambda)$ is a fuzzy open fibre (resp. fuzzy closed fibre) in (X, τ_1).

Proposition 3.1. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and $(Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). Let $(A_{\tau_A}, p_3, B_{\tau_3})$ and $(D_{\tau_D}, p_4, B_{\tau_3})$ be any two $\mathcal{FFB}$ substructures of $(X_{\tau_1}, p_1, B_{\tau_3})$ over the fuzzy topological space (B, τ_3). Let $1_X = (\chi_A \vee \chi_D)$, where χ_A and χ_D are the fuzzy closed fibres in (X, τ_1). Let $\phi : (A_{\tau_A}, p_3, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ and $\psi : (D_{\tau_D}, p_4, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions. If $\phi|_{A\cap D} = \psi|_{A\cap D}$, then $\varphi : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ defined by

$$\varphi(x) = \begin{cases} \phi(x), & x \in A; \\ \psi(x), & x \in D. \end{cases}$$

is a $\mathcal{FFB}$ continuous function.

Proof : Let λ be any fuzzy closed fibres in (Y, τ_2). Let χ_A, χ_D be any two fuzzy closed fibres in (X, τ_1). Now,

$$\begin{aligned}\varphi^{-1}(\lambda) &= \varphi^{-1}(\lambda) \wedge 1_X \\ &= \varphi^{-1}(\lambda) \wedge (\chi_A \vee \chi_D) \\ &= (\varphi^{-1}(\lambda) \wedge \chi_A) \vee (\varphi^{-1}(\lambda) \wedge \chi_D)\end{aligned}$$

$$\varphi^{-1}(\lambda) = \phi^{-1}(\lambda) \vee \psi^{-1}(\lambda).$$

Since ϕ and ψ are $\mathcal{FFB}$ continuous functions, $\phi^{-1}(\lambda)$ and $\psi^{-1}(\lambda)$ are fuzzy closed fibres of (A, T_A) and (D, T_D) respectively. Thus $\varphi^{-1}(\lambda)$ is a fuzzy closed fibres in (X, τ_1). Hence φ is a $\mathcal{FFB}$ continuous function.

Remark 3.1. Let I be the unit interval. Let ξ be an Euclidean subspace topology on I and $(I, \tilde{\xi})$ be a fuzzy topological space introduced by the (usual) topological space (I, ξ). Then $((I, \tilde{\xi}), p, (B, \sigma))$ (simply $((I_{\tilde{\xi}}, p, B_\sigma))$ is a fuzzy fibrewise structure over (B, σ).

Proposition 3.2. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B, τ_3). Let I_1 and I_2 be (usual) closed in (I, ξ) and $I_1 \cup I_2 = I$. Let $f : (I_{1_{\tilde{\xi}_1}}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ and $g : (I_{2_{\zeta_2}}, p_4, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions. If $f|_{I_1 \cap I_2} = g|_{I_1 \cap I_2}$, then $h : (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ defined by

$$h(r) = \begin{cases} f(r), & r \in I_1; \\ g(r), & r \in I_2. \end{cases}$$

is a $\mathcal{FFB}$ continuous function.

Proof : The proof is apparent.

The definition of $\mathcal{FFB}$ homotopy involves the $\mathcal{FFB}$ cylinder

$$\begin{aligned}(I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) &= ((I, \tilde{\xi}) \times (B, \tau_3), p_4, (B, \tau_3)) \\ &\quad \times (X_{\tau_1}, p_1, B_{\tau_3})\end{aligned}$$

on the domain, which comes equipped with a family of $\mathcal{FFB}$ embeddings

$$\sigma_t : (X_{\tau_1}, p_1, B_{\tau_3}) \to ((I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) (0 \le t \le 1),$$

where $\sigma_t(x_\lambda) = (t, x_\lambda)$ for every fuzzy fibre point $x_\lambda \in FFBP(X)$.

Definition 3.6. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, $(Y_{\tau_2}, p_2, B_{\tau_3})$ and $(I_{\tilde{\xi}}, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). Let $\theta, \phi : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be $\mathcal{FFB}$ continuous functions. A $\mathcal{FFB}$ homotopy of θ into ϕ is a $\mathcal{FFB}$ continuous function

$$f : (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$$

such that $f \circ \sigma_0(x_\lambda) = \theta(x_\lambda)$ and $f \circ \sigma_1(x_\lambda) = \phi(x_\lambda)$ for each fuzzy point $x_\lambda \in FFBP(X)$.

If there exists a $\mathcal{FFB}$ homotopy of θ into ϕ, then we say that θ is $\mathcal{FFB}$ homotopic to ϕ denoted by $\theta \simeq_{FB} \phi$.

Proposition 3.3. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, $(Y_{\tau_2}, p_2, B_{\tau_3})$ and $(I_{\tilde{\xi}}, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). Then $\simeq_{FB}$ is an equivalence relation.

Proof : **(i)** Let $\theta : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be a $\mathcal{FFB}$ continuous function and $f : (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be a $\mathcal{FFB}$ function such that $f \circ \sigma_t(x_\lambda) = \theta(x_\lambda)$ for each fuzzy fibre point $x_\lambda \in FFP(X)$ and $t \in I$. Then f is a $\mathcal{FFB}$ continuous function and $f \circ \sigma_0(x_\lambda) = \theta(x_\lambda) = f \circ \sigma_1(x_\lambda)$. Hence $\theta \simeq_{FB} \theta$.

