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Draining of Water Tank using Runge-Kutta Methods

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Abstract: ***The use of water tanks as a tool for storing water before being distributed for daily use has become a widely used system today. Among the attempts to develop a water distribution system is optimization in terms of system and operating costs. In this study, four methods of the Runge Kutta method are the Implicit such as Explicit Euler method, Implicit Euler method, Implicit Midpoint Rule, Runge Kutta Fourth-order method are used and compared with the exact solution method. The method will be compared in terms of accuracy and efficiency in solving differential equations based on set parameters for optimum design of water tank. The accuracy and efficiency of each method can be determined based on error graph. At the end of the study, numerical results obtained indicate that the Implicit Midpoint Rule provides greater stability and accuracy for the fixed stepsize given compared to other numerical methods.***

Keywords: ***Runge Kutta, Tank, Stability, Design, Numerical Method.***

I. INTRODUCTION

The water distribution system is a physical work where the water will be distributed from the water source that has been provided to the user. The system should be carefully designed to provide sufficient quantity and quality of water to meet the needs of consumers. In general, providing quality water with enough quantities has been an issue that is often emphasized in our daily use. Every day, demand for water supply is increasing in tandem with the increase in consumers. As the consumer population grows, challenges to meet consumer demands are also increasing. This has led to various efforts to develop the water supply system through sustainable water supply. However, the application of this system is very limited due to difficult use. Among the attempts to develop a water distribution system is optimization in terms of system development and required operating costs. Today, the water supply system consists of infrastructure that collects, treats, stores, and distributes

water between water sources and consumers. Due to this, complexity in the water supply system and water distribution system exists because it involves many components and connections in the construction of a new system.

The purpose of this study is to present the results of the analysis in optimizing the design of high tank and different tank diameters. The use of water in a building can be optimized if the design and operation consider the features of green building which involve the efficiency of water use. This is reflected in the DPN 18, "Enhancing efficient water management with emphasis on demand, alternative and non-conventional (rainwater harvesting, water recycling) and reduced wastage" (National Urbanization Policy, 2006). The most common problem facing today's consumers is that users often use uncontrolled water. Therefore, lack of awareness in this case will cause water supply problems to emerge. The scarcity of resources followed by high demand will cause the cost of gaining water supply increasingly on the day. The lack of electricity also causes difficulty in supplying water through pumps especially at peak times. In such a situation, water tank become much needed in life. Given that the demand for water tanks will continue to rise in the years to come, the prediction of the cost of the water tank for the design of the tank before it can help in the selection of tanks for the actual design. Providing optimum tank design in water drainage systems is able to avoid or control wastage of water resources.

In an attempt to develop a water distribution system, Lansey (Lansey & Larry 1989) conducted a study in which he formed a methodology to determine the optimum (low cost) water distribution system design. Through this study, he has maintained the size of the hydraulic simulation model, so the problem is only limited to the simulation model's ability rather than the optimization model. This methodology uses a generalized reduced gradient model to solve the problem of reduced in size and complexity implicitly by solving the preservation of mass and energy equations using a hydraulic simulator and Lagrangian approach. This methodology combines nonlinear programming techniques with existing water distribution simulation models in contrast to previous studies (Alperovits & Shamir 1977), (Bhave & Sonak 1992), and (Morgan & Goulter 1985) which do not take into account other design components other than pipe design.

Morgan (1985) conducted a study in which the study procedure was based on a linear programming developed for the cost reduction and water distribution network design.

The methodology has been demonstrated by applying the method for the new network design and expansion of the existing network.

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In 2015, a study was made on the optimization of water distribution networks using linear integer program (Samani & Mottaghi 2006). In this study, the optimum design of the municipal water distribution network is determined by the branch and is associated with linear programming integer techniques. Hydraulic and optimization analysis is linked through recurring procedures to develop models in designing a water distribution system that meets all the required constraints with minimum cost. These constraints include pipe size, reservoir height, pipe flow velocity and node pressure.

