



# Intuitionistic Fuzzy Soft Cubic Sets

V. Amarendra Babu, V. Siva Naga Malleswari

**Abstract**—We propose the definition of Intuitionistic fuzzy soft cubic set (IFSCS) and define few operations on IFSCS and deliberated its properties.

**Keywords:** Intuitionistic fuzzy soft set (IFSS), Interval-valued Intuitionistic fuzzy soft set (IVIFSS).

## I. INTRODUCTION

In 1999, the conception of a soft set was designed by Molodstov[2]. Zadeh[6] deals with the conceit of a fuzzy set in 1965 and also introduces the theory of interval valued fuzzy sets[7].K.Attansov[1][3][4] suggested the conceit of intuitionistic fuzzy set and also introduces the conceit of interval valued intuitionistic fuzzy sets. The perception of a fuzzy soft set was initiated by Maji et al [8] and he enlarged the idea of soft sets to the intuitionistic fuzzy system [9]. The definition of the cubic set was initiated by Jun et al[5].

Here we propose the definition of IFSCS with the consolidation of interval-valued intuitionistic fuzzy set and intuitionistic fuzzy set. We define some operations on IFSCS and deliberated its properties.

## II. PRELIMINARIES

### Definition 2.1[3]:

A mapping given by  $S: X \rightarrow P(\chi)$ , then a pair  $(S, X)$  is called a soft set over  $\chi$ . Then a soft set on  $\chi$  is characterized by a class of subsets of  $\chi$ .

### Definition 2.2:

A soft fuzzy set is defined by  $S = \{(\tau_\theta, \mu(\tau_\theta)) : \tau_\theta \in \chi, \theta \in P\}$  on  $\chi$ . Here the mapping  $\mu: \chi \rightarrow [0,1]P$  represents the association and the level of association of  $\tau_\theta$  in  $S$  is given by  $\mu(\tau_\theta)$ .

**Definition 2.3:** A set  $E = \{ \langle \tau, I(\tau), \mu(\tau) \rangle : \tau \in \chi \}$  is said to be a cubic set, where  $I$  is an IVFS and  $\mu$  indicates fuzzy set. The cubic set is symbolized by  $(I, \mu)$ .

**Definition 2.4:** An IFS  $\mathcal{J}$  is characterised by  $\mathcal{J} = \{ \langle T(\tau), F(\tau) \rangle : \tau \in \chi \}$ . Here  $T: \chi \rightarrow [0,1]$  represents the level of association and  $F: \chi \rightarrow [0,1]$  represents the level of non-association of  $\tau \in \chi$  to the set  $\mathcal{J}$ . Here  $\chi$  indicates the universal set.

**Definition 2.5:** Let  $P(\chi)$  symbolizes the class of all IFSSs

of  $\chi$  and  $F: \xi_1 \rightarrow P(\chi)$ . The set  $(F, \xi_1)$  is defined as an IFSS on  $\chi$ . Here  $\chi$  indicates the universal set,  $\xi$  indicates the class of parameters and  $\xi_1 \subset \xi$ .  $P(\chi)$  indicates the class of all the IFSSs of  $\chi$ .

**Definition 2.6:** An IVIFS  $I$  in  $\chi$  is constituted by association function  $I_T$  and non association function  $I_F$ .  $I_T(\tau)$ ,

$I_F(\tau) \subseteq [0, 1]$  for  $\tau \in \chi$ , where  $\chi$  is a universal set.

## III. INTUITIONISTIC FUZZY SOFT CUBIC SET (IFSCS)

**Definition 3.1:** The pair  $(\widehat{\mathcal{M}}, \xi)$  is said to be an IFSCS on  $\chi$  and  $\widehat{\mathcal{M}}: \xi \rightarrow IC(\chi)$ .

Here  $(\widehat{\mathcal{M}}, \xi) = \{ \widehat{\mathcal{M}}(\theta_i) = \{ \langle \rho, \widehat{\mathcal{M}}_{\theta_i}(\rho), \mu_{\theta_i}(\rho) \rangle : \rho \in \chi, \theta_i \in \xi \}$ , where

$\widehat{\mathcal{M}}_{\theta_i}(\rho) = \{ \langle \rho, [\widehat{\mathcal{M}}_{\theta_i}^{-T}(\rho), \widehat{\mathcal{M}}_{\theta_i}^{+T}(\rho)], [\widehat{\mathcal{M}}_{\theta_i}^{-F}(\rho), \widehat{\mathcal{M}}_{\theta_i}^{+F}(\rho)] \rangle / \rho \in \chi \}$  is an IVIFSS and  $\mu_{\theta_i}(\rho) = \{ \langle \rho, \mu_{\theta_i}^T(\rho), \mu_{\theta_i}^F(\rho) \rangle / \rho \in \chi \}$  is an IFSS and  $IC(\chi)$  indicates the collection of all IFSCSs,  $\xi$  indicates the set of parameters.

**Definition 3.2:** An IFSCS  $(\widehat{\mathcal{M}}, \xi)$  in  $\chi$  is said to be

- 1) Truth internal (shortly, T-internal) if  $(\forall \rho \in \chi, \theta_i \in \xi), (\widehat{\mathcal{M}}_{\theta_i}^{-T}(\rho) \leq \mu_{\theta_i}^T(\rho) \leq \widehat{\mathcal{M}}_{\theta_i}^{+T}(\rho))$
- 2) Falsity internal (shortly, F-internal) if  $(\forall \rho \in \chi, \theta_i \in \xi), (\widehat{\mathcal{M}}_{\theta_i}^{-F}(\rho) \leq \mu_{\theta_i}^F(\rho) \leq \widehat{\mathcal{M}}_{\theta_i}^{+F}(\rho))$

If IFSCS in  $\chi$  satisfies (1) and (2) then  $(\widehat{\mathcal{M}}, \xi)$  is an internal IFSCS in  $\chi$ .