(ii) Let $\theta_1, \theta_2 : (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions. If $\theta_1 \simeq_{FB} \theta_2$, then there exists a $\mathcal{FFB}$ continuous function $f : (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ such that $f \circ \sigma_0(x_\lambda) = \theta_1(x_\lambda)$ and $f \circ \sigma_1(x_\lambda) = \theta_2(x_\lambda)$. Let $g : (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be a $\mathcal{FFB}$ function such that $g \circ \sigma_t(x_\lambda) = f \circ \sigma_{1-t}(x_\lambda)$ for each fuzzy fibre point $x_\lambda \in FFBP(X)$ and $t \in I$. Then g

is a $\mathcal{FFB}$ continuous function and $g\circ\sigma_0(x_\lambda)=f\circ\sigma_1(x_\lambda)=\theta_2(x_\lambda)$, $g\circ\sigma_1(x_\lambda)=f\circ\sigma_0(x_\lambda)=\theta_1(x_\lambda)$. Hence $\theta_2\simeq_{FB}\theta_1$.

(iii) Let $\theta_1,\theta_2,\theta_3:(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ be any three $\mathcal{FFB}$ continuous functions such that $\theta_1\simeq_{FB}\theta_2$ and $\theta_2\simeq_{FB}\theta_3$. Since $\theta_1\simeq_{FB}\theta_2$, there exists $\mathcal{FFB}$ continuous function $f:(I_\xi,p_3,B_{\tau_3})\times(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ such that $f\circ\sigma_0(x_\lambda)=\theta_1(x_\lambda)$ and $f\circ\sigma_1(x_\lambda)=\theta_2(x_\lambda)$ for each fuzzy fibre point $x_\lambda\in FFBP(X)$. Similarly, since $\theta_2\simeq_{FB}\theta_3$, there exists a fuzzy fibrewise continuous function $g:(I_\xi,p_3,B_{\tau_3})\times(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ such that $g\circ\sigma_0(x_\lambda)=\theta_2(x_\lambda)$ and $g\circ\sigma_1(x_\lambda)=\theta_3(x_\lambda)$ for each fuzzy fibre point $x_\lambda\in FFBP(X)$.

Let $h:(I_\xi,p_3,B_{\tau_3})\times(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ be a fibrewise function such that

$$h\circ\sigma_t(x_\lambda)=\begin{cases} f\circ\sigma_{2t}(x_\lambda), & 0\le t\le 1/2;\\ g\circ\sigma_{2t-1}(x_\lambda), & 1/2\le t\le 1,\end{cases}$$

for each fuzzy fibre point $x_\lambda\in FFBP(X)$. Since f and g are $\mathcal{FFB}$ continuous functions and by Proposition 3.2, h is a $\mathcal{FFB}$ continuous function. Also $h\circ\sigma_0(x_\lambda)=f\circ\sigma(x_\lambda)=\theta_1(x_\lambda)$ and $h\circ\sigma_1(x_\lambda)=g\circ\sigma(x_\lambda)=\theta_3(x_\lambda)$. Hence $\theta_1\simeq_{FB}\theta_3$. Thus $\simeq_{FB}$ is an equivalence relation.

Proposition 3.4. Let $(X_{\tau_1},p_1,B_{\tau_3})$, $(Y_{\tau_2},p_2,B_{\tau_3})$ and $(Z_{\tau_4},p_3,B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B,τ_3). Let $\theta,\phi:(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions such that $\theta\simeq_{FB}\phi$. If $\varphi:(Y_{\tau_2},p_2,B_{\tau_3})\to(Z_{\tau_4},p_3,B_{\tau_3})$ is a $\mathcal{FFB}$ continuous function, then $\varphi\circ\theta,\varphi\circ\phi:(X_{\tau_1},p_1,B_{\tau_3})\to(Z_{\tau_4},p_3,B_{\tau_3})$ are $\mathcal{FFB}$ continuous functions and $\varphi\circ\theta\simeq_{FB}\varphi\circ\phi$.

Proof : Let (I_ξ,p_4,B_{τ_3}) be a $\mathcal{FFB}$ structure over (B,τ_3). Since φ,θ,ϕ are $\mathcal{FFB}$ continuous functions, $\varphi\circ\theta$ and $\varphi\circ\phi$ are $\mathcal{FFB}$ continuous functions. Also $\theta\simeq_{FB}\phi$ implies that there exists a $\mathcal{FFB}$ continuous function $f:(I_\xi,p_3,B_{\tau_3})\times(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ such that $f\circ\sigma_0(x_\lambda)=\theta(x_\lambda)$ and $f\circ\sigma_1(x_\lambda)=\phi(x_\lambda)$ for each fuzzy fibre point $x_\lambda\in FFBP(X)$. Let $g:(I_\xi,p_3,B_{\tau_3})\times(X_{\tau_1},p_1,B_{\tau_3})\to(Z_{\tau_4},p_3,B_{\tau_3})$ be a $\mathcal{FFB}$ function such that $g\circ\sigma_t(x_\lambda)=(\varphi(f\circ\sigma_t))(x_\lambda)$ for each fuzzy fibre point $x_\lambda\in FFBP(X)$ and $t\in I$. Since φ and f are $\mathcal{FFB}$ continuous functions, $g=\varphi\circ f$ is a $\mathcal{FFB}$ continuous function. Further, g satisfies the following conditions :

$$g\circ\sigma_0(x_\lambda)=(\varphi(f\circ\sigma_0))(x_\lambda)=\varphi(\theta(x_\lambda))=(\varphi\circ\theta)(x_\lambda),$$

$$g\circ\sigma_1(x_\lambda)=(\varphi(f\circ\sigma_1))(x_\lambda)=\varphi(\phi(x_\lambda))=(\varphi\circ\phi)(x_\lambda)$$

for each fuzzy fibre point $x_\lambda\in FFBP(X)$. Hence $\varphi\circ\theta\simeq_{FB}\varphi\circ\phi$.