Based on Josué Njock's study in 2003 (Josue´ Njock Libii n.d.), he states that if the ratio between the cross-sectional area and the opening of the hole below the tank is greater, the drainage will be slower. In his study, he used different parameters of different openings. (Forbes & Hocking 1990) used an integral equation method to calculate the steady stream of water in a tank hole with a horizontal water surface and found that the maximum steady rate of extraction was characterized by the formation of round circles of free surface stagnation points. A few years later, (Forbes & Hocking 2010) has conducted a study on non-permanent drainage from a round tank where he used a two-dimensional three-fluid drainage system from a round tank. He found that for a larger radius drainage, the interface portion above the end of the drain would be withdrawn first. This solution is done using non-linear solution techniques.

In addition, (Chan, R. P. K., & Razali 2014) conducted a study on the effect of the smoothing effects on the Implicit Midpoint Rule (IMR) and the Implicit Trapezoid Rule (ITR) with the implications for extrapolating the numerical solutions of common differential equations. The study was conducted using continuous measure size for three problems, namely PR, LAMPS and HIRES. The comparison is done between the basic method, the one-step symmetrizer in active and passive mode. The results show that the Implicit Midpoint Rule and the Implicit Trapezoid Rule have high efficiency for both stiffness and absorption systems. However, the Implicit Trapezoid Rule is more accurately than any other method for a stiffness system although there is little oscillation in the case resolution.

In 2018, (Razali et al. 2018) has made improvements to the study (Chan, R. P. K., & Razali 2014) where he found it unrealistic to use continuous steps in almost all applications. Therefore, he has varied the size of the step in the problem that must be solved for active symmetrization. In this study, two-step symmetrizers has been implemented in the same method that is the Implicit Midpoint Rule (IMR) and the Implicit Trapezoid Rule (ITR). The results of the study for non-linear problems using variable size measures indicate that two-step symmetrizer has higher efficiency than one step's symmetrizer

II. METHODOLOGY

In this study, the main purpose is to identify which water tank has an optimistic design in the discharge of the water tank. Based on this study, the time taken to empty the water tank is a parameter to be searched and compared in terms of speed of time. Designs for the preparation of experiments to be carried out will be presented in this chapter. Fig. 1 shows the research process and Fig. 2 shows the model used in deriving the mathematical model.

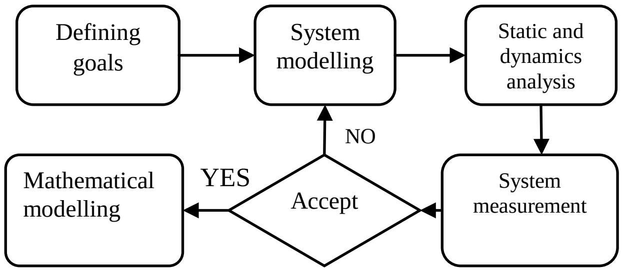


Fig. 1. Research process

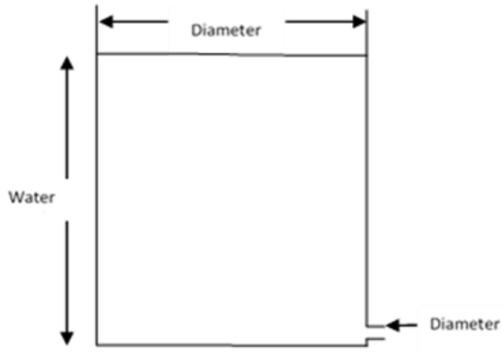


Fig. 2. Model

Runge-Kutta methods are used to solve the ordinary differential equation obtain from the model. Here is the Runge-Kutta method with initial value $y_0 = y(x_0)$ as;

$$y'(x_n) = f(x_n, y(x_n))$$

A. Explicit Euler

This is the simplest numerical method for initial value problems. This method is called implicitly because y_n can be explicitly calculated in terms of y_{n-1}; in other words, y_n is implicitly defined as the solution of some equations. There are various ways to build numerical methods for initial value problems; the most important are numerical integration and numerical differentiation (D. Akrivis Georgios 2012).