**Definition 3.3:** An IFSCS  $(\widehat{\mathcal{M}}, \xi)$  in  $\chi$  is said to be

- 1) Truth external (shortly, T-external) if  $(\forall \rho \in \chi, \theta_i \in \xi), (\mu_{\theta_i}^T(\rho) \notin (\widehat{\mathcal{M}}_{\theta_i}^{-T}(\rho), \widehat{\mathcal{M}}_{\theta_i}^{+T}(\rho))$
- 2) Falsity external (shortly, F-external) if  $(\forall \rho \in \chi, \theta_i \in \xi), (\mu_{\theta_i}^F(\rho) \notin (\widehat{\mathcal{M}}_{\theta_i}^{-F}(\rho), \widehat{\mathcal{M}}_{\theta_i}^{+F}(\rho))$

If IFSCS in  $\chi$  satisfies (1) and (2) then  $(\widehat{\mathcal{M}}, \xi)$  is an external IFSCS in  $\chi$ .

**Definition 3.4:**

If  $(\widehat{\mathcal{M}}, \xi_1) = \{ \widehat{\mathcal{M}}(\theta_i) = \{ \langle \rho, \widehat{\mathcal{M}}_{\theta_i}(\rho), \lambda_{\theta_i}(\rho) \rangle : \rho \in \chi, \theta_i \in \xi_1 \}$ ,  $(\widehat{\mathcal{N}}, \xi_2) = \{ \widehat{\mathcal{N}}(\theta_i) = \{ \langle \rho, \widehat{\mathcal{N}}_{\theta_i}(\rho), \mu_{\theta_i}(\rho) \rangle : \rho \in \chi, \theta_i \in \xi_2 \}$  are two IFSCSs in  $\chi$  and  $\xi_1$  and  $\xi_2$  are any two subsets of  $\xi$ , then

$(\widehat{\mathcal{M}}, \xi_1) = (\widehat{\mathcal{N}}, \xi_2)$  if and only if

- i.  $\xi_1 = \xi_2$  and
- ii.  $\widehat{\mathcal{M}}(\theta_i) = \widehat{\mathcal{N}}(\theta_i) \Leftrightarrow \widehat{\mathcal{M}}_{\theta_i}(\rho) = \widehat{\mathcal{N}}_{\theta_i}(\rho)$  and  $\lambda_{\theta_i}(\rho) = \mu_{\theta_i}(\rho) \forall \rho \in \chi$  corresponding to each  $\theta_i \in \xi_1$ .

**Definition 3.5:** An IFSCS  $(\widehat{\mathcal{M}}, \xi_1)$  is called a subset of an IFSCS  $(\widehat{\mathcal{N}}, \xi_2)$  if  $\xi_1 \subseteq \xi_2$  and  $\widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta)$  if and only if  $\widehat{\mathcal{M}}_{\theta}(\rho) \subseteq \widehat{\mathcal{N}}_{\theta}(\rho)$  and  $\lambda_{\theta}(\rho) \subseteq \mu_{\theta}(\rho)$  for all  $\rho \in \chi$ . We denote it by  $(\widehat{\mathcal{M}}, \xi_1) \subseteq (\widehat{\mathcal{N}}, \xi_2)$

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$(\widehat{\mathcal{N}}, \xi_2)$ .

$$\widehat{\mathcal{M}}_\theta(\rho) \subseteq \widehat{\mathcal{N}}_\theta(\rho) \implies \widehat{\mathcal{M}}_\theta^{-T}(\rho) \leq \widehat{\mathcal{N}}_\theta^{-T}(\rho)$$

$$\widehat{\mathcal{M}}_\theta^{+T}(\rho) \leq \widehat{\mathcal{N}}_\theta^{+T}(\rho)$$

$$\widehat{\mathcal{M}}_\theta^{-F}(\rho) \geq \widehat{\mathcal{N}}_\theta^{-F}(\rho)$$

$$\widehat{\mathcal{M}}_\theta^{+F}(\rho) \geq \widehat{\mathcal{N}}_\theta^{+F}(\rho)$$

$$\lambda_\theta(\rho) \subseteq \mu_\theta(\rho) \implies \lambda_\theta^T(\rho) \leq \mu_\theta^T(\rho)$$

$$\lambda_\theta^F(\rho) \geq \mu_\theta^F(\rho)$$

**Definition 3.6:** An IFSCS  $(\widehat{\mathcal{M}}, \xi)^c$  is said to be the complement of an IFSCS  $(\widehat{\mathcal{M}}, \xi)$  if  $(\widehat{\mathcal{M}}, \xi)^c = (\widehat{\mathcal{M}}^c, \overline{\xi})$  where  $\widehat{\mathcal{M}}^c: \overline{\xi} \rightarrow \widehat{\mathcal{M}}(\chi)$ .

$$\begin{aligned} \text{i.e } (\widehat{\mathcal{M}}, \xi)^c &= \{ \langle \rho, \widehat{\mathcal{M}}_\theta^c(\rho), \lambda_\theta^c(\rho) \rangle : \rho \in \chi, \theta \in \xi \} \\ &= \{ \langle \rho, 1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{M}}_\theta^F(\rho), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) \rangle : \rho \in \chi, \theta \in \xi \} \end{aligned}$$

$$= \{ \langle \rho, 1 - \widehat{\mathcal{M}}_\theta^{+T}(\rho), 1 - \widehat{\mathcal{M}}_\theta^{-T}(\rho), 1 - \widehat{\mathcal{M}}_\theta^{-F}(\rho), 1 - \widehat{\mathcal{M}}_\theta^{+F}(\rho), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) \rangle : \rho \in \chi, \theta \in \xi \}$$

**Remark 3.7:**  $(\widehat{\mathcal{M}}, \xi)^c = \chi - \widehat{\mathcal{M}}(\theta)$  for all  $\theta \in \xi$ .

**Definition 3.8:** If

$$(\widehat{\mathcal{M}}, \xi_1) = \{ \langle \rho, \widehat{\mathcal{M}}_\theta(\rho), \lambda_\theta(\rho) \rangle : \rho \in \chi, \theta \in \xi_1 \},$$

$$(\widehat{\mathcal{N}}, \xi_2) = \{ \langle \rho, \widehat{\mathcal{N}}_\theta(\rho), \lambda_\theta(\rho) \rangle : \rho \in \chi, \theta \in \xi_2 \}$$

are two IFSCSs on  $(\chi, \xi)$  then  $(\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  and  $C = \xi_1 \cup \xi_2$ , for all  $\theta \in C$

$$H(e) = \begin{cases} \widehat{\mathcal{M}}(\theta) & \text{if } \theta \in \xi_1 - \xi_2 \\ \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_2 - \xi_1 \\ \widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