Proposition 3.5. Let $(X_{\tau_1},p_1,B_{\tau_3})$, $(Y_{\tau_2},p_2,B_{\tau_3})$ and $(Z_{\tau_4},p_3,B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over (B,τ_3). Let $\theta_1,\theta_2:(X_{\tau_1},p_1,B_{\tau_3})\to(Y_{\tau_2},p_2,B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions such that $\theta_1\simeq_{FB}\theta_2$ and let $\phi_1,\phi_2:(Y_{\tau_2},p_2,B_{\tau_3})\to(Z_{\tau_4},p_3,B_{\tau_3})$ be any two $\mathcal{FFB}$ continuous functions such that $\phi_1\simeq_{FB}\phi_2$. Then $\phi_1\circ\theta_1,\phi_2\circ\theta_2:(X_{\tau_1},p_1,B_{\tau_3})\to(Z_{\tau_4},p_3,B_{\tau_3})$ are $\mathcal{FFB}$ continuous functions and $\phi_1\circ\theta_1\simeq_{FB}\phi_2\circ\theta_2$.

Proof :
The proof is apparent from the following steps:

(i) $\phi_1\circ\theta_1\simeq_{FB}\phi_1\circ\theta_2$
(ii) $\phi_1\circ\theta_2\simeq_{FB}\phi_2\circ\theta_2$.
(iii) Transitivity of (i) and (ii).

IV. FUZZY FIBREWISE PATH HOMOTOPY

This section studies that the set of all $\mathcal{FFB}$ path homotopy equivalence claases on the collection of $\mathcal{FFB}$ loops forms a fundamental group by showing they satisfy group properties of associative, identity and inverse.

Definition 4.1. Let $(X_{\tau_1},p_1,B_{\tau_3})$ and (I_ξ,p_2,B_{τ_3}) be any two $\mathcal{FFB}$ structures over the fuzzy topological space (B,τ_3). Let the fuzzy fibre points be $x_\mu,x'_\lambda\in FFBP(X)$. If $\beta:(I_\xi,p_2,B_{\tau_3})\to(X_{\tau_1},p_1,B_{\tau_3})$ is a $\mathcal{FFB}$ continuous function such that $\beta(0)=x_\mu$ and $\beta(1)=x'_\lambda$, then β is said to be a $\mathcal{FFB}$ path in $(X_{\tau_1},p_1,B_{\tau_3})$ from x_μ to x'_λ. Here x_μ is called the initial point and x'_λ is called the end point of the fuzzy fibrewise path β.

Proposition 4.1. Let $(X_{\tau_1},p_1,B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B,τ_3) and the fuzzy fibre points be $x_\mu,x'_\lambda\in FFP(X)$. If there exists a $\mathcal{FFB}$ path in $(X_{\tau_1},p_1,B_{\tau_3})$ with initial and end points x_μ,x'_λ respectively, then there exists a $\mathcal{FFB}$ path in $(X_{\tau_1},p_1,B_{\tau_3})$ with initial and end points x'_λ,x_μ respectively.
Proof :

Let (I_ξ,p_2,B_{τ_3}) be a $\mathcal{FFB}$ structure over the fuzzy topological space (B,τ_3). Let α be a $\mathcal{FFB}$ path in $(X_{\tau_1},p_1,B_{\tau_3})$ with initial and end points x_μ,x'_λ respectively. That is $\alpha:(I_\xi,p_2,B_{\tau_3})\to(X_{\tau_1},p_1,B_{\tau_3})$ is a $\mathcal{FFB}$ continuous function with $\alpha(0)=x_\mu$ and $\alpha(1)=x'_\lambda$. Let $\beta:(I_\xi,p_2,B_{\tau_3})\to(X_{\tau_1},p_1,B_{\tau_3})$ be a $\mathcal{FFB}$ function such that $\beta(t)=\alpha(1-t)$ for all $t\in I$. Hence β is a fuzzy fibrewise continuous function. Then $\beta(0)=\alpha(1-0)=\alpha(1)=x'_\lambda$ and $\beta(1)=\alpha(1-1)=\alpha(0)=x_\mu$. Hence β is a fuzzy fibrewise path in $(X_{\tau_1},p_1,B_{\tau_3})$ with initial and end points x_μ,x'_λ respectively.

Definition 4.2. Let $(X_{\tau_1},p_1,B_{\tau_3})$, (I_ξ,p_2,B_{τ_3}) and (I_ζ,p_3,B_{τ_3}) be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B,τ_3) and let $x_\mu,x'_\lambda\in FFBP(X)$. Let α and β be two $\mathcal{FFB}$ paths in $(X_{\tau_1},p_1,B_{\tau_3})$. Then α is said to be $\mathcal{FFB}$ path homotopic to β, denoted by $\alpha\simeq_{FP}\beta$ if they have the same initial point x_μ and the same end point x'_λ and if there exists a $\mathcal{FFB}$ continuous function $F:(I_\xi,p_2,B_{\tau_3})\times(I_\zeta,p_3,B_{\tau_3})\to(X_{\tau_1},p_1,B_{\tau_3})$ such that

$$F(r,0)=\alpha(r)\ \text{ and }\ F(r,1)=\beta(r),\quad (0\le r\le 1).$$

$$F(0,s)=x_\mu\quad \text{and}$$

$$F(1,s) = x'_\lambda, \qquad (0 \le s \le 1).$$

The $\mathcal{FFB}$ function F is called the $\mathcal{FFB}$ path homotopy between α and β.