We consider differential equations at that point x_{n-1}.

$$y'(x_n) = f(x_n, y(x_n))$$

and the estimated $y'(x_n)$ uses the

$$\frac{1}{h}[y(x_{n+1}) - y(x_n)],$$

$$\frac{1}{h}[y(x_{n+1}) - y(x_n)] \approx f(x_n, y(x_n)).$$

Final equation is

$$y_{n+1} = y_n + hf(x_n, y_n).$$

B. Implicit Euler

For the Implicit Euler method, the unknown y_n value is implicitly defined as the solution of the equation, and therefore this method is called implicit.

This is a computational weakness for the Implicit Euler method compared to the Explicit Euler method. The Implicit Euler method can be obtained using a completely analogue way as in the explicit method. For example, by using numerical differentiation, we consider the differential equations at points x_n;

$$y'(x_n) = f(x_n, y(x_n)),$$

and approximation $y'(x_n)$

$$\frac{1}{h}[y(x_{n+1}) - y(x_n)],$$

$$\frac{1}{h}[y(x_{n+1}) - y(x_n)] \approx f(x_n, y(x_n)).$$

Final equation is

$$y_n = y_{n+1} + hf(x_n, y_n)$$

C. Implicit Midpoint Rule

To obtain Runge-Kutta's second order method, Taylor's expansion of the series can be used to derive the equation from the Euler method.

$$\begin{aligned} y_{n+1} &= y_n + \left.\frac{dy}{dx}\right|_{x_n,y_n}(x_{n+1}-x_n) \\ &+ \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_n,y_n}(x_{n+1}-x_n)^2 \\ &+ \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_n,y_n}(x_{n+1}-x_n)^3 + \cdots \\ &= y_n + f(x_n,y_n)(x_{n+1}-x_n) \\ &+ \frac{1}{2!}f'(x_n,y_n)(x_{n+1}-x_n)^2 \\ &+ \frac{1}{3!}f''(x_n,y_n)(x_{n+1}-x_n)^3 + \cdots \end{aligned}$$

As you can see from equation above,

$$y_{n+1} = y_n + f(x_n, y_n)h$$

By considering only the second order for Taylor's series of developments, the last equation we have is;

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, Y\right)$$

$$Y = y_n + \frac{h}{2}f\left(x_n + \frac{h}{2}, Y\right)$$

D. Runge Kutta Fourth Order

The Runge-Kutta method is an effective method for solving common differential equations. Widely used Runge-Kutta formula is the fourth order formula RK4 (Yang & Shen 2015).

For typical differential equations with initial values, X_0

$$dX_t = F(t, X_t)dt$$

The method uses the following formula:

$$X(t_{n+1}) = X(t_n) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where k_i is

$$k_1 = hF(t_n, X_n)$$

$$k_2 = hF\left(t_n + \frac{1}{2}h, X_n + \frac{1}{2}hk_1\right)$$

$$k_3 = hF\left(t_n + \frac{1}{2}h, X_n + \frac{1}{2}hk_2\right)$$

$$k_4 = hF(t_n + h, X_n + hk_3)$$

$t_n = nh$ and fixed stepsize h is used.

E. MATLAB

In this study, the following parameters will be used in MATLAB as an accurate solution for this study;

Table-I: Parameter of tank

	Description
Parameter	h_0 initial water level
Variable	D is tank diameter d is diameter of pipe
Diferential Equation	$t_e = \frac{D^2}{d^2}\sqrt{\frac{2h_0}{g}}$
Condition	h_0
Input	D and d
Output	t_e

F. Stability Region

Stability region is a standard tool in the analysis of numerical formulas for the initial problem of ODE value. A small stability region indicates that a very small stepsize should be used in ODE solutions to obtain a high stability region.

