$\mathcal{C}$-Ideal Via $\mathcal{C}$-Topological Space

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Abstract: ***The purpose of this paper is to present a new family called $\mathcal{C}$-ideal by using the center set and study the most important properties of this family, in addition, to define new points via $\mathcal{C}$-ideal called, $\mathcal{C}$-turning and $\mathcal{C}$-bench points with respect to $\mathcal{C}$-topological space.***

Keywords ***: Center set, $\mathcal{C}$-topology, $\mathcal{C}$-ideal, $\mathcal{C}$-turning point and $\mathcal{C}$-bench point***

I. INTRODUCTION

The concept of proximity space is one of the important concepts introduced by mathematician Efremovice in 1951 [1], which was considered an axiom of geometry and suitable instruments for the investigation of topology. More comprehensive work on the proximity space was carried out by Naimpally and Warrack [2]. Many authors have worked on the concept of proximity space.

In [4],Focus on different approaches to distinctive patterns in non-empty sets that have to do with proximity theory and the discovery of new patterns in proximity, In [5],they discussed the spatial and descriptive relationships of proximity theory. In [6], an explicit description of the properties of separation axioms T_0 and T_1 was derived at the p point in the surface category of proximity space. In [7],an explicit description has been given of the separation properties for T_0, T_1, $Pre - T_2$ $(pre - Hausdorff)$ and T_2 $(Hausdorff)$ is given in the topological category of proximity spaces. In [8],Presented planar shape signatures derived from homology nerves, which are intersecting 1-cycles in a collection of homology groups endowed with a proximal relator (set of nearness relations) that includes a descriptive proximity. In [9],It's been deduced with showing that every connected asymptotic resemblance space induces a coarse proximity if and only if the connected asymptotic resemblance space is asymptotically normal. In the extreme importance shown above, the authors D. A. Abdulsada and L. A.A. Al-Swidi in 2019 [10] used proximity space to define a new set called center set. In [11], used center set to define a new topology called $\mathcal{C}$-topology. In this paper, we will explain the concept of $\mathcal{C}$-ideal by relying on the center set and explain the most important theories with studying $\mathcal{C}$-turning, point.

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II. PRELIMINARIES

In this section we will describe the basic concepts that we will use in this paper.

Definition 2.1.[2].
Let X be anonempty set and $\delta \subseteq P(X) \times P(X)$ a binary relation, then (X, δ) is proximity space iff for each $A', A'', A''' \subseteq X$,

1. $A'\delta A''$ iff $A''\delta A'$;
2. $A'\delta(A'' \cup A''')$ iff $A'\delta A''$ or $A'\delta A'''$;
3. $X\bar{\delta}\emptyset$;
4. $\{x\}\delta\{x\}$ for each $x \in X$;
5. $A'\bar{\delta}A''$ implies there is an $E \subseteq X$ such that $A'\bar{\delta}E$ and $(X - E)\bar{\delta}A''$;

where $A'\bar{\delta}A''$ means it is not true that $A'\delta A''$.

Proposition 2.2.[3] .
Let (X, δ) be a proximity space. Then
If $A'\delta A''$ and $A'' \subseteq A'''$, then $A'\delta A'''$;

1. If $A'\bar{\delta}A''$ and $A''' \subseteq A''$, then $A'\bar{\delta}A'''$;
2. If there exists a point $x \in X$ such that $A'\delta\{x\}$ and $\{x\}\delta A''$, then $A'\delta A''$;
3. If $A' \cap A'' \neq \emptyset$;, then $A'\delta A''$;
4. $A'\bar{\delta}\emptyset$ for every $A' \subseteq X$;
5. If $A'\delta A''$, then $A' \neq \emptyset$; and $A'' \neq \emptyset$.

Definition 2.3.[10].
Let (X, δ) be a proximity space and $A \subseteq X$. A center set is $\mathcal{C}_A = \{\langle A, A'\rangle : A\delta A'\}$.

Definition 2.4.[10].
For two center sets $\mathcal{C}_A$ and $\mathcal{C}_B$ over a proximity space (X, δ). Then

1. $\mathcal{C}_X = \{\langle X, A\rangle : \emptyset \neq A \subseteq X\}$ is called universe center set.
2. $\mathcal{C}_\emptyset = \emptyset$ is called Null center set.
3. $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ iff $A \subseteq B$, and $\langle A, C\rangle \in \mathcal{C}_A \Longrightarrow \langle B, C\rangle \in \mathcal{C}_B$.
4. $\mathcal{C}_A = \mathcal{C}_B$ iff $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$.
5. $\mathcal{C}_A \sqsubseteq_{\mathcal{C}} \mathcal{C}_B$ iff $\langle A, C\rangle \in \mathcal{C}_A \Longrightarrow \langle B, C\rangle \in \mathcal{C}_B$.