Definition 4.3. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and (I_ξ, p_2, B_{τ_3}) be any two fuzzy fibrewise structures over the fuzzy topological space (B, τ_3). Let $x_\mu, x'_\lambda, x''_\mu \in FFP(X)$. If α is a fuzzy fibrewise path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x_μ to x'_λ and if β is a fuzzy fibrewise path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x'_λ to x''_μ, then the fuzzy fibrewise path product $\alpha * \beta$ is the fuzzy fibrewise path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x_μ to x''_μ defined by

$$(\alpha * \beta)(r) = \begin{cases} \alpha(2r), & if\ 0 \le r \le 1/2 \\ \beta(2r-1), & if\ 1/2 \le r \le 1. \end{cases}$$

Definition 4.4. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and (I_ξ, p_2, B_{τ_3}) be any two $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. A $\mathcal{FFB}$ path $\alpha: (I_\xi, p_2, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ in $(X_{\tau_1}, p_1, B_{\tau_3})$ that begins and ends at x_λ (that is $\alpha(0) = \alpha(1) = x_\lambda$) is called a $\mathcal{FFB}$ loop in $(X_{\tau_1}, p_1, B_{\tau_3})$ at x_λ. The collection of all $\mathcal{FFB}$ loops in $(X_{\tau_1}, p_1, B_{\tau_3})$ is denoted by $\Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Then the fuzzy fibre point x_λ is said to be fuzzy fibre base point of $(X_{\tau_1}, p_1, B_{\tau_3})$.

Proposition 4.2. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. Then the relation $\simeq_{FP}$ is an equivalence relation on $\Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Proof : The proof is vivid.

Notation 4.1. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. Let $\alpha \in \Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and let $[\alpha]$ denotes the $\mathcal{FFB}$ path homotopy equivalence class that contains α. Then the set of all $\mathcal{FFB}$ path homotopy equivalence classes on $\Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ is denoted by $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Now an operation $\circ$ on $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ is defined by $[\alpha] \circ [\beta] = [\alpha * \beta]$.

Proposition 4.3. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. Also let $\gamma_1, \gamma_2, \eta_1, \eta_2 \in \Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. If $\gamma_1 \simeq_{FP} \gamma_2$ and $\eta_1 \simeq_{FP} \eta_2$, then $\gamma_1 * \eta_1 \simeq_{FP} \gamma_2 * \eta_2$.

Proof : Let (I_ξ, p_2, B_{τ_3}) and $(I_\zeta, p_3, B_{\tau_3})$ be any two $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). As $\gamma_1 \simeq_{FP} \gamma_2$ and $\eta_1 \simeq_{FP} \eta_2$, there exist $\mathcal{FFB}$ continuous functions $F, G: (I_\xi, p_2, B_{\tau_3}) \times (I_\zeta, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ such that

$$F(r,0) = \gamma_1(r), \quad F(r,1) = \gamma_2(r),$$
$$G(r,0) = \eta_1(r), \quad F(r,1) = \eta_2(r), \quad \text{where } (0 \le r \le 1),$$
$$F(0,s) = F(1,s) = G(0,s) = G(1,s) = x_\lambda, \quad \text{where } (0 \le s \le 1),$$

Let $H: (I_\xi, p_2, B_{\tau_3}) \times (I_\zeta, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be defined by

$$H(r,s) = \begin{cases} F(2r,s), & if\ 0 \le r \le \frac{1}{2}\ and 0 \le s \le 1 \\ G(2r-1,s), & if\ 1/2 \le r \le 1, 0 \le s \le 1. \end{cases}$$

Hence H is well defined and it is a $\mathcal{FFB}$ continuous function. Moreover,

$$H(r,0) = \begin{cases} F(2r,0), & \text{if } 0 \le r \le \frac{1}{2} \\ G((2r-1,0) & \text{if } \frac{1}{2} \le r \le 1 \end{cases}$$

$$= \begin{cases} \gamma_1((2r), & \text{if } 0 \le r \le 1/2 \\ \eta_1(2r-1), & \text{if } 1/2 \le r \le 1 \end{cases}$$

$$= (\gamma_1 * \eta_1)(r)$$

and

$$H(r,1) = \begin{cases} F(2r,1), & \text{if } 0 \le r \le \frac{1}{2} \\ G((2r-1,1) & \text{if } \frac{1}{2} \le r \le 1 \end{cases}$$

$$= \begin{cases} \gamma_2((2r), & \text{if } 0 \le r \le 1/2 \\ \eta_2(2r-1), & \text{if } 1/2 \le r \le 1 \end{cases}$$

$$= (\gamma_2 * \eta_2)(r)$$

for all $r \in [0,1]$ and $H(0,s) = F(0,s) = x_\lambda$, $H(1,s) = G(1,s) = x_\lambda$, for all $(0 \le s \le 1)$. Hence $\gamma_1 * \eta_1 \simeq_{FP} \gamma_2 * \eta_2$.

By the above proposition, it is apparent that the operation $\circ$ is well defined.

Proposition 4.4. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, (I_ξ, p_2, B_{τ_3}) and $(I_\zeta, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFP(X)$. Also let $[\gamma_1], [\gamma_2]$ and $[\gamma_3] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Then $([\gamma_1] \circ [\gamma_2]) \circ [\gamma_3] = [\gamma_1] \circ ([\gamma_2] \circ [\gamma_3])$.