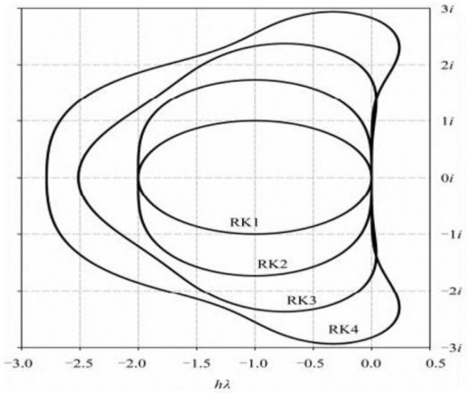


Fig. 3. Stability region for Runge Kutta Method

To investigate the stability of numerical methods for initial ODE value settlement, model problems should be considered as follows

$y'(x) = \lambda . y(x)$

$y(0) = 1$

Where it has the following analytical solutions

$y(x) = e^{\lambda x}, \lambda$

Where λ is a constant.

The analytic solution for the model is

$$y(x) = e^{\lambda x} = 1 + \lambda x + \frac{(\lambda x)^2}{2!} + \frac{(\lambda x)^3}{3!} + \cdots + \frac{(\lambda x)^n}{n!}$$

Replacing λx with λh, we will get a solution of the difference equation Euler method (RK1) with the first two terms, Heun method (RK2) with the first three terms, RK3 with the first four terms, and finally RK4 with the first five terms:

$$G = 1 + \lambda x + \frac{(\lambda x)^2}{2!} + \frac{(\lambda x)^3}{3!} + \cdots + \frac{(\lambda x)^n}{n!}$$

III. RESULT AND DISCUSSION

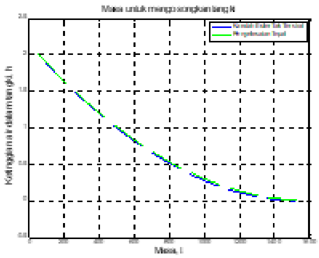
Some MATLAB program code was written to compare the accuracy and stability of each method used. This study is based on the design parameters of the water tank which is taken into account in the discharge of the water tank. In this study, 3 different sets of parameters were used to compare the time taken in the discharge of the water tank.

This set of parameters is composed of different diameter and water tank height where 1500-gallon water tank having the same pipe diameter has been used throughout the study of 25 mm.

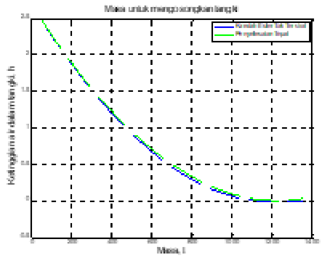
Table-II. Set Parameter for design of tank

Parameter	Diameter	Height
First	1230 mm	2140 mm
Second	1020 mm	2680 mm
Third	1530 mm	1250 mm

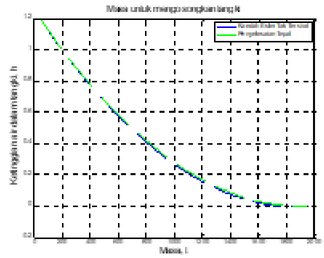
A. Comparison of the time required for the draining of the tank



Graph 1 Draining of water tank set 1



Graph 2 Draining of water tank 2



Graph 3 Draining of water tank set 3

The graph above shows a comparison of the Explicit Euler Methods and the exact solution to find time to empty the water tank. Detailed information for each time and height of the water level for each graph can be seen carefully in the table above. This method has a fixed stepsize of 50 seconds. Based on the first parameter, the time to empty the water tank is about 25 minutes, 1501 up to 1551 seconds. The difference range for the Implicit Euler Method and the exact solution for this set is between 0.0005-0.0242.