Where  $H(\theta) = \{ \rho, [\max(\inf \widehat{\mathcal{M}}_\theta(\rho), \inf \widehat{\mathcal{N}}_\theta(\rho)), \max(\sup \widehat{\mathcal{M}}_\theta(\rho), \sup \widehat{\mathcal{N}}_\theta(\rho))] \}$ ,

$[\min(\inf \widehat{\mathcal{M}}_\theta(\rho), \inf \widehat{\mathcal{N}}_\theta(\rho)), \min(\sup \widehat{\mathcal{M}}_\theta(\rho), \sup \widehat{\mathcal{N}}_\theta(\rho))], \max(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi$

**Definition 3.9:**

If  $(\widehat{\mathcal{M}}, \xi_1) = \{ \langle \rho, \widehat{\mathcal{M}}_\theta(\rho), \lambda_\theta(\rho) \rangle : \rho \in \chi, \theta \in \xi_1 \}$  and  $(\widehat{\mathcal{N}}, \xi_2) = \{ \langle \rho, \widehat{\mathcal{N}}_\theta(\rho), \lambda_\theta(\rho) \rangle : \rho \in \chi, \theta \in \xi_2 \}$  are two IFSCSs on  $(\chi, \xi)$  and  $(\widehat{\mathcal{M}}, \xi_1) \cap (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  where  $C = \xi_1 \cap \xi_2$ , for all  $\theta \in C$ .

$$H(\theta) = \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta)$$

Where  $H(\theta) = \{ t, [\min(\inf \widehat{\mathcal{M}}_\theta(\rho), \inf \widehat{\mathcal{N}}_\theta(\rho)), \min(\sup \widehat{\mathcal{M}}_\theta(\rho), \sup \widehat{\mathcal{N}}_\theta(\rho))] \}$ ,

$[\max(\inf \widehat{\mathcal{M}}_\theta(\rho), \inf \widehat{\mathcal{N}}_\theta(\rho)), \max(\sup \widehat{\mathcal{M}}_\theta(\rho), \sup \widehat{\mathcal{N}}_\theta(\rho))], \min(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi$

**Definition 3.10:** If  $(\widehat{\mathcal{M}}, \xi_1)$  and  $(\widehat{\mathcal{N}}, \xi_2)$  are two IFSCSs then  $(\widehat{\mathcal{M}}, \xi_1)$  AND  $(\widehat{\mathcal{N}}, \xi_2)$  is an IFSCS symbolized by  $(\widehat{\mathcal{M}}, \xi_1) \wedge (\widehat{\mathcal{N}}, \xi_2) = (Q, \xi_1 \times \xi_2)$  where

$$Q^T(a_i, b_i)(q)$$

$$= \{ \min(\widehat{\mathcal{M}}^T(a)(q), \widehat{\mathcal{M}}^T(b)(q)), \min(\lambda^T(a)(q), \lambda^T(b)(q)) \}$$

$$Q^F(a_i, b_i)(q) =$$

$$\{ \min(\widehat{\mathcal{M}}^F(a)(q), \widehat{\mathcal{M}}^F(b)(q)), \min(\lambda^F(a)(q), \lambda^F(b)(q)) \}$$

For all  $(a_i, b_i) \in \xi_1 \times \xi_2$

**Definition 3.11:** If  $(\widehat{\mathcal{M}}, \xi_1)$  and  $(\widehat{\mathcal{N}}, \xi_2)$  are two IFSCSs then  $(\widehat{\mathcal{M}}, \xi_1)$  OR  $(\widehat{\mathcal{N}}, \xi_2)$  is an IFSCS symbolized by  $(\widehat{\mathcal{M}}, \xi_1) \vee (\widehat{\mathcal{N}}, \xi_2) = (Q, \xi_1 \times \xi_2)$  where

$$Q^T(a_i, b_i)(q)$$

$$= \{ \max(\widehat{\mathcal{M}}^T(a)(q), \widehat{\mathcal{M}}^T(b)(q)), \max(\lambda^T(a)(q), \lambda^T(b)(q)) \}$$

$$Q^F(a_i, b_i)(q) =$$

$$\{ \max(\widehat{\mathcal{M}}^F(a)(q), \widehat{\mathcal{M}}^F(b)(q)), \max(\lambda^F(a)(q), \lambda^F(b)(q)) \}$$

For all  $(a_i, b_i) \in \xi_1 \times \xi_2$

#### IV. SOME RESULTS ON INTUITIONISTIC FUZZY SOFT CUBIC SETS

**Proposition 4.1:** If  $(\widehat{\mathcal{M}}, \xi)$  and  $(\widehat{\mathcal{N}}, \xi)$  are two IFSCS on  $\chi$  then

$$(i) (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ iff } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = (\widehat{\mathcal{M}}, \xi)$$

$$(ii) (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ iff } (\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi) = (\widehat{\mathcal{N}}, \xi)$$

Proof:

(i) Suppose that  $(\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)$  then  $\widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta)$  for all  $\theta \in \xi$ .

$$\text{Let } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = (H, \xi)$$

$$\text{Then } H(\theta) = \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta) = \widehat{\mathcal{M}}(\theta) \text{ for every } \theta \in \xi.$$

$$\text{Hence } (H, \xi) = (\widehat{\mathcal{M}}, \xi).$$

$$\text{Consider } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = (\widehat{\mathcal{M}}, \xi)$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) = \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta) \text{ for every } \theta \in \xi$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta) \text{ for all } \theta \in \xi$$

$$\text{Hence } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi).$$

(ii) Suppose that  $(\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)$  then  $\widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta)$  for all  $\theta \in \xi$ .