6. $\mathcal{C}_A =_{\mathcal{C}} \mathcal{C}_B$ iff $\mathcal{C}_A \subseteq_{\mathcal{C}} \mathcal{C}_B$ and $\mathcal{C}_B \subseteq_{\mathcal{C}} \mathcal{C}_A$.
7. $\mathrm{Cop}.\mathcal{C}_A = \{\langle X, B\rangle \in \mathbb{C}(X) : \langle A, B\rangle \notin \mathcal{C}_A\}$
8. $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B = \{\langle A \cup B, C\rangle : \langle A, C\rangle \in \mathcal{C}_A$ or $\langle B, C\rangle \in \mathcal{C}B$
9. $\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_B = \{\langle A \cup B, C\rangle : \langle A, C\rangle \in \mathcal{C}_A$ and $\langle B, C\rangle \in \mathcal{C}B$.
10. $\curlyvee_{\mathcal{C}} \mathcal{C}_{A_i} = \{(\langle \cup A_i, C\rangle : \langle A_i, C\rangle \in \mathcal{C}_{A_i}$ for some $i \in I$.
11. $\curlywedge_{\mathcal{C}} \mathcal{C}_{A_i} = \{(\langle \cup A_i, C\rangle : \langle A_i, C\rangle \in \mathcal{C}_{A_i}$ for each $i \in I\}$.

Definition 2.4.[10].
Let (X, δ) be a proximity space and $\{x\}, B \subseteq X$ such that $\{x\}\delta B$. Then $x_B = \{\langle \{x\}, B\rangle\}$ is called a center point in X.

Definition 2.4.[10].
Let x_B be a center point in (X, δ) and $\mathcal{C}_A$ center set in (X, δ). Then $x_B \epsilon \mathcal{C}_A$ iff $x \in A$ and $(A, B) \in \mathcal{C}_A$.
$\mathbb{P}_{\mathcal{C}}(X) = \{\mathcal{C}_A : \mathcal{C}_A$ center set of X$\}$

Proposition 2.5.[10].
Let $\{\mathcal{C}_{A_i} : i \in I\}$ be a family of center sets in (X, δ). Then

1. If $x_B \epsilon \mathcal{C}_{A_i}$ for each $i \in I$, then $x_B \epsilon \curlywedge_{\mathcal{C}} \mathcal{C}_{A_i}$.
2. If $\exists i \in I$ such that $x_B \epsilon \mathcal{C}_{A_i}$, then $x_B \epsilon \curlyvee_{\mathcal{C}} \mathcal{C}_{A_i}$.
3. $\mathcal{C}_{A_1} \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_2}$ iff for each $x_B \epsilon \mathcal{C}_{A_1} \Longrightarrow x_B \epsilon \mathcal{C}_{A_2}$.
4. $(\curlyvee_{\mathcal{C}})_{x_B \epsilon \mathcal{C}_A} x_B = \mathcal{C}_A$

Proposition 2.6.[10].
If $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ and $\{\mathcal{C}_{A_i} : i \in I\}$ be a center sets. Then

1. $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_C \Longrightarrow \mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_C$
2. $\mathcal{C}_{A_i} \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ for each $i \in I \Longrightarrow \curlyvee_{\mathcal{C}} \mathcal{C}_{A_i} \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$
3. $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A_i}$ for each $i \in I \Longrightarrow \mathcal{C}_B \preccurlyeq_{\mathcal{C}} \curlywedge_{\mathcal{C}} \mathcal{C}_{A_i}$
4. $\mathrm{Co}.(\curlyvee_{\mathcal{C}} \mathcal{C}_{A_i}) = \curlywedge_{\mathcal{C}} (\mathrm{Cop}.\mathcal{C}_{A_i})$
5. $\mathrm{Co}.(\curlywedge_{\mathcal{C}} \mathcal{C}_{A_i}) = \curlyvee_{\mathcal{C}} (\mathrm{Co}.\mathcal{C}_{A_i})$
6. $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B \Longrightarrow \mathrm{Cop}.\mathcal{C}_B \subseteq_{\mathcal{C}} \mathrm{Cop}.\mathcal{C}_A$
7. $(\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B) \curlyvee_{\mathcal{C}} \mathcal{C}_C = \mathcal{C}_A \curlyvee_{\mathcal{C}} (\mathcal{C}_B \curlyvee_{\mathcal{C}} \mathcal{C}_C)$,
$(\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_B) \curlywedge_{\mathcal{C}} \mathcal{C}_C = \mathcal{C}_A \curlywedge_{\mathcal{C}} (\mathcal{C}_B \curlywedge_{\mathcal{C}} \mathcal{C}_C)$
8. $(\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B) \curlywedge_{\mathcal{C}} \mathcal{C}_C = (\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_B) \curlyvee_{\mathcal{C}} (\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_C)$,
$(\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_B) \curlyvee_{\mathcal{C}} \mathcal{C}_C = (\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B) \curlywedge_{\mathcal{C}} (\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_C)$
9. $\mathrm{Cop}.(\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B) = \mathrm{Cop}.\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathrm{Cop}.\mathcal{C}_B$,
$\mathrm{Cop}.(\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_B) = \mathrm{Cop}.\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathrm{Cop}.\mathcal{C}_B$
10. $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_A = \mathcal{C}_A$, $\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_A = \mathcal{C}_A$, $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_\emptyset = \mathcal{C}_A$
11. $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathrm{Co}.\mathcal{C}_A = \mathcal{C}_X$, $\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathrm{Co}.\mathcal{C}_A = \mathcal{C}_\emptyset$.