Meanwhile, the time to empty the water tank for the second set of parameters is about 19 to 21 minutes which is 1151 to 1251 seconds. The difference range for the set of second parameters is between 0.0002-0.0394. The time to empty the water tank for the third set of parameters is about 30 minutes to 1801 to 1851 seconds. The difference range for this set is between 0.0004-0.0179. Based on the results obtained, the second set of parameters has the ideal time to empty the water tank. One of the characteristics of an optimum water tank design is that the system is capable of supplying the amount of water needed with sufficient pressure. The fast time to empty the tank proves that the system has enough pressure to discharge the water out of the tank.

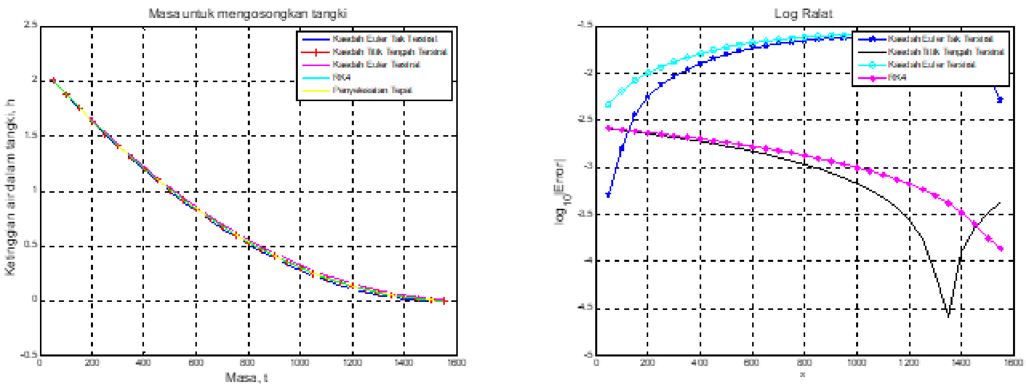
The mechanical energy distribution during the evacuation of these water tanks can be used to explain the differences that occur for these three sets of parameters. In the case of particles that fall freely without air resistance, gravity is the only force acting on it; because gravity is a conservative force, the amount of mechanical energy of the particles is kept at all times.

This is because it consists of only kinetic energy and potentially gravitational energy, the amount of mechanical energy is only distributed in the form of two such energy. The following is the ratio between potentially kinetic energy and instantaneous gravity for falling particles:

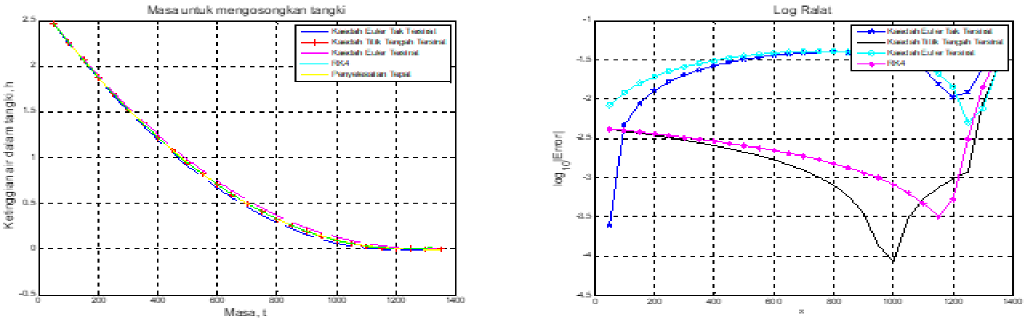
$$R_f = \frac{E_k}{E_f}$$

This ratio varies by time and as expected, it is increasing as potential energy continues to be converted into kinetic energy (Josué Njock Libii 2003).