$$\text{Let } (\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi) = (H, \xi)$$

$$\text{Then } H(\theta) = \widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta) = \widehat{\mathcal{N}}(\theta) \text{ for all } \theta \in \xi.$$

$$\text{Then } (H, \xi) = (\widehat{\mathcal{N}}, \xi).$$

$$\text{Suppose that } (\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi) = (\widehat{\mathcal{N}}, \xi)$$

$$\text{Then } \widehat{\mathcal{N}}(\theta) = \widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta) \text{ for all } \theta \in \xi$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta) \text{ for all } \theta \in \xi$$

$$\text{Hence } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi).$$

**Proposition 4.2:** Let  $\chi$  be an initial universe set and  $\xi$  be a set of parameters. If  $(\widehat{\mathcal{M}}, \xi)$ ,  $(\widehat{\mathcal{N}}, \xi)$ ,  $(\widehat{H}, \xi)$  and  $(\widehat{K}, \xi)$  are IFSCSs on  $\chi$ . Then

$$(i) \text{ If } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = \emptyset \text{ then } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)^c$$

(ii) If  $(\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)$  and  $(\widehat{\mathcal{N}}, \xi) \subseteq (\widehat{H}, \xi)$  then  $(\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{H}, \xi)$ .

$$(iii) \text{ If } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ and } (\widehat{H}, \xi) \subseteq (\widehat{K}, \xi)$$

$$(iv)$$

$$(v)$$

$$\text{then } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{H}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \cap (\widehat{K}, \xi).$$

$$(vi) \text{ If } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ iff } (\widehat{\mathcal{N}}, \xi)^c \subseteq (\widehat{\mathcal{M}}, \xi)^c$$

Proof:

$$(i) \text{ Suppose } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = \emptyset \text{ then } \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta) = \emptyset$$

$$\text{So } \widehat{\mathcal{M}}(\theta) \subseteq \chi - \widehat{\mathcal{N}}(\theta) = \widehat{\mathcal{N}}^c(\theta) \text{ for all } \theta \in \xi$$

$$\text{Then we have } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)^c$$

$$(ii) \text{ Suppose } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ and } (\widehat{\mathcal{N}}, \xi) \subseteq (\widehat{H}, \xi)$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta) \text{ and } \widehat{\mathcal{N}}(\theta) \subseteq \widehat{H}(\theta)$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta) \subseteq \widehat{H}(\theta) \text{ for all } \theta \in \xi$$

$$\text{Then } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{H}, \xi).$$

$$(iii) \text{ Suppose } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \text{ and } (\widehat{H}, \xi) \subseteq (\widehat{K}, \xi)$$

$$\text{Then } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta) \text{ and } \widehat{H}(\theta) \subseteq \widehat{K}(\theta) \text{ for every } \theta \in \xi$$

$$\text{Then } (\widehat{\mathcal{M}}, \xi) \cap (\widehat{H}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi) \cap (\widehat{K}, \xi).$$

$$(iv) \text{ Suppose } (\widehat{\mathcal{M}}, \xi) \subseteq (\widehat{\mathcal{N}}, \xi)$$

$$\text{iff } \widehat{\mathcal{M}}(\theta) \subseteq \widehat{\mathcal{N}}(\theta)$$

$$\text{iff } (\widehat{\mathcal{N}}(\theta))^c \subseteq (\widehat{\mathcal{M}}(\theta))^c$$

$$\text{iff } \widehat{\mathcal{N}}^c(\theta) \subseteq \widehat{\mathcal{M}}^c(\theta) \text{ for all } \theta \in \xi$$

$$\text{iff } (\widehat{\mathcal{N}}, \xi)^c \subseteq (\widehat{\mathcal{M}}, \xi)^c$$

**Definition 4.3:** If  $\{(\widehat{\mathcal{M}}_i, \xi) : i \in I\}$  is a class of IFSCS on  $\chi$  then

$$(i) \bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi) =$$

$$(\widehat{H}, \xi)$$

Where

$$\widehat{H}(\theta) = \bigcup_{i \in I} \widehat{\mathcal{M}}_i(\theta) \text{ for } \theta \in \xi.$$



(ii)  $\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi) = (\widehat{K}, \xi)$  Where  $\widehat{K}(\theta) = \bigcap_{i \in I} \widehat{\mathcal{M}}_i(\theta)$  for  $\theta \in \xi$ .

**Proposition 4.4:** If  $\{(\widehat{\mathcal{M}}_i, \xi) / i \in I\}$  is a class of IFSCS on  $\mathcal{X}$ , then

- (i)  $(\bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi))^c = \bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c$
- (ii)  $(\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi))^c = \bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c$

Proof:

(i) Let  $(\bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi)) = (\widehat{H}, \xi)$

Then  $(\bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi))^c = (\widehat{H}, \xi)^c$

Since  $H^c(\theta) = \mathcal{X} \setminus \widehat{H}(\theta) = \mathcal{X} \setminus \bigcup_{i \in I} \widehat{\mathcal{M}}_i(\theta) = \bigcap_{i \in I} (\mathcal{X} \setminus \widehat{\mathcal{M}}_i(\theta))$  for all  $\theta \in \xi$  ..... (1)

On the other hand,  $\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c = (K, \xi)$

Then  $K(\theta) = \bigcap_{i \in I} \widehat{\mathcal{M}}_i^c(\theta) = \bigcap_{i \in I} (\mathcal{X} \setminus \widehat{\mathcal{M}}_i(\theta))$  for all  $\theta \in \xi$  ..... (2)

From (1) and (2),  $(\bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi))^c = \bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c$ .