Definition 2.7.[11].
Let(X, δ) be a proximity space and $\mathfrak{J}_{\mathcal{C}} \subseteq \mathbb{P}_{\mathcal{C}}(X)$, then $\mathfrak{J}_{\mathcal{C}}$ said to be a $\mathcal{C}$-topology if

1. $\mathcal{C}_\emptyset, \mathcal{C}_X \in \mathfrak{J}_{\mathcal{C}}$
2. $\{\mathcal{C}_{A_i} : i \in I\} \in \mathfrak{J}_{\mathcal{C}} \Longrightarrow \curlyvee_{\mathcal{C}} \{\mathcal{C}_{A_i} : i \in I\} \in \mathfrak{J}_{\mathcal{C}}$.
3. $\mathcal{C}_{A_1}, \mathcal{C}_{A_2} \in \mathfrak{J}_{\mathcal{C}} \Longrightarrow \mathcal{C}_{A_1} \curlywedge_{\mathcal{C}} \mathcal{C}_{A_2} \in \mathfrak{J}_{\mathcal{C}}$

The triplet $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ is called a $\mathcal{C}$-topological space and the members of $\mathfrak{J}_{\mathcal{C}}$ are said to be $\mathcal{C}$-open. $\mathfrak{J}_{\mathcal{C}}$ called indiscrete $\mathcal{C}$-topology if $\mathfrak{J}_{\mathcal{C}} = \{\mathcal{C}_X, \mathcal{C}_\emptyset\}$ and called discrete $\mathcal{C}$-topology if $\mathfrak{J}_{\mathcal{C}} \subseteq \mathbb{P}_{\mathcal{C}}(X)$.

Definition 2.8.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space. A center set $\mathcal{C}_B$ over (X, δ) is said to be $\mathcal{C}$-closed set, if there exists $\mathcal{C}$-open $\mathcal{C}_A$ such that $cop.(\mathcal{C}_A) = \mathcal{C}_B$.

Since $cop.(\mathcal{C}_A) \subseteq \mathcal{C}_X$ for each $A \subseteq X$, then we denote $\mathcal{C}$-closed set by $\mathcal{C}_{XA}$

Proposition 2.9.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be $\mathcal{C}$-topological space and $\mathfrak{H}_{\mathcal{C}}$ be acollection of all $\mathcal{C}$-closed sets of $(X, \delta, \mathfrak{J}_{\mathcal{C}})$. Then

1. $\mathcal{C}_X, \mathcal{C}_\emptyset \in \mathfrak{H}_{\mathcal{C}}$
2. $\{\mathcal{C}_{XA_i} : i \in I\} \in \mathfrak{H}_{\mathcal{C}} \Longrightarrow \curlywedge_{\mathcal{C}} \{\mathcal{C}_{XA_i} : i \in I\} \in \mathfrak{H}_{\mathcal{C}}$.
3. $\mathcal{C}_{XA_1}, \mathcal{C}_{XA_2} \in \mathfrak{H}_{\mathcal{C}} \Longrightarrow \mathcal{C}_{XA_1} \curlyvee_{\mathcal{C}} \mathcal{C}_{XA_2} \in \mathfrak{H}_{\mathcal{C}}$.

Proposition 2.10.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space and $\mathfrak{H}_{\mathcal{C}}$ be acollection of all $\mathcal{C}$-closed center sets of $(X, \delta, \mathfrak{J}_{\mathcal{C}})$. Then the following hold,

1. If $\mathfrak{J}_{\mathcal{C}}$ $\mathcal{C}$-topology, then $\mathfrak{J}_{\mathcal{C}} = \{\mathcal{C}_A : cop.(\mathcal{C}_A) \in \mathfrak{H}\mathcal{C}$.
2. If $\mathfrak{J}_{\mathcal{C}}$ indiscrete $\mathcal{C}$-topology, then $\mathfrak{J}_{\mathcal{C}} = \{cop.\mathcal{C}_{XA} : \mathcal{C}_{XA} \in \mathfrak{H}_{\mathcal{C}}\}$.
3. If $\mathfrak{J}_{\mathcal{C}}$ non-indiscrete $\mathcal{C}$-topology on $(X, \delta) \Longrightarrow \mathcal{C} - \mathfrak{J}_{\mathcal{C}} = \{\mathcal{C}_{XA'} = cop.\mathcal{C}_{XA} : \mathcal{C}_{XA} \in \mathfrak{H}_{\mathcal{C}}\}$ is $\mathcal{C}$-topology it is not necessary $\mathfrak{J}_{\mathcal{C}} = \mathcal{C} - \mathfrak{J}_{\mathcal{C}}$, But for each $\mathcal{C}_{XA'} \in \mathcal{C} - \mathfrak{J}_{\mathcal{C}}$ there is $\mathcal{C}_A \in \mathfrak{J}_{\mathcal{C}}$ so that $\mathcal{C}_{XA'} =_{\mathcal{C}} \mathcal{C}_A$ and for each $\mathcal{C}_A \in \mathfrak{J}_{\mathcal{C}}$ there is $\mathcal{C}_{XA'} \in \mathcal{C} - \mathfrak{J}_{\mathcal{C}}$ so that $\mathcal{C}_A =_{\mathcal{C}} \mathcal{C}_{XA'}$.