Proof : It is enough to prove that $(\gamma_1 * \gamma_2) * \gamma_3 \simeq_{FP} \gamma_1 * (\gamma_2 * \gamma_3)$. For all $r \in [0,1]$, it is clear that,

$$[(\gamma_1 * \gamma_2) * \gamma_3](r) = \begin{cases} (\gamma_1 * \gamma_2)(2r) & if\ 0 \le r \le 1/2 \\ \gamma_3(2r-1) & if\ 1/2 \le r \le 1 \end{cases}$$

This implies

$$[(\gamma_1 * \gamma_2) * \gamma_3](r) =$$

$$\begin{cases} \gamma_1(4r), & if \quad 0 \le r \le 1/4 \\ \gamma_2(4r-1), & if \quad 1/4 \le r \le 1/2 \\ \gamma_3(2r-1), & if \quad 1/2 \le r \le 1 \end{cases}$$

Also

$$[\gamma_1 * (\gamma_2 * \gamma_3)](r) = \begin{cases} \gamma_1(2r) & if \quad 0 \le r \le 1/2 \\ (\gamma_2 * \gamma_3)(2r-1) & if \quad 1/2 \le r \le 1 \end{cases}$$

This implies

$$[\gamma_1 * (\gamma_2 * \gamma_3)](r) = \begin{cases} \gamma_1(2r), & if \quad 0 \le r \le 1/2 \\ \gamma_2(4r-2), & if \quad 1/2 \le r \le 3/4 \\ \gamma_3(4r-3), & if \quad 3/4 \le r \le 1 \end{cases}$$

Now let $H: (I_{\xi}, p_2, B_{\tau_3}) \times (I_{\xi}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be defined by

$$H(r,s) =$$

Thus H is fuzzy fibrewise continuous function. Also $H(r,0) = [(\gamma_1 * \gamma_2) * \gamma_3](r)$ and $H(r,1) = [\gamma_1 * (\gamma_2 * \gamma_3)](r)$ for all $r \in [0,1]$ and $H(0,s) = H(1,s) = x_\lambda$ for all $s \in [0,1]$. Therefore $([\gamma_1] \circ [\gamma_2]) \circ [\gamma_3] = [\gamma_1] \circ ([\gamma_2] \circ [\gamma_3])$.

So it is clear from the above proposition that the operation $\circ$ is associative on $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Proposition 4.5. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, $(I_{\xi}, p_2, B_{\tau_3})$ and $(I_{\xi}, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. Let $e: (I_{\xi}, p_2, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be the $\mathcal{FFB}$ loop defined by $e(r) = x_\lambda$ for all $r \in [0,1]$. Then $[\gamma] \circ [e] = [e] \circ [\gamma] = [\gamma]$ for each $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Proof : To prove if $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$, then $\gamma * e \simeq_{FP} \gamma$ and $e * \gamma \simeq_{FP} \gamma$. Since $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$, $\gamma(1) = \gamma(0) = x_\lambda$. Let $H: (I_{\xi}, p_2, B_{\tau_3}) \times (I_{\xi}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be defined by

$$H(r,s) = \begin{cases} \gamma(\frac{2r}{1+s}), & if \ 0 \le s \le 1 \ and \ 0 \le r \le \frac{1+s}{2} \\ x_\lambda = e(r), & if \ 0 \le s \le 1 \ and \ \frac{1+s}{2} \le r \le 1. \end{cases}$$

Hence H is a $\mathcal{FFB}$ continuous function and

$$H(r,0) = \begin{cases} \gamma(2r), & if \quad 0 \le r \le 1/2 \\ x_\lambda = e(r), & if \quad 1/2 \le r \le 1 \end{cases}$$

$$= (\gamma * e)(r),$$

and $H(r,1) = \gamma(r)$ for all $r \in [0,1]$. Also $H(0,s) = H(1,s) = x_\lambda$ for all $s \in [0,1]$. Hence $(\gamma * e) \simeq_{FP} \gamma$. That is $[\gamma] \circ [e] = [\gamma]$.

Similarly let H be defined by

$$H(r,s)$$

Hence H is a $\mathcal{FFB}$ continuous function and

$$H(r,0) = \begin{cases} x_\lambda = e(r), & if \quad 0 \le r \le 1/2 \\ \gamma(2r-1), & if \quad 1/2 \le r \le 1 \end{cases}$$

$$= (e * \gamma)(r),$$

and $H(r,1) = \gamma(r)$ for all $r \in [0,1]$. Also $H(0,s) = H(1,s) = x_\lambda$ for all $s \in [0,1]$. Hence $(e * \gamma) \simeq_{FP} \gamma$. That is $[e] \circ [\gamma] = [\gamma]$. Therefore $[\gamma] \circ [e] = [e] \circ [\gamma] = [\gamma]$ for each $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Hence $[e]$ is the identity element of $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Definition 4.5. Let γ be the $\mathcal{FFB}$ path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x_λ to x'_μ, where $x_\lambda, x'_\mu \in FFBP(X)$. Let $\bar{\gamma}$ be the $\mathcal{FFB}$ path from x'_μ to x_λ defined by $\bar{\gamma}(r) = \gamma(1-r)$, for all $r \in I$. Then $\bar{\gamma}$ is called the inverse of γ.

Proposition 4.6. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, $(I_{\xi}, p_2, B_{\tau_3})$ and $(I_{\xi}, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. If $\delta \in \Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$, then there exists a $\mathcal{FFB}$ continuous function $\bar{\delta}: (I_{\xi}, p_2, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ such that $\bar{\delta} \in \Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Proof : By definitio of $\bar{\delta}$, it is apparent that $\bar{\delta}$ is a $\mathcal{FFB}$ continuous function. Also $\bar{\delta}(0) = \delta(1)$ and $\bar{\delta}(1) = \delta(0)$. Hence $\bar{\delta} \in \Gamma((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$.