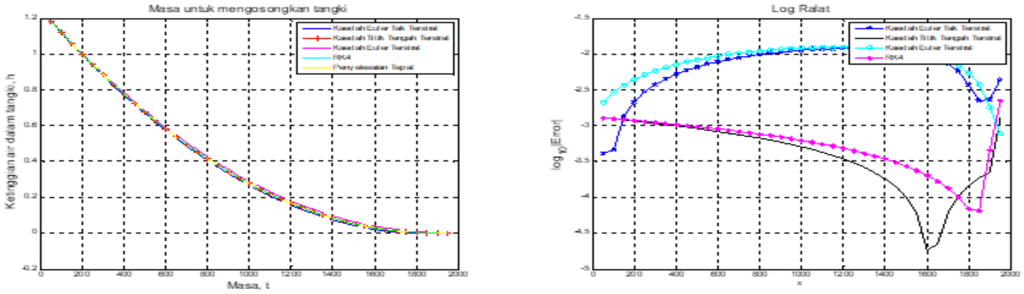
B. Comparison of accuracy of Runge Kutta Method



Graph 4 Accuracy of Runge Kutta Method for Set 1



Graph 5 Accuracy of Runge Kutta Method for Set 2



Graph 6 Accuracy of Runge Kutta Method for Set 3

The numerical results of the non-stiffness problems for the water tank system have been given in Graph 4-6. Graph 4 to Graph 6 is a comparison of the accuracy and efficiency of each numerical method used in this study. Graph 4 shows the precision of the numerical method for the first parameter set while Graph 5 for the second parameter set and Graph 6 for the third parameter set. The results for all sets show the same trends in precision for each method. Based on the diagram for the error log, the graph at the bottom shows the highest precision. Based on the graph, the Implicit Midpoint Rule shows the lowest plot graph compared to the other method. This proves that the Implicit Midpoint Rule has the smallest error.

Based on the results in the diagram above, the Explicit and Implicit Euler methods show different accuracy and efficiency where the Implicit Euler method has higher accuracy than the Explicit Euler method. For Explicit Euler method, if the stepsize in Explicit Euler is too large, the solution becomes unstable. For a robust ODE with a rapid shrinkage solution (Real (λ) << -1) or high oscillation mode (Im (λ) >> 1), Explicit Euler method requires a small step size (Lei & Hongzhou 2012).

However, when compared with other implicit numerical methods such as the Midpoint Method and Runge Kutta Fourth Order method, the Implicit Euler method has lower accuracy while the Implicit Midpoint Method shows high accuracy than the Runge Kutta Fourth Order method. The stability of the Runge Kutta method can usually be stabilized using different size steps according to the order of the Runge Kutta method. The Implicit Euler method requires a very small step size to achieve high accuracy while the 4th Runge Kutta method requires a large size measure to achieve high accuracy. In this study, the

Midpoint method succeeded in achieving high accuracy due to the appropriate size measure of $h = 50$.

C. Stability Region

Here are the results of the study for stability region for Runge Kutta Method.