Example 2.11.[11].
Let (X, δ) be a proximity space such that $X = \{a, b, c\}$ and for each $, B \subseteq X$ $(A\delta B \Leftrightarrow A \cap B \neq \emptyset)$. Then $\mathfrak{J}_{\mathcal{C}} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{a\}}, \mathcal{C}_{X \curlywedge \{a\}}, \mathcal{C}_X\}$ is a $\mathcal{C}$-topology on (X, δ). Hence $\mathfrak{H}_{\mathcal{C}} = \{\mathcal{C}_X, \mathcal{C}_{X\{a\}}, \mathcal{C}_{X(X \curlywedge \{a\})}, \mathcal{C}_\emptyset\}$ is the collection of all $\mathcal{C}$-closed sets. Not that $\mathcal{C}_{X\{a\}}$ is a $\mathcal{C}$-closed and
$\mathcal{C}_{\{a\}} \neq cop.\mathcal{C}_{X\{a\}} = cop.\{\langle X, \{b\}\rangle, \langle X, \{c\}\rangle, \langle X, \{b, c\}\rangle\} = \{\langle X, \{a\}\rangle, \langle X, \{a, b\}\rangle, \langle X, \{a, c\}\rangle, \langle X, X\rangle\} =_{\mathcal{C}} \mathcal{C}_{\{a\}}$.

Definition 2.12.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space and $\mathcal{C}_B$ be a center set. Then the $\mathcal{C}$-closure of $\mathcal{C}_B$, denoted by $cl_{\mathcal{C}}(\mathcal{C}_B)$ is,

$$cl_{\mathcal{C}}(\mathcal{C}_B) = \curlywedge_{\mathcal{C}} \{\mathcal{C}_{XA} : \mathcal{C}_{XA}\ \mathcal{C} - \text{closed and } \mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_{XA}\}$$

Definition 2.13.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space , $\mathcal{C}_A$ be a center set and x_B be a center point. Then $\mathcal{C}_A$ is said to be a $\mathcal{C}$-neighborhood of x_B, if there exists a $\mathcal{C}$-open set $\mathcal{C}_{A'}$ such that $x_B \in_{\mathcal{C}} \mathcal{C}_{A'} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$.

Proposition 2.14.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space . Then

1. each center point x_B has a $\mathcal{C}$-neighborhood.
2. if $\mathcal{C}_A$ and $\mathcal{C}_{A'}$ are $\mathcal{C}$-neighborhoods of some x_B, then $\mathcal{C}_A \curlywedge_{\mathcal{C}} \mathcal{C}_{A'}$ is also a $\mathcal{C}$-neighborhood of x_B.
3. if $\mathcal{C}_A$ is a $\mathcal{C}$-neighborhood of x_B and $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_{A'}$, then $\mathcal{C}_{A'}$ is also a $\mathcal{C}$-neighborhood of x_B.

Definition 2.15.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space, $\mathcal{C}_A$ be a center set and x_B be a center point. Then x_B is said to be an $\mathcal{C}$-interior point of $\mathcal{C}_A$, if there exists a $\mathcal{C}$-open set $\mathcal{C}_{A'}$ such that $x_B \in_{\mathcal{C}} \mathcal{C}_{A'} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$.

Definition 2.16.[11].
Let $(X, \delta, \mathfrak{J}_{\mathcal{C}})$ be a $\mathcal{C}$-topological space and $\mathcal{C}_A$ be a center set. Then the $\mathcal{C}$-interior of $\mathcal{C}_A$, denoted by $\mathrm{int}_{\mathcal{C}}(\mathcal{C}_A)$ is,

$$\mathrm{int}_{\mathcal{C}}(\mathcal{C}_A) = \curlyvee_{\mathcal{C}} \{\mathcal{C}_{A'} : \mathcal{C}_{A'}\ \mathcal{C}-\text{open and } \mathcal{C}_{A'} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A\}$$

III. $\mathcal{C}$-IDEAL SPACE

In this section, we shall recall some basic concepts about "ideal" in topological space.

Definition 3.1.
Let (X,δ) be a proximity space. A family $\mathcal{J}_{\mathcal{C}}$ of subsets center sets is an "$\mathcal{C}$-ideal" if

1. $\mathcal{C}_A, \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$ implies $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$
2. $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$implies $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$.
3. $\mathcal{C}_X \notin \mathcal{J}_{\mathcal{C}}$.

Theorem 3.2.
Let $\{(\mathcal{J}_{\mathcal{C}})_\alpha : \alpha \in \Delta\}$ be any $\mathcal{C}$-ideals family on proximity space (X,δ). Then $\mathcal{J}_{\mathcal{C}} = \cap \{(\mathcal{J}_{\mathcal{C}})_\alpha : \alpha \in \Delta\}$ is also $\mathcal{C}$-ideal on (X,δ).
Proof.

1. Let $\mathcal{C}_A$ and $\mathcal{C}_A$ in $\mathcal{J}_{\mathcal{C}}$. Then Let $\mathcal{C}_A$ and $\mathcal{C}_A$ in $(\mathcal{J}_{\mathcal{C}})_\alpha$, for each $\alpha \in \Delta$. Since $(\mathcal{J}_{\mathcal{C}})_\alpha$ is an $\mathcal{C}$-ideal on (X,δ)., then $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in (\mathcal{J}_{\mathcal{C}})_\alpha$, for each $\alpha \in \Delta$. So $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$.
2. Let $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$. Then $\mathcal{C}_A \in (\mathcal{J}_{\mathcal{C}})_\alpha$ for each $\alpha \in \Delta$. Since$(\mathcal{J}_{\mathcal{C}})_\alpha$ is an ideal on (X,δ), $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$, then $\mathcal{C}_B \in (\mathcal{J}_{\mathcal{C}})_\alpha$ for each $\alpha \in \Delta$. So $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$.