(ii) Let  $(\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi)) = (\widehat{H}, \xi)$  implies  $\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c = (\widehat{H}, \xi)^c$

Since  $H^c(\theta) = \mathcal{X} \setminus \widehat{H}(\theta) = \mathcal{X} \setminus \bigcap_{i \in I} \widehat{\mathcal{M}}_i(\theta) = \bigcup_{i \in I} (\mathcal{X} \setminus \widehat{\mathcal{M}}_i(\theta))$  for all  $\theta \in \xi$  ..... (1)

On the other hand,  $\bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c = (K, \xi)$

Then  $K(\theta) = \bigcup_{i \in I} \widehat{\mathcal{M}}_i^c(\theta) = \bigcup_{i \in I} (\mathcal{X} \setminus \widehat{\mathcal{M}}_i(\theta))$  for all  $\theta \in \xi$  ..... (2)

(iii) From (1) and (2),  $(\bigcap_{i \in I} (\widehat{\mathcal{M}}_i, \xi))^c = \bigcup_{i \in I} (\widehat{\mathcal{M}}_i, \xi)^c$

**Proposition 4.5:** Show that

- (i)  $(\emptyset, \xi)^c = (\mathcal{X}, \xi)$
- (ii)  $(\mathcal{X}, \xi)^c = (\emptyset, \xi)$

Proof:

(i) Let  $(\emptyset, \xi) = (\widehat{\mathcal{M}}, \xi)$  then for all  $\theta \in \xi$ ,

$$\widehat{\mathcal{M}}(\theta) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$

$$(\emptyset, \xi)^c = (\widehat{\mathcal{M}}, \xi)^c$$

$$= \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}^c$$

$$= \{(\rho, 1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{M}}_\theta^F(\rho)), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, [1 - \widehat{\mathcal{M}}_\theta^{+T}, 1 - \widehat{\mathcal{M}}_\theta^{-T}](\rho), [1 - \widehat{\mathcal{M}}_\theta^{-F}, 1 - \widehat{\mathcal{M}}_\theta^{+F}](\rho)) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \mathcal{X}\}$$

$$\text{Thus } (\emptyset, \xi)^c = (\mathcal{X}, \xi).$$

(ii) Let  $(\mathcal{X}, \xi) = (\widehat{\mathcal{M}}, \xi)$  then for all  $\theta \in \xi$ ,

$$\widehat{\mathcal{M}}(\theta) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, ([1,1][1,1]), (1,1)) / \rho \in \mathcal{X}\}$$

$$(\mathcal{X}, \xi)^c = (\widehat{\mathcal{M}}, \xi)^c$$

$$= \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}^c$$

$$= \{(\rho, 1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{M}}_\theta^F(\rho)), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, [1 - \widehat{\mathcal{M}}_\theta^{+T}, 1 - \widehat{\mathcal{M}}_\theta^{-T}](\rho), [1 - \widehat{\mathcal{M}}_\theta^{-F}, 1 - \widehat{\mathcal{M}}_\theta^{+F}](\rho)) / \rho \in \mathcal{X}\}$$

$$= \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$

$$\text{Thus } (\mathcal{X}, \xi)^c = (\emptyset, \xi).$$

**Proposition 4.6:** If  $(\widehat{\mathcal{M}}, \xi)$  is an IFSCS then

- (i)  $(\widehat{\mathcal{M}}, \xi) \cup (\emptyset, \xi) = (\widehat{\mathcal{M}}, \xi)$ .
- (ii)  $(\widehat{\mathcal{M}}, \xi) \cup (\mathcal{X}, \xi) = (\mathcal{X}, \xi)$ .

Proof:

$$(i) (\widehat{\mathcal{M}}, \xi) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$(\emptyset, \xi) = \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$

$$(\widehat{\mathcal{M}}, \xi) \cup (\emptyset, \xi)$$

=

$$\{(\rho, \max(\widehat{\mathcal{M}}_\theta^T(\rho), 0), \max(\widehat{\mathcal{M}}_\theta^F(\rho), 0), \max(\lambda_\theta^T(\rho), 0), \max(\lambda_\theta^F(\rho), 0)) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= (\widehat{\mathcal{M}}, \xi).$$

$$\text{Thus } (\widehat{\mathcal{M}}, \xi) \cup (\emptyset, \xi) = (\widehat{\mathcal{M}}, \xi).$$

$$(ii) (\widehat{\mathcal{M}}, \xi) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$(\mathcal{X}, \xi) = \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \mathcal{X}\}$$

$$(\widehat{\mathcal{M}}, \xi) \cup (\mathcal{X}, \xi)$$

=

$$\{(\rho, \max(\widehat{\mathcal{M}}_\theta^T(\rho), 1), \max(\widehat{\mathcal{M}}_\theta^F(\rho), 1), \max(\lambda_\theta^T(\rho), 1), \max(\lambda_\theta^F(\rho), 1)) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= (\mathcal{X}, \xi).$$

$$\text{Thus } (\widehat{\mathcal{M}}, \xi) \cup (\mathcal{X}, \xi) = (\mathcal{X}, \xi).$$

**Proposition 4.7:** If  $(\widehat{\mathcal{M}}, \xi)$  is an IFSCS then

$$(i) (\widehat{\mathcal{M}}, \xi) \cap (\emptyset, \xi) = (\emptyset, \xi).$$

$$(ii) (\widehat{\mathcal{M}}, \xi) \cap (\mathcal{X}, \xi) = (\widehat{\mathcal{M}}, \xi).$$

Proof:

$$(i) (\widehat{\mathcal{M}}, \xi) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$(\emptyset, \xi) = \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$

$$(\widehat{\mathcal{M}}, \xi) \cap (\emptyset, \xi)$$

=

$$\{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), 0), \min(\widehat{\mathcal{M}}_\theta^F(\rho), 0), \min(\lambda_\theta^T(\rho), 0), \min(\lambda_\theta^F(\rho), 0)) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$

$$= (\emptyset, \xi).$$

$$\text{Thus } (\widehat{\mathcal{M}}, \xi) \cap (\emptyset, \xi) = (\emptyset, \xi).$$

$$(ii) (\widehat{\mathcal{M}}, \xi) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$(\mathcal{X}, \xi) = \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \mathcal{X}\}$$

$$(\widehat{\mathcal{M}}, \xi) \cap (\mathcal{X}, \xi)$$

=

$$\{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), 1), \min(\widehat{\mathcal{M}}_\theta^F(\rho), 1), \min(\lambda_\theta^T(\rho), 1), \min(\lambda_\theta^F(\rho), 1)) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi\}$$

$$= (\widehat{\mathcal{M}}, \xi).$$

$$\text{Thus } (\widehat{\mathcal{M}}, \xi) \cap (\mathcal{X}, \xi) = (\widehat{\mathcal{M}}, \xi).$$