Proposition 4.7. Let $(X_{\tau_1}, p_1, B_{\tau_3})$, $(I_{\xi}, p_2, B_{\tau_3})$ and $(I_{\xi}, p_3, B_{\tau_3})$ be any three $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3) and let $x_\lambda \in FFBP(X)$. If $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$, then $[\bar{\gamma}] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ such that $[\gamma] \circ [\bar{\gamma}] = [\bar{\gamma}] \circ [\gamma] = [e]$.

Proof : It is enough to prove that $\gamma * \bar{\gamma} \simeq_{FP} e$ and $\bar{\gamma} * \gamma \simeq_{FP} e$. Now let

$H: (I_{\xi}, p_2, B_{\tau_3}) \times (I_{\xi}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ be defined by

$$H(r,s) = \begin{cases} x_\lambda, & if \quad 0 \le s \le 1 \quad and \quad 0 \le r \le s/2, \\ \gamma(2r-s), & if \quad 0 \le s \le 1, \ s/2 \le r \le 1/2, \\ \gamma(2-2r-s), & if \ 0 \le s \le 1, 1/2 \le r \le 1-(s/2), \\ x_\lambda, & if \ 0 \le s \le 1, 1-(s/2) \le r \le 1. \end{cases}$$

Hence H is a $\mathcal{FFB}$ continuous function and

$$H(r,0) = \begin{cases} \gamma(2r), & if \quad 0 \le r \le 1/2 \\ \gamma(2-2r), & if \quad 1/2 \le r \le 1 \end{cases}$$

$$= (\gamma * \bar{\gamma})(r),$$

and $H(r,1) = e(r)$ for all $r \in [0,1]$. Also $H(0,s) = H(1,s) = x_\lambda$ for all $s \in [0,1]$.

Hence $(\gamma * \bar{\gamma}) \simeq_{FP} e$. That is $[\gamma] \circ [\bar{\gamma}] = [e]$.

Similarly let H be defined by

$$H(r,s) = \begin{cases} x_\lambda, & \text{if } 0 \le s \le 1 \text{ and } 0 \le r \le s/2, \\ \gamma(1-2r+s), & \text{if } 0 \le s \le 1 \text{ and } s/2 \le r \le 1/2, \\ \gamma(2r+s-1), & \text{if } 0 \le s \le 1, 1/2 \le r \le 1-(s/2), \\ x_\lambda, & \text{if } 0 \le s \le 1, 1-(s/2) \le r \le 1. \end{cases}$$

Hence H is a $\mathcal{FFB}$ continuous function and

$$H(r,0) = \begin{cases} \gamma(1-2r), & \text{if } 0 \le r \le 1/2 \\ \gamma(2r-1), & \text{if } 1/2 \le r \le 1 \end{cases}$$

$$= (\bar{\gamma} * \gamma)(r),$$

and $H(r,1) = e(r)$ for all $r \in [0,1]$. Also $H(0,s) = H(1,s) = x_\lambda$ for all $s \in [0,1]$. Hence $(\bar{\gamma} * \gamma) \simeq_{FP} e$. That is $[\bar{\gamma}] \circ [\gamma] = [e]$. Therefore $[\gamma] \circ [\bar{\gamma}] = [\bar{\gamma}] \circ [\gamma] = [e]$.

From the Propositions 4.4, 4.5 and 4.7, it is vivid that $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ is a group under the operation $\circ$. Also $\Upsilon(((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda), \circ)$ is said to be the $\mathcal{FFB}$ fundamental group of $(X_{\tau_1}, p_1, B_{\tau_3})$ at x_λ.

V. FUZZY FIBREWISE FUNDAMENTAL GROUP

In this section, some interesting properties of $\mathcal{FFB}$ fund amental group are established. It is shown that there exists a $\mathcal{FFB}$ isomorphism between two $\mathcal{FFB}$ fundamental groups provided they are $\mathcal{FFB}$ path connected structures.

Definition 5.1. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ structure over the fuzzy topological space (B, τ_3). Then $(X_{\tau_1}, p_1, B_{\tau_3})$ is said to be $\mathcal{FFB}$ path connected if for every pair of fuzzy fibre points $x_\lambda, x_\mu \in FFBP(X)$, there is a $\mathcal{FFB}$ path δ in $(X_{\tau_1}, p_1, B_{\tau_3})$ such that $\delta(0) = x_\lambda$ and $\delta(1) = x_\mu$.

Definition 5.2. Let $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$ be any two $\mathcal{FFB}$ fundamental groups. A $\mathcal{FFB}$ function $f: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$ is said to be a $\mathcal{FFB}$ homomorphism if

$$f([\gamma_1] \circ [\gamma_2]) = f([\gamma_1]) \circ f([\gamma_2])$$

for all $[\gamma_1], [\gamma_2] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Further, the $\mathcal{FFB}$ homomorphism is said to be $\mathcal{FFB}$ isomorphism if it is bijective.

Proposition 5.1. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ path connected structure over the fuzzy topological space (B, τ_3) Let $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$ be any two $\mathcal{FFB}$ fundamental groups where $x_\lambda, x_\mu \in FFBP(X)$. Then $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and $\Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$ are $\mathcal{FFB}$ isomorphic.

Proof : Let δ be a $\mathcal{FFB}$ path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x_λ to x_μ and hence $\bar{\delta}$ is a $\mathcal{FFB}$ path in $(X_{\tau_1}, p_1, B_{\tau_3})$ from x_μ to x_λ defined by $\bar{\delta}(r) = \delta(1-r)$ for all $r \in I$. If $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$, then $[\bar{\delta}] * [\gamma] * [\delta] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$. Define a $\mathcal{FFB}$ function $\hat{\delta}: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\mu)$ by

$$\hat{\delta}([\gamma]) = [\bar{\delta}] * [\gamma] * [\delta].$$

It is apparent that $\hat{\delta}$ is well defined since the operation $*$ is well defined.