Remark 3.3.

1. The union of two $\mathcal{C}$-ideals on (X,δ) is not necessary $\mathcal{C}$-ideal, for example: Let(X,δ) be a proximity space where $X = \{x,y\}$ and $(\forall A, B \subseteq X, A\delta B \Leftrightarrow A \cap B \neq \emptyset)$, then $\mathcal{J}_{\mathcal{C}1} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{x\}}\}$ and $\mathcal{J}_{\mathcal{C}2} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{y\}}\}$ be $\mathcal{C}$-ideals $\mathcal{J}_{\mathcal{C}1} \cup \mathcal{J}_{\mathcal{C}2} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{x\}}, \mathcal{C}_{\{y\}}\}$ is not an $\mathcal{C}$-ideal.
2. The intersection of all $\mathcal{C}$-ideals on (X,δ) is the $\mathcal{C}$-ideal $\{\mathcal{C}_\emptyset\}$.

Example 3.4.
Let (X,δ) be a proximity space. Then $\mathcal{J}_{\mathcal{C}} = \{\mathcal{C}_A : \mathcal{C}_A\ is\ finite\ center\ of\ X\}$ is an $\mathcal{C}$-ideal on (X,δ) called finite $\mathcal{C}$-ideal.

Definition 3.5.
Let (X,δ) be a proximity space. A family $\mathcal{J}_{\mathcal{C}0}$ of subsets center sets is an "$\mathcal{C}$-ideal base" if,

1. $\mathcal{C}_X \notin \mathcal{J}_{\mathcal{C}0}$.
2. If $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}0}$ and $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}0}$, then there exists $\mathcal{C}_C \in \mathcal{J}_{\mathcal{C}0}$ such that $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_C$.

Observe that if $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}0}$, for each $\mathcal{C}_A$ and $\mathcal{C}_B$ in $\mathcal{J}_{\mathcal{C}0}$, then $\mathcal{J}_{\mathcal{C}0}$ is an $\mathcal{C}$-ideal base on X and so any $\mathcal{C}$-ideal on X is $\mathcal{C}$-ideal base.

Example 3.6.
Let (X,δ) be a proximity space where $X = \{x,y\}$ and $(\forall A, B \subseteq X, A\delta B \Leftrightarrow A \cap B \neq \emptyset)$, then $\mathcal{J}_{\mathcal{C}0} = \{\mathcal{C}_{\{x\}}, \mathcal{C}_{\{y\}}, \mathcal{C}_{\{x\}} \curlyvee_{\mathcal{C}} \mathcal{C}_{\{y\}}\}$ is an $\mathcal{C}$-ideal base.

Example 3.7.
Let $\mathcal{J}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal on (X,δ) and $\mathcal{C}_A$ center set, such that $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$. Then, $\mathcal{J}_{\mathcal{C}0} = \{\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B : \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}\}$ is an $\mathcal{C}$-ideal base on X.
Solution.
Let $\mathcal{C}_{K1}$ and $\mathcal{C}_{K2}$ in $\mathcal{J}_{\mathcal{C}0}$, then there exists $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ in $\mathcal{J}_{\mathcal{C}}$ such that $\mathcal{C}_{K1} = \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_{B1}$ and $\mathcal{C}_{K2} = \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_{B2}$. Then $\mathcal{C}_{K1} \curlyvee_{\mathcal{C}} \mathcal{C}_{K2} = \mathcal{C}_A \curlyvee_{\mathcal{C}} (\mathcal{C}_{B1} \curlyvee_{\mathcal{C}} \mathcal{C}_{B2}) \in \mathcal{J}_{\mathcal{C}0}$ because $\mathcal{C}_{B1} \curlyvee_{\mathcal{C}} \mathcal{C}_{B2} \in \mathcal{J}_{\mathcal{C}}$. So that $\mathcal{J}_{\mathcal{C}0}$ is an $\mathcal{C}$-ideal base.

Proposition 3.8.
Let $\mathcal{J}_{\mathcal{C}0}$ be an $\mathcal{C}$-ideal base on (X,δ), then $\mathcal{J}_{\mathcal{C}} = \{\mathcal{C}_A : \mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B\ for\ some\ \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}0}\}$ is an $\mathcal{C}$-ideal on (X,δ) generated by $\mathcal{J}_{\mathcal{C}0}$.
Proof.
Let $\mathcal{J}_{\mathcal{C}0}$ be an $\mathcal{C}$-ideal base on (X,δ).

1. Let $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$. Then there exists $\mathcal{C}_C \in \mathcal{J}_{\mathcal{C}0}$ lo such that $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_C$, so $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_C$. Thus $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$.
2. Let $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ and $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$. Then there exists $\mathcal{C}_C \in \mathcal{J}_{\mathcal{C}0}$, and $\mathcal{C}_D \in \mathcal{J}_{\mathcal{C}0}$ such that $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_C$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_D$.