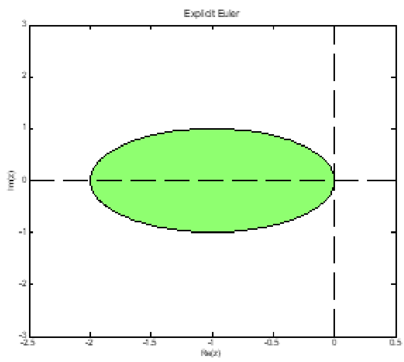


Figure 1.4 Stability Region for Explicit Euler

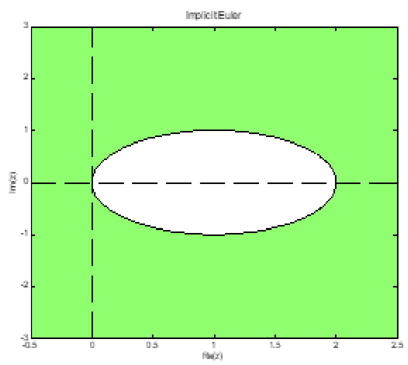


Figure 1.5 Stability Region for Implicit Euler

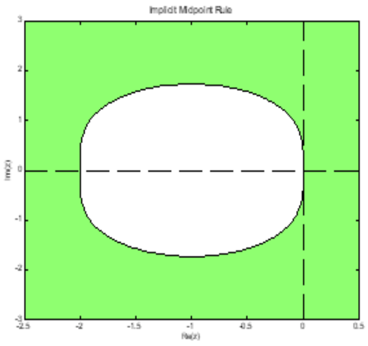


Figure 1.6 Stability Region for Implicit Midpoint

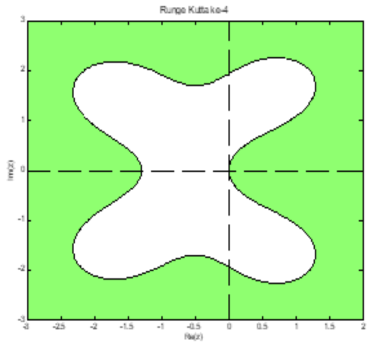


Figure 1.7 Stability Region for RK4

Note that in this case, the stability area of S in the method consists of all points $z = h\lambda$ in the complex plane. The green shaded region shows λh values in a complex plane where the method is completely stable (Rabiei et al. 2013). As illustrated in the above plot, the Implicit Euler method has a far greater stability area than Explicit Euler method. For the implicit method, the shaded green area extends across the complex plane and beyond the plot range. The implicit Runge Kutta method has a very wide area of stability and is not limited to a particular area while explicit Runge Kutta method has limited stability methods. The Implicit Euler method and the Runge Kutta Fourth Order method will only be stable if the measure size, h corresponds to the limit for both of these methods.

The above diagram shows the stability plot of the first order formula for Runge Kutta which the stability region for the Explicit Euler method is. The absolute stability measures obtained are approximately (-2, 0) located in the immediate area in Figure 1.4. Based on the figure, S stability region is

$$S = \{z \in \mathbb{C}: |1 + z| \leq 1\}$$

where the outside of the unit disk is open in a complex plane centered on the coordinates -1. In the case of the Explicit Euler method, if λ is located right in the left half of the plane, you can see that for a small h, we will have $Re\ (z) < 1$, since the adjustment of the value of λ using the smaller value of h will causing the $h\lambda$ point to move towards the point of origin.

Figure 1.5 shows that the Implicit Euler method has a higher stability region. The absolute stability steps obtained for this method are outside the area (0.2) where it has no boundary or boundary. Based on the figure, S stability region is

$$S = \{z \in \mathbb{C}: |z - 1| \geq 1\}$$

where the outside of the unit disk is open in a complex plane centered on coordinates 1.

The findings show that the implicit Runge Kutta method is more efficient in the search for stability region for a fixed point in the water tank model. Numerical methods such as the Euler and Runge Kutta methods maintain a local stability for a point if the size of the steps used is smaller while the Implicit Midpoint Rule always maintains the stability area for a fixed point.

IV. CONCLUSION

The objective of this study is to compare the time required to empty the water tank in the optimization of water tank parameters. Among the optimum tank design features, it is supposed to be able to supply water to all the points that are subjected to adequate pressure and are capable of supplying the amount of water required when firing. This is closely related to the studies that have been done because the water should be dispersed quickly with sufficient pressure to meet the optimum water tank characteristics.

The comparison of these three sets of parameters can help to determine the simulation of the actual condition of the water tank drainage to obtain optimum design. The Runge Kutta method is one of the numerical methods commonly used to solve the initial value problem of differential equations. Simulations for different water tank model studies using numerical computing software indicate that higher stability region can be achieved despite using different values with fixed size measures.

Some suggestions are suggested in order to develop the findings of this study in the future and can be used as a reference in understanding the issues and issues that occur. Among the recommendations are to diversify parameters in water tank modeling to get the most appropriate optimum parameter in the water tank drainage system and also use variable stepsizes according to the order of Runge Kutta.

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