First to prove $\hat{\delta}$ is a $\mathcal{FFB}$ homomorphism, let $\gamma_1, \gamma_2 \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Then

$$\hat{\delta}([\gamma_1]) * \hat{\delta}([\gamma_2]) =$$

$$[\bar{\delta}] * [\gamma_1] * [\delta]) * ([\bar{\delta}] * [\gamma_2] * [\delta]).$$

$$= [\bar{\delta}] * [\gamma_1] * [\gamma_2] * [\delta].$$

$$= \hat{\delta}([\gamma_1] * [\gamma_2]).$$

Next to prove $\hat{\delta}$ is bijective, it is enough to show that if α denotes the fuzzy fibrewise path $\bar{\delta}$, then $\hat{\alpha}$ is an inverse of $\hat{\delta}$. Now,

$$\hat{\alpha}([\beta]) = [\bar{\alpha}] * [\beta] * [\alpha],$$

where $[\beta] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ implies $\hat{\alpha}([\beta]) = [\delta] * [\beta] * [\bar{\delta}]$, since $\alpha = \bar{\delta}$. Then

$$\hat{\delta}(\hat{\alpha}([\beta])) = [\bar{\delta}] * (\hat{\alpha}([\beta])) * [\delta].$$

$$= [\bar{\delta}] * ([\delta] * [\beta] * [\bar{\delta}]) * [\delta].$$

$$= [\beta].$$

Similarly, $\hat{\alpha}(\hat{\delta}([\beta])) = [\beta]$, for each $[\beta] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Therefore $\hat{\delta}$ is bijective and hence $\mathcal{FFB}$ isomorphism.

Notation 5.1. Let $h: (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be a $\mathcal{FFB}$ continuous function that carries the fuzzy fibre point x_λ to the fuzzy fibre point y_μ, where $x_\lambda \in FFBP(X)$ and $y_\mu \in FFBP(Y)$. In this case, let h be denoted by $h: ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to ((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$.

Proposition 5.2. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and $(Y_{\tau_2}, p_2, B_{\tau_3})$ be any two $\mathcal{FFB}$ path connected structures. Let $(I_{\xi}, p_3, B_{\tau_3})$ and $(I_{\xi}, p_4, B_{\tau_3})$ be any two $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). Let $h: ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to ((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$ be a $\mathcal{FFB}$ continuous function. Then h induces a $\mathcal{FFB}$ homomorphism $h_*: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$.

Proof : Let $[\gamma_1] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Then $\gamma_1: (I_{\xi}, p_3, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ is a $\mathcal{FFB}$ continuous function such that $\gamma_1(0) = \gamma_1(1) = x_\lambda$. Hence $h \circ \gamma_1: (I_{\xi}, p_3, B_{\tau_3}) \to ((Y_{\tau_2}, p_2, B_{\tau_3})$ is a $\mathcal{FFB}$ continuous function such that $(h \circ \gamma_1)(0) = (h \circ \gamma_1)(1) = y_\mu$. Hence $[h \circ \gamma_1] \in \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$. Define $h_*: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$ by $h_*([\gamma_1]) = [h \circ \gamma_1]$.

To show h_* is well defined it is enough to show that if $\gamma_1, \gamma_2 \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and $\gamma_1 \simeq_{FP} \gamma_2$, then $h \circ \gamma_1 \simeq_{FP} h \circ \gamma_2$. Since

$\gamma_1 \simeq_{FP} \gamma_2$, there exists a $\mathcal{FFB}$ continuous function $F: (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (I_{\tilde{\zeta}}, p_4, B_{\tau_3}) \to (X_{\tau_1}, p_1, B_{\tau_3})$ such that

$$F(r,0) = \gamma_1(r) \text{ and } F(r,1) = \gamma_2(r),$$

where $(0 \le r \le 1)$,

$$F(0,s) = F(1,s) = x_\lambda, \text{ where } (0 \le s \le 1),$$

Let $G: (I_{\tilde{\xi}}, p_3, B_{\tau_3}) \times (I_{\tilde{\zeta}}, p_4, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ be defined by

$$G(r,s) = (h \circ F)(r,s).$$

Then

$$G(r,0) = (h \circ \gamma_1)(r), \quad G(r,1) = (h \circ \gamma_2)(r) = x_\lambda, \text{ where } (0 \le r \le 1), and$$

$$G(0,s) = G(1,s) = y_\mu \text{ where } (0 \le s \le 1).$$

Hence $(h \circ \gamma_1) \simeq_{FP} (h \circ \gamma_2)$ and so h_* is well defined.

Next to show h_* is a $\mathcal{FFB}$ homomorphism, let $\gamma_1, \gamma_2 \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. It is clear that

$$(\gamma_1 * \gamma_2)(r) = \begin{cases} \gamma_1(2r), & if\ 0 \le r \le 1/2, \\ \gamma_2(2r-1), & if\ 1/2 \le r \le 1. \end{cases}$$

Then

$$(h \circ (\gamma_1 * \gamma_2))(r) = \begin{cases} (h \circ \gamma_1)(2r), & if\ 0 \le r \le 1/2 \\ (h \circ \gamma_2)(2r-1), & if\ 1/2 \le r \le 1 \end{cases}$$

$$= (h \circ \gamma_1) * (h \circ \gamma_2)(r).$$

Therefore,

$$\begin{aligned} h_*([\gamma_1] \circ [\gamma_2]) &= h_*([\gamma_1 * \gamma_2]) \\ &= [h \circ (\gamma_1 * \gamma_2)] \\ &= [(h \circ \gamma_1) * (h \circ \gamma_2)] \\ &= [h \circ \gamma_1] \circ [h \circ \gamma_2] \\ &= (h_*[\gamma_1]) \circ (h_*[\gamma_2]). \end{aligned}$$

Hence h_* is a fuzzy fibrewise homomorphism.