Since $\mathcal{J}_{\mathcal{C}0}$ is an $\mathcal{C}$-ideal base, then there exists $\mathcal{C}_E \in \mathcal{J}_{\mathcal{C}0}$, such that $\mathcal{C}_C \curlyvee_{\mathcal{C}} \mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_E$. So $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_E$. Therefore $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$. From 1 and 2, $\mathcal{J}_{\mathcal{C}}$ is an $\mathcal{C}$-ideal on (X,δ).

Proposition 3.9.
Let (X,δ) be a proximity space and $Y \subseteq X$. If $\mathcal{J}_{\mathcal{C}0}$, is an $\mathcal{C}$-ideal base on (Y,δ), then it's $\mathcal{C}$-ideal base on (X,δ).
Proof.
Direct by using definition of ideal base.

Corollary 3.10.
Let (X,δ) be a proximity space and $Y \subseteq X$. If $\mathcal{J}_{\mathcal{C}}$, is an $\mathcal{C}$-ideal on (Y,δ), then it's $\mathcal{C}$-ideal on (X,δ).

Definition 3.11.
Let $\mathcal{J}_{\mathcal{C}}$ and $\mathscr{J}_{\mathcal{C}}$ be two $\mathcal{C}$-ideals on the proximity space (X,δ). Then $\mathcal{J}_{\mathcal{C}}$ is said to be "finer than" $\mathscr{J}_{\mathcal{C}}$ if and only if $\mathscr{J}_{\mathcal{C}} \preccurlyeq_{\mathcal{C}} \mathcal{J}_{\mathcal{C}}$.

Example 3.12.
Let (X,δ) be a proximity space where $X = \{1,2\}$ and $(\forall A, B \subseteq X, A\delta B \Leftrightarrow A \cap B \neq \emptyset)$, . Then $\mathcal{J}_{\mathcal{C}1} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{1\}}\}$ and $\mathcal{J}_{\mathcal{C}1} = \{\mathcal{C}_\emptyset, \mathcal{C}_{\{2\}}, \mathcal{C}_{\{1\}} \curlyvee_{\mathcal{C}} \mathcal{C}_{\{2\}}\}$ are $\mathcal{C}$-ideals on (X,δ) and $\mathcal{J}_{\mathcal{C}2}$ finer than $\mathcal{J}_{\mathcal{C}1}$.

Remark 3.13.
Let (X,δ) be a proximity space. Any $\mathcal{C}$-ideal on (X,δ) is finer than $\{\mathcal{C}_\emptyset\}$.

Theorem 3.14.
Let (X,δ) be a proximity space and let $\mathcal{J}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal on (X,δ) such that $\curlyvee_{\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}} \mathcal{C}_B = \mathcal{C}_X$. Then $\mathcal{J}_{\mathcal{C}}$ is finer than the finite $\mathcal{C}$-ideal on (X,δ).
Proof.

Let $\mathcal{J}_{\mathcal{C}}$) be the finite $\mathcal{C}$-ideal on (X,δ). To show that $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$. Suppose if possible $\mathcal{J}_{\mathcal{C}} \nsubseteq \mathcal{I}_{\mathcal{C}}$.
Then there exists $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ such that $\mathcal{C}_A \notin \mathcal{I}_{\mathcal{C}}$. Then $\mathcal{C}_A$ is finite center set of (X,δ), Le $\mathcal{C}_A = \{\langle A, K_1\rangle, \langle A, K_2\rangle, \dots, \langle A, K_n\rangle\}$.
Now $\curlyvee_{\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}} \mathcal{C}_B = \mathcal{C}_X$. Then $\langle B_i, K_i\rangle \in \mathcal{C}_{B_i}$ for some $\mathcal{C}_{B_i} \in \mathcal{I}_{\mathcal{C}}, (i = 1,\dots,n)$. Since $\mathcal{I}_{\mathcal{C}}$ is an $\mathcal{C}$-ideal, $\mathcal{C}_D = \curlyvee_{i=1}^{n} \mathcal{C}_{K_i} \in \mathcal{I}_{\mathcal{C}}$.
Thus, $\langle \cup_{i=1}^{n} B_i, K_i\rangle \in \mathcal{C}_D$element of $\{i = 1,2,\dots,n\}$. Hence $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_D$. Since$\mathcal{I}_{\mathcal{C}}$ is an $\mathcal{C}$-ideal, implies $\mathcal{C}_A \in \mathcal{I}_{\mathcal{C}}$.
But this is a contradiction with $\mathcal{C}_A \notin \mathcal{I}_{\mathcal{C}}$. Hence $\mathcal{I}_{\mathcal{C}}$ must be finer than the finite $\mathcal{C}$-ideal.

Definition 3.15.
Let $\mathcal{I}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal on (X,δ). Then $\mathcal{I}_{\mathcal{C}}$ is said to be "maximal $\mathcal{C}$-ideal" on (X,δ) if and only if $\mathcal{I}_{\mathcal{C}}$ is not contained in any other $\mathcal{C}$-ideal on (X,δ). i.e. $\mathcal{I}_{\mathcal{C}}$ is a maximal $\mathcal{C}$-ideal on (X,δ) if and only if for every $\mathcal{C}$-ideal $\mathcal{J}_{\mathcal{C}}$ on (X,δ) such that $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$, then $\mathcal{I}_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}}$.