**Proposition 4.8:** If  $(\widehat{\mathcal{M}}, \xi_1)$  is an IFSCS and  $\xi_1, \xi_2 \subseteq \xi$ , then

$$(i) (\widehat{\mathcal{M}}, \xi_1) \cup (\emptyset, \xi_2) = (\widehat{\mathcal{M}}, \xi_1) \text{ iff } \xi_2 \subseteq \xi_1,$$

$$(ii) (\widehat{\mathcal{M}}, \xi_1) \cup (\mathcal{X}, \xi_2) = (\mathcal{X}, \xi_2) \text{ iff } \xi_1 \subseteq \xi_2,$$

$$\text{Proof: } (\widehat{\mathcal{M}}, \xi_1) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \mathcal{X}, \theta \in \xi_1\}$$

$$\text{Also let } (\emptyset, \xi_2) = (\widehat{\mathcal{N}}, \xi_2), \text{ then}$$

$$\widehat{\mathcal{N}}(\theta) = \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \mathcal{X}\}$$



$\chi\}$   
 Let  $(\widehat{\mathcal{M}}, \xi_1) \cup (\emptyset, \xi_2) = (\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  and  $C = \xi_1 \cup \xi_2$

$$H(\theta) = \begin{cases} \widehat{\mathcal{M}}(\theta) & \text{if } \theta \in \xi_1 - \xi_2 \\ \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_2 - \xi_1 \\ \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$H(\theta) = \begin{cases} \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, (\widehat{\mathcal{N}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)), \mu_\theta^T(\rho), \mu_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, \max(\widehat{\mathcal{M}}_\theta(\rho), \widehat{\mathcal{N}}_\theta(\rho)), \max(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$H(\theta) = \begin{cases} \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, \max(\widehat{\mathcal{M}}_\theta(\rho), 0), \max(\lambda_\theta(\rho), 0), \max(\mu_\theta(\rho), 0)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

Let  $E_2 \subseteq E_1$ , then

$$H(\theta) = \begin{cases} \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \mathbb{U}\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \mathbb{U}\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$= \widehat{\mathcal{M}}(e)$  for all  $\theta \in \xi_1$ .

Conversely, let  $(\widehat{\mathcal{M}}, \xi_1) \cup (\emptyset, \xi_2) = (\widehat{\mathcal{M}}, \xi_1)$

Then  $\xi_1 \cup \xi_2 = \xi_1$  implies  $\xi_2 \subseteq \xi_1$ .

(i)  $(\widehat{\mathcal{M}}, \xi_1) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \chi, \theta \in \xi_1\}$

Also let  $(\chi, E_2) = (\widehat{\mathcal{N}}, \xi_2)$ , then

$$\widehat{\mathcal{N}}(\theta) = \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \chi\}$$

Let  $(\widehat{\mathcal{M}}, \xi_1) \cup (\chi, \xi_2) = (\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  and  $C = \xi_1 \cup \xi_2$

$$H(\theta) = \begin{cases} \widehat{\mathcal{M}}(\theta) & \text{if } \theta \in \xi_1 - \xi_2 \\ \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_2 - \xi_1 \\ \widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta) & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$H(\theta) = \begin{cases} \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, (\widehat{\mathcal{N}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)), \mu_\theta^T(\rho), \mu_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, \max(\widehat{\mathcal{M}}_\theta(\rho), \widehat{\mathcal{N}}_\theta(\rho)), \max(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$H(\theta) = \begin{cases} \{(\rho, (\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, \max(A_\theta(\rho), 1), \max(\lambda_\theta(\rho), 1), \max(\mu_\theta(\rho), 1)) / \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

Let  $\xi_1 \subseteq \xi_2$ , then

$$H(\theta) = \begin{cases} \{(t, (1,1), (1,1)) : \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(t, (1,1), (1,1)) : \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$= \widehat{\mathcal{M}}(e)$  For all  $\theta \in \xi_1$ .

Conversely, let  $(\widehat{\mathcal{M}}, \xi_1) \cup (\chi, \xi_2) = (\chi, \xi_2)$

Then  $\xi_2 = \xi_1 \cup \xi_2$  implies  $\xi_1 \subseteq \xi_2$ .

**Proposition 4.9:** If  $(\mathcal{M}, \xi_1)$  is an IFSCS and  $\xi_1, \xi_2 \subseteq \xi$ , then

(i)  $(\widehat{\mathcal{M}}, \xi_1) \cap (\emptyset, \xi_2) = (\emptyset, \xi_1 \cap \xi_2)$ .

(ii)  $(\widehat{\mathcal{M}}, \xi_1) \cap (\chi, \xi_2) = (\widehat{\mathcal{M}}, \xi_1 \cap \xi_2)$ .

Proof:

(i)  $(\widehat{\mathcal{M}}, \xi_1) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \chi, \theta \in \xi_1\}$

Also let  $(\emptyset, \xi_2) = (\widehat{\mathcal{N}}, \xi_2)$ , then

$$\widehat{\mathcal{N}}(\theta) = \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \chi\}$$

Let  $(\widehat{\mathcal{M}}, \xi_1) \cap (\emptyset, \xi_2) = (\widehat{\mathcal{M}}, \xi_1) \cap (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  where  $C = \xi_1 \cap \xi_2$  for all  $\theta \in C$

$$H(\theta) = \{(\rho, \min(\widehat{\mathcal{M}}_\theta(\rho), \widehat{\mathcal{N}}_\theta(\rho)), \min(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi\}$$

$$= \{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), 0), \min(\widehat{\mathcal{M}}_\theta^F(\rho), 0), \min(\lambda_\theta^T(\rho), 0), \min(\lambda_\theta^F(\rho), 0)) / \rho \in \chi\}$$

$$= \{(\rho, ([0,0], [0,0]), (0,0)) / \rho \in \chi\}$$

$= (\widehat{\mathcal{N}}, C) = (\emptyset, C)$

Hence  $(\widehat{\mathcal{M}}, \xi_1) \cap (\emptyset, \xi_2) = (\emptyset, \xi_1 \cap \xi_2)$ .