Proposition 5.3. If $h: ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to ((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$ and $k: ((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu) \to ((Z_{\tau_4}, p_3, B_{\tau_3}), z_\delta)$ are the $\mathcal{FFB}$ continuous functions, then $(k \circ h)_* = k_* \circ h_*$.

Proof : Let $[\gamma] \in \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$. Now

$$\begin{aligned} (k \circ h)_*([\gamma]) &= [(k \circ h) \circ \gamma] \\ &= [k \circ (h \circ \gamma)] \\ (k_* \circ h_*)([\gamma]) &= k_*(h_*([\gamma]) \\ &= k_*[h \circ \gamma] \\ &= [k \circ (h \circ \gamma)]. \end{aligned}$$

Hence $(k \circ h)_* = k_* \circ h_*$.

Proposition 5.4. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ be a $\mathcal{FFB}$ path connected structure and let $x_\lambda \in FFBP(X)$. If $i: ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ is the $\mathcal{FFB}$ identity function, then i_* is the $\mathcal{FFB}$ identity homomorphism.

Proof : The proof is similar to the above proposition, since $i_*([f]) = [i \circ f] = [f]$.

Definition 5.3. Let $(X_{\tau_1}, p_1, B_{\tau_3})$ and $(Y_{\tau_2}, p_2, B_{\tau_3})$ be $\mathcal{FFB}$ structures over the fuzzy topological space (B, τ_3). A $\mathcal{FFB}$ function $f: (X_{\tau_1}, p_1, B_{\tau_3}) \to (Y_{\tau_2}, p_2, B_{\tau_3})$ is said to be $\mathcal{FFB}$ homeomorphism if f is bijective, f and f^{-1} are $\mathcal{FFB}$ continuous.

Proposition 5.5. Let $h: ((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to ((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$ be a $\mathcal{FFB}$ homeomorphism between two $\mathcal{FFB}$ path connected structures. Then $h_*: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$ is a $\mathcal{FFB}$ isomorphism.

Proof : Let $i_*: \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda) \to \Upsilon((X_{\tau_1}, p_1, B_{\tau_3})$, and $j_*: \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu) \to \Upsilon((Y_{\tau_1}, p_2, B_{\tau_3}), y_\mu$ Let $k: \Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu) \to \Upsilon((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ be the inverse of h. Since h is a $\mathcal{FFB}$ homeomorphism, k is $\mathcal{FFB}$ continuous.Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the $\mathcal{FFB}$ identity function of $((X_{\tau_1}, p_1, B_{\tau_3}), x_\lambda)$ and $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the $\mathcal{FFB}$ identity function of $\Upsilon((Y_{\tau_2}, p_2, B_{\tau_3}), y_\mu)$. Hence k_* is the inverse of h_*. Thus h_* is a $\mathcal{FFB}$ isomorphism.

VI. CONCLUSION

In this paper, it is investigated that the group properties are satisfied by the set of all $\mathcal{FFB}$ path homotopy equivalence classes on the collection of $\mathcal{FFB}$ loops. Further, the results in this treatise motivate to study the applications of $\mathcal{FFB}$ fundamental groups.

REFERENCES

1. G. Balasubramanian, " Fuzzy β-open sets and fuzzy β-seperation axioms", KYBERNETIKA, vol. 35, 1999, pp. 215-223.
2. C. L. Chang, "Fuzzy topological Spaces", J. Math. Anal. Appl., vol. 24, 1968, pp.182-190.
3. G. Culvacioglu and M. Citil, "On fuzzy homotopy theory, Adv. Stud. Contemp. Math.(Kyungshang), vol. 12, 2006, pp. 163-166.
4. I. M. James, Fibrewise topology, Cambridge university press, 1989.
5. James R. Munkres, Topology, Second edition,Prentice hall of India, 2004.
6. V. Madhuri and B. Amudhambigai, "The fuzzy F〗^*-fundamental group of fuzzy F^*-structure spaces", Int. J. Comput. Appl. Math., vol. 12, 2017,pp. 40-53.
7. A. R. Salleh, A. O. Md. Tap, "The fundamental Group of fuzzy topological spaces", Sains Malaysiana, vol. 5, 1986, pp. 397-407.
8. A. R. Salleh, A. O. Md. Tap, "The fundamental groupoid of fuzzy topological spaces", Sains Malaysiana, vol. 4, 1987, pp. 338-353.
9. T. Yogalakshmi, E. Roja and M. K. Uma, "Soft fuzzy Soft homotopy and its topological foldings of a soft Fuzzy soft manifold", Ann. Fuzzy Math. Inform.,vol.8,2014, pp. 125-139.
10. C. Y. Zheng, "Fuzzy path and fuzzy connectedness", Fuzzy sets and systems, vol. 14, 1984, pp. 273-280.
11. L.A.Zadeh, "Fuzzy sets", Information and control,vol. 8, 1965, pp. 338-353.

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Retrieval Number: F1156038620/2020©BEIESP
DOI:10.35940/ijrte.F1156.038620
Journal Website: www.ijrte.org

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