Theorem 3.16.
Let (X,δ) be a proximity space. Every $\mathcal{C}$-ideal on (X,δ) is contained in a maximal $\mathcal{C}$-ideal.
Proof.
Let $\mathcal{I}_{\mathcal{C}}$ be any $\mathcal{C}$-ideal on (X,δ) and let W be the class of all $\mathcal{C}$-ideals on (X,δ) containing $\mathcal{I}_{\mathcal{C}}$.
Then W is non-empty because $\mathcal{I}_{\mathcal{C}} \in W$. Also, W is partially orderd by the inclusion relation $\subseteq$.
Now let K ben linearly ordered subset of W. Then by definition of linear ordering for any two members $\mathcal{I}_{\mathcal{C}1}, \mathcal{I}_{\mathcal{C}2}$ of K, we have either $\mathcal{I}_{\mathcal{C}1} \subseteq \mathcal{I}_{\mathcal{C}2}$ or la $\mathcal{I}_{\mathcal{C}2} \subseteq \mathcal{I}_{\mathcal{C}1}$.
Let $S = \cup\{(\mathcal{I}_{\mathcal{C}})_\gamma : (\mathcal{I}_{\mathcal{C}})_\gamma \in K$. To show that S is an $\mathcal{C}$-ideal on (X,δ).

1. Since each $(\mathcal{I}_{\mathcal{C}})_\gamma$ is an $\mathcal{C}$-ideal, we have $\mathcal{C}_X \notin (\mathcal{I}_{\mathcal{C}})_\gamma$, for each an $\mathcal{C}$-ideal $(\mathcal{I}_{\mathcal{C}})_\gamma \in S$ and so $\mathcal{C}_X \notin S$.
2. Let $\mathcal{C}_A \in S$ and $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A$. Then $\mathcal{C}_A \in (\mathcal{I}_{\mathcal{C}})_\gamma$ for some $(\mathcal{I}_{\mathcal{C}})_\gamma \in S$. Since $(\mathcal{I}_{\mathcal{C}})_\gamma$, is an $\mathcal{C}$-ideal, then $\mathcal{C}_B \in (\mathcal{I}_{\mathcal{C}})_\gamma$. It follows that $\mathcal{C}_B \in S$.
3. Let $\mathcal{C}_A \in S$ and $\mathcal{C}_B \in S$. Then $\mathcal{C}_A \in (\mathcal{I}_{\mathcal{C}})_\gamma$, and $\mathcal{C}_B \in (\mathcal{I}_{\mathcal{C}})_\alpha$for some $(\mathcal{I}_{\mathcal{C}})_\gamma, (\mathcal{I}_{\mathcal{C}})_\alpha \in S$.
   Since S is linearly ordered, we have either $(\mathcal{I}_{\mathcal{C}})_\gamma \subseteq (\mathcal{I}_{\mathcal{C}})_\alpha$ or $(\mathcal{I}_{\mathcal{C}})_\alpha \subseteq (\mathcal{I}_{\mathcal{C}})_\gamma$. Hence both $\mathcal{C}_A$ and $\mathcal{C}_B$ belong either to $(\mathcal{I}_{\mathcal{C}})_\gamma$ or to $(\mathcal{I}_{\mathcal{C}})_\alpha$ and so $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ belongs either to $(\mathcal{I}_{\mathcal{C}})_\gamma$ or to $(\mathcal{I}_{\mathcal{C}})_\alpha$. It follows that $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \in S$.

Further S is finer than every member of K and so S is upper bound of K.
Thus, we have shown that W is a non-empty partially ordered set in which every linearly ordered subset has an upper bound. Hence by Zorn's lemma W contains a maximal element $\mathcal{J}_{\mathcal{C}}$. This maximal element) is by definition, maximal ideal on (X,δ) containing $\mathcal{I}_{\mathcal{C}}$.

Proposition 3.17.
Let (X,δ) be a proximity space. An $\mathcal{C}$-ideal $\mathcal{I}_{\mathcal{C}}$ on (X,δ) is a maximal $\mathcal{C}$-ideal if and only if for each center set $\mathcal{C}_A$ of X, then either $\mathcal{C}_A \in \mathcal{I}_{\mathcal{C}}$ or $cop.(\mathcal{C}_A) \in \mathcal{I}_{\mathcal{C}}$.
Proof.
If $cop.(\mathcal{C}_A) \notin \mathcal{I}_{\mathcal{C}}$, then $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$, because if there exists $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$ such that $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B = \mathcal{C}_X$, then $cop.(\mathcal{C}_A) \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ and so by definition $\mathcal{C}$-ideal we have $cop.(\mathcal{C}_A) \in \mathcal{I}_{\mathcal{C}}$ contradiction.
Let $\mathcal{J}_{\mathcal{C}}$, be an $\mathcal{C}$-ideal generated by $\mathcal{C}$-ideal base $\{\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B : \mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}\}$ then

$$\mathcal{J}_{\mathcal{C}} = \{\mathcal{C}_D : \mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B, for\ some\ \mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}\}$$