(ii)  $(\widehat{\mathcal{M}}, \xi_1) = \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \chi, \theta \in \xi_1\}$

Also let  $(\chi, \xi_2) = (\widehat{\mathcal{N}}, \xi_2)$ , then

$$\widehat{\mathcal{N}}(e) = \{(\rho, ([1,1], [1,1]), (1,1)) / \rho \in \chi\}$$

Let  $(\widehat{\mathcal{M}}, \xi_1) \cap (\chi, \xi_2) = (\widehat{\mathcal{M}}, \xi_1) \cap (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  and  $C = \xi_1 \cap \xi_2$

$$H(\theta) = \{(\rho, \min(\widehat{\mathcal{M}}_\theta(\rho), \widehat{\mathcal{N}}_\theta(\rho)), \min(\lambda_\theta(\rho), \mu_\theta(\rho)) / \rho \in \chi\}$$

$$= \{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), 1), \min(\widehat{\mathcal{M}}_\theta^F(\rho), 1), \min(\lambda_\theta^T(\rho), 1), \min(\lambda_\theta^F(\rho), 1)) / \rho \in \chi\}$$

$$= \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) / \rho \in \chi\}$$

for all  $\theta \in C$

$= (\widehat{\mathcal{M}}, C)$

Thus  $(\widehat{\mathcal{M}}, \xi_1) \cap (\chi, \xi_2) = (\widehat{\mathcal{M}}, \xi_1 \cap \xi_2)$ .

**Proposition 4.10:** If  $(\widehat{\mathcal{M}}, \xi_1)$  and  $(\widehat{\mathcal{N}}, \xi_2)$  are two IFSCSs and  $\xi_1, \xi_2 \subseteq \xi$  then  $((\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2))^c \subseteq (\widehat{\mathcal{M}}, \xi_1)^c \cup (\widehat{\mathcal{N}}, \xi_2)^c$

Proof:

Let  $(\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2) = (H, C)$  where  $C = \xi_1 \cup \xi_2$  and for all  $\theta \in C$

$$H(e) = \begin{cases} \{(\rho, \widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{M}}_\theta^F(\rho)), \lambda_\theta^T(\rho), \lambda_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, \widehat{\mathcal{N}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)), \mu_\theta^T(\rho), \mu_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, \max(\widehat{\mathcal{M}}_\theta(\rho), \widehat{\mathcal{N}}_\theta(\rho)), \max(\lambda_\theta(\rho), \mu_\theta(\rho)) : \rho \in \chi\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

Thus  $((\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2))^c = (H, C)^c$  where  $C = \xi_1 \cup \xi_2$  and for all  $\theta \in C$

$$(H(\theta))^c = \begin{cases} (\widehat{\mathcal{M}}(\theta))^c & \text{if } \theta \in \xi_1 - \xi_2 \\ (\widehat{\mathcal{N}}(\theta))^c & \text{if } \theta \in \xi_2 - \xi_1 \\ ((\widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta))^c) & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$= \begin{cases} \{(\rho, 1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{M}}_\theta^F(\rho)), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, 1 - \widehat{\mathcal{N}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)), 1 - \mu_\theta^T(\rho), 1 - \mu_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)), 1 - \max(\lambda_\theta^T(\rho), \lambda_\theta^F(\rho)), 1 - \max(\mu_\theta^T(\rho), \mu_\theta^F(\rho)))\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

Again  $(\widehat{\mathcal{M}}, \xi_1)^c \cup (\widehat{\mathcal{N}}, \xi_2)^c = (I, J)$  say  $J = \xi_1 \cup \xi_2$  and for all  $\theta \in J$

$$I(\theta) = \begin{cases} (\widehat{\mathcal{M}}(\theta))^c & \text{if } \theta \in \xi_1 - \xi_2 \\ (\widehat{\mathcal{N}}(\theta))^c & \text{if } \theta \in \xi_2 - \xi_1 \\ ((\widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta))^c) & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$$= \begin{cases} \{(\rho, 1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{M}}_\theta^F(\rho)), 1 - \lambda_\theta^T(\rho), 1 - \lambda_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_1 - \xi_2 \\ \{(\rho, 1 - \widehat{\mathcal{N}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)), 1 - \mu_\theta^T(\rho), 1 - \mu_\theta^F(\rho) : \rho \in \chi\} & \text{if } \theta \in \xi_2 - \xi_1 \\ \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)), 1 - \max(\lambda_\theta^T(\rho), \lambda_\theta^F(\rho)), 1 - \max(\mu_\theta^T(\rho), \mu_\theta^F(\rho)))\} & \text{if } \theta \in \xi_1 \cap \xi_2 \end{cases}$$

$C \subseteq J$  for all  $\theta \in J$ .

$$(H(\theta))^c \subseteq I(\theta).$$

Thus  $((\widehat{\mathcal{M}}, \xi_1) \cup (\widehat{\mathcal{N}}, \xi_2))^c \subseteq (\widehat{\mathcal{M}}, \xi_1)^c \cup (\widehat{\mathcal{N}}, \xi_2)^c$

$(\widehat{\mathcal{N}}, \xi_2)^c$

**Proposition 4.11:** If  $(\widehat{\mathcal{M}}, \xi)$  and  $(\widehat{\mathcal{N}}, \xi)$  are two IFSCSs then

- (i)  $((\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi))^c = (\widehat{\mathcal{M}}, \xi)^c \cap (\widehat{\mathcal{N}}, \xi)^c$
- (ii)  $((\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi))^c = (\widehat{\mathcal{M}}, \xi)^c \cup (\widehat{\mathcal{N}}, \xi)^c$

Proof:

(i) Consider  $(\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi) = (H, \xi)$  and  $H(\theta) = \widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta) = \{(\rho, \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$

Thus  $((\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi))^c = (H, \xi)^c$

$(H(\theta))^c = (\widehat{\mathcal{M}}(\theta) \cup \widehat{\mathcal{N}}(\theta))^c$

$= \{(\rho, \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$   
 $= \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$

Again  $(\widehat{\mathcal{M}}, \xi)^c \cap (\widehat{\mathcal{N}}, \xi)^c = (I, \xi)$

$I(\theta) = (\widehat{\mathcal{M}}(\theta))^c \cap (\widehat{\mathcal{N}}(\theta))^c$   
 $= \{(\rho, \min(1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^T(\rho)), \min(1 - \widehat{\mathcal{M}}_\theta^F(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$

$= \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$

Thus  $((\widehat{\mathcal{M}}, \xi) \cup (\widehat{\mathcal{N}}, \xi))^c = (\widehat{\mathcal{M}}, \xi)^c \cap (\widehat{\mathcal{N}}, \xi)^c$ .

- (ii) Let  $(\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi) = (H, \xi)$

$H(\theta) = \widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta) = \{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$

Thus  $((\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi))^c = (H, \xi)^c$

$(H(\theta))^c = (\widehat{\mathcal{M}}(\theta) \cap \widehat{\mathcal{N}}(\theta))^c$

$= \{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$   
 $= \{(\rho, 1 - \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$

Again  $(\widehat{\mathcal{M}}, \xi)^c \cup (\widehat{\mathcal{N}}, \xi)^c = (I, \xi)$

$I(\theta) = (\widehat{\mathcal{M}}(\theta))^c \cup (\widehat{\mathcal{N}}(\theta))^c$   
 $= \{(\rho, \max(1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^T(\rho)), \max(1 - \widehat{\mathcal{M}}_\theta^F(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$   
 $= \{(\rho, 1 - \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}^c$

Thus  $((\widehat{\mathcal{M}}, \xi) \cap (\widehat{\mathcal{N}}, \xi))^c = (\widehat{\mathcal{M}}, \xi)^c \cup (\widehat{\mathcal{N}}, \xi)^c$ .

**Proposition 4.12:** If  $(\widehat{\mathcal{M}}, \xi_1)$  and  $(\widehat{\mathcal{N}}, \xi_2)$  are two IFSCSs and  $\xi_1, \xi_2 \subseteq \xi$ , then

- (i)  $((\widehat{\mathcal{M}}, \xi_1) \wedge (\widehat{\mathcal{N}}, \xi_2))^c = (\widehat{\mathcal{M}}, \xi_1)^c \vee (\widehat{\mathcal{N}}, \xi_2)^c$
- (ii)  $((\widehat{\mathcal{M}}, \xi_1) \vee (\widehat{\mathcal{N}}, \xi_2))^c = (\widehat{\mathcal{M}}, \xi_1)^c \wedge (\widehat{\mathcal{N}}, \xi_2)^c$

Proof:

- (i) Let  $(\widehat{\mathcal{M}}, \xi_1) \wedge (\widehat{\mathcal{N}}, \xi_2) = (H, \xi_1 X \xi_2)$  and

$H(m, n) = \{(\rho, \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

$((\widehat{\mathcal{M}}, \xi_1) \wedge (\widehat{\mathcal{N}}, \xi_2))^c = (H, \xi_1 X \xi_2)^c$  for all  $(a, b) \in \xi_1 X \xi_2$

$(H(m, n))^c = \{(\rho, 1 - \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

Let  $(\widehat{\mathcal{M}}, \xi_1)^c \vee (\widehat{\mathcal{N}}, \xi_2)^c = (R, \xi_1 X \xi_2)$  and

$R(m, n) = \{(\rho, \max(1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^T(\rho)), \max(1 - \widehat{\mathcal{M}}_\theta^F(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all

$m \in \xi_1, n \in \xi_2$

$= \{(\rho, 1 - \min(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \min(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

Thus  $((\widehat{\mathcal{M}}, \xi_1) \wedge (\widehat{\mathcal{N}}, \xi_2))^c = (\widehat{\mathcal{M}}, \xi_1)^c \vee (\widehat{\mathcal{N}}, \xi_2)^c$ .

- (ii) Let  $(\widehat{\mathcal{M}}, \xi_1) \vee (\widehat{\mathcal{N}}, \xi_2) = (H, \xi_1 X \xi_2)$  where

$H(m, n) = \{(\rho, \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

$((\widehat{\mathcal{M}}, \xi_1) \vee (\widehat{\mathcal{N}}, \xi_2))^c = (H, \xi_1 X \xi_2)^c$  for all  $(a, b) \in \xi_1 X \xi_2$

$(H(m, n))^c = \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

Let  $(\widehat{\mathcal{M}}, \xi_1)^c \wedge (\widehat{\mathcal{N}}, \xi_2)^c = (R, \xi_1 X \xi_2)$  Where

$R(m, n) = \{(\rho, \min(1 - \widehat{\mathcal{M}}_\theta^T(\rho), 1 - \widehat{\mathcal{N}}_\theta^T(\rho)), \min(1 - \widehat{\mathcal{M}}_\theta^F(\rho), 1 - \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

$= \{(\rho, 1 - \max(\widehat{\mathcal{M}}_\theta^T(\rho), \widehat{\mathcal{N}}_\theta^T(\rho)), 1 - \max(\widehat{\mathcal{M}}_\theta^F(\rho), \widehat{\mathcal{N}}_\theta^F(\rho)))\}$  for all  $m \in \xi_1, n \in \xi_2$

Thus  $((\widehat{\mathcal{M}}, \xi_1) \vee (\widehat{\mathcal{N}}, \xi_2))^c = (\widehat{\mathcal{M}}, \xi_1)^c \wedge (\widehat{\mathcal{N}}, \xi_2)^c$

## V. CONCLUSION

The concept of an IFSCS is proposed and some operations are defined on IFSCS. Some properties are deliberated regarding the concept of IFSCS. We expect the future work on topological spaces upon this concept.

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