Since $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$, then$\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ ... (1).
To show that $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$
Let $\mathcal{C}_K \in \mathcal{I}_{\mathcal{C}}$. Since $\mathcal{C}_K \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_K$, then $\mathcal{C}_K \in \mathcal{J}_{\mathcal{C}}$ and we have $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$. But $\mathcal{I}_{\mathcal{C}}$ is maximal $\mathcal{C}$-ideal, then $\mathcal{I}_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}}$. By (1) $\mathcal{C}_A \in \mathcal{I}_{\mathcal{C}}$.
Conversely. Let, $\mathcal{J}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal on (X,δ) and $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$. To prove that $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$. Suppose$\mathcal{J}_{\mathcal{C}} \nsubseteq \mathcal{I}_{\mathcal{C}}$, then there exists $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$ such that $\mathcal{C}_B \notin \mathcal{I}_{\mathcal{C}}$.
Then by hypothesis $cop.(\mathcal{C}_B) \in \mathcal{I}_{\mathcal{C}}$. But $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$, so $cop(\mathcal{C}_B) \in \mathcal{J}_{\mathcal{C}}$. Thus$\mathcal{C}_B \curlyvee_{\mathcal{C}} cop.(\mathcal{C}_B) = \mathcal{C}_X \in \mathcal{J}_{\mathcal{C}}$
But this contradiction because $\mathcal{C}_X \notin \mathcal{J}_{\mathcal{C}}$, so that $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$and we have $\mathcal{J}_{\mathcal{C}} = \mathcal{I}_{\mathcal{C}}$. Therefore, that $\mathcal{I}_{\mathcal{C}}$ is maximal $\mathcal{C}$-ideal.

Theorem 3.18.
Let (X,δ) be a proximity space. An $\mathcal{C}$-ideal $\mathcal{I}_{\mathcal{C}}$ on X is maximal $\mathcal{C}$-ideal if and only if $\mathcal{I}_{\mathcal{C}}$ contains all those center sets which $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$.
Proof.
Let $\mathcal{I}_{\mathcal{C}}$ be a maximal $\mathcal{C}$-ideal and let $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_X$ such that $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$. Let

$$\mathcal{J}_{\mathcal{C}} = \{\mathcal{C}_D : \mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B\ for\ each\ \mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}\}$$

Observe that $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$, because $\mathcal{C}_B \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$. To show that $\mathcal{J}_{\mathcal{C}}$ is $\mathcal{C}$-ideal on (X,δ).

1. To prove $\mathcal{C}_X \notin \mathcal{J}_{\mathcal{C}}$. Let $\mathcal{C}_D \in \mathcal{J}_{\mathcal{C}}$, then $\mathcal{C}_D \in \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$. But $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$. So $\mathcal{C}_D \neq \mathcal{C}_X$. Therefore, $\mathcal{C}_X \notin \mathcal{J}_{\mathcal{C}}$.
2. Let $\mathcal{C}_D \in \mathcal{J}_{\mathcal{C}}$ and $\mathcal{C}_K \preccurlyeq_{\mathcal{C}} \mathcal{C}_D$, then $\mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$. So that $\mathcal{C}_K \preccurlyeq_{\mathcal{C}} \mathcal{C}_D \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$. Hence $\mathcal{C}_K \in \mathcal{J}_{\mathcal{C}}$.
3. Let $\mathcal{C}_{D1}, \mathcal{C}_{D2} \in \mathcal{J}_{\mathcal{C}}$), then $\mathcal{C}_{D1} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$and $\mathcal{C}_{D2} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$, so $\mathcal{C}_{D1} \curlyvee_{\mathcal{C}} \mathcal{C}_{D2} \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$. Thus $\mathcal{C}_{D1} \curlyvee_{\mathcal{C}} \mathcal{C}_{D2} \in \mathcal{J}_{\mathcal{C}}$ for each. Therefore $\mathcal{J}_{\mathcal{C}}$ is an $\mathcal{C}$-ideal on (X,δ). Since $\mathcal{I}_{\mathcal{C}}$ is maximal $\mathcal{C}$-ideal, then $\mathcal{I}_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}}$. Since $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$ so that $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$ so also $\mathcal{C}_A \in \mathcal{I}_{\mathcal{C}}$.

Conversely. Let $\mathcal{I}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal satisfying the condition and let $\mathcal{J}_{\mathcal{C}}$ be an $\mathcal{C}$-ideal on (X,δ) such that $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$. To prove, $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$, let $\mathcal{C}_A \in \mathcal{J}_{\mathcal{C}}$, then $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{J}_{\mathcal{C}}$. Since $\mathcal{I}_{\mathcal{C}} \subseteq \mathcal{J}_{\mathcal{C}}$, then $\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B \neq \mathcal{C}_X$ for each $\mathcal{C}_B \in \mathcal{I}_{\mathcal{C}}$, so $\mathcal{C}_A \in \mathcal{I}_{\mathcal{C}}$ and so also $\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{I}_{\mathcal{C}}$. Thus $\mathcal{I}_{\mathcal{C}} = \mathcal{J}_{\mathcal{C}}$ and we have $\mathcal{I}_{\mathcal{C}}$ is a maximal $\mathcal{C}$-ideal on (X,δ).

IV. CONCLUSION

In the current work, we continue to study the properties of $\mathcal{C}$-ideal via $\mathcal{C}$-topology. We also offer $\mathcal{C}$-turning (bench) points and have established many interesting properties. We trust that the discoveries in this paper will enable analyst to upgrade and advance further investigation on the $\mathcal{C}$-ideal to complete a general system for their applications in down to practical life.

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