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Co-Secure Set Domination in Graphs

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Abstract**: *Throughout this paper, consider G = (V,E) as a connected graph. A subset D of V(G) is a set dominating set of G if for every $M \subseteq V/D$ there exists a non-empty set N of D such that the induced sub graph <MUN> is connected. A subset D of the vertex set of a graph G is called a co-secure dominating set of a graph if D is a dominating set, and for each $u' \in D$ there exists a vertex $v' \in V/D$ such that $u'v'$ is an edge and $(D \setminus \{u'\}) \cup \{v'\}$ is a dominating set. A co-secure dominating set D is a co-secure set dominating set of G if D is also a set dominating set of G. The co-secure set domination number $\gamma_{cs}^{s}(G)$ is the minimum cardinality of a co-secure set dominating set. In this paper we initiate the study of this new parameter & also determine the co-secure set domination number of some standard graphs and obtain its bounds.***

Keywords: Co-secure dominating set, Dominating set, Set dominating set.

I. INTRODUCTION

Let G = (V,E) be a connected, finite, undirected graph without loops and multiple edges. The order and size of the graph G is m and n respectively. For $v \in G$, the number of incident edges in v is the degree of a vertex v and is written as d(v) or deg(v). The $N_G(v)$ is an open neighborhood of a vertex $v \in V$ of a graph G is defined as $N_G(v) = \{u \in V \setminus \text{u is adjacent to v}\}$ and $N_G[v] = N_G(v) \cup \{v\}$ is a closed neighborhood of a graph G. For a set, $I \subseteq V$, $N_G(I) = \bigcup_{v \in I} N_G(v)$ and $N_G[I] = N_G(I) \cup I$ is a closed neighborhood of a graph G.

The path of m vertices is denoted by P_m and cycle of m vertices is denoted as C_m. The complete graph of m vertices is a graph having every vertices of degree m-1 and is denoted by K_m. The wheel of m vertices is a graph obtained by joining all the vertices of cycle C_{m-1} to the vertex at the centre of the cycle and is denoted by W_m, $m \geq 4$. The $K_{r,s}$ is a complete Bipartite graph with the partite sets R and S having cardinality r and s respectively. The graph $K_{1,m}$ is star[4],[5]. The Friendship graph F_m is also a graph with 2m+1 vertices having m copies of K_3 with central vertex of degree 2m and all other vertices of degree 2.

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A subset D of V(G) is a dominating set of G if for every vertex $u' \in V \setminus D$ there exist atleast one vertex $v' \in D$ such that u' is adjacent to v'. The domination number of a graph is denoted by $\gamma(G)$ which is minimum cardinality of a dominating set of G[4],[5].

For any connected graph G, a subset D of V(G) is a set dominating set of G if for every M⊆V/D there exists a non-empty set N⊆D such that the sub graph <MUN> induced by MUN is connected and its minimum cardinality of a set dominating set of G is the set domination number and is denoted by $\gamma_s(G)$ and is abbreviated as a SD-set (set dominating set)[6].

For any non-trivial graphs G with a subset D of the vertex set V(G) is a co-secure dominating set of a graph G if D is a dominating set and for every $u' \in D$ there exists a vertex $v' \in V \setminus D$ such that $u'v'$ is an edge and $(D \setminus \{u'\}) \cup \{v'\}$ is a dominating set[1]. The minimum cardinality of a co-secure dominating set (or CSDS) of G is the co-secure domination number of G and its denoted by $\gamma_{cs}(G)$[1].

The concept of co-secure domination in graphs was introduced and the co-secure domination number of path and a cycle was determined by Arumugan S., KaramEbadi and Martin Manrique [1]. Aleena Joseph and Sangeetha determined the co-secure domination number of Friendship graph, Jahangir graph and Helm graph and also obtained the bounds[2]. For any graph G with isolated vertices, there will be no CSDS for G.

We introduce a new variant of co-secure domination namely the co-secure set domination set(abbreviated as CSSDS) and we initiate the study of CSSD set. A co-secure dominating set D is a co-secure set dominating set of G if D is also a set dominating set of G. The minimum cardinality of CSSD set is the co-secure set domination number and is denoted as a $\gamma_{cs}^{s}(G)$. Throughout this paper, we consider the finite, connected, nontrivial graphs and undefined terms are in[4],[5].

We need the following result in[1].

Corollary 1.1

For every integer $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

II. MAIN RESULT

In this Section, we analyze the new parameter called the co-secure set domination number of various other graphs such as Path graph, Cycle graph, Complete graph, Wheel graph, Complete Bipartite graph, Star graph and Friendship graph. Moreover, we obtain the bounds for co-secure set domination number for any nontrivial graph with order m.

A. Definition

Let G = (V,E) be any non-trivial graph. A Co-secure dominating set of G is called a Co-secure set dominating set if for every M⊆V/D there exists a set N⊆D which is non empty such that the sub graph <MUN> induced by MUN is connected. The minimum cardinality of co-secure set dominating set of G is defined as the co-secure set domination number of G and is denoted as $\gamma_s^{cs}(G)$.

From this, we say that a dominating set D is a co-secure set dominating set if and only if D is co-secure as well as set dominating set.

B. Observation

If Δ(G) = m-1, then $\gamma(G)\le\gamma_s(G)\le\gamma^{cs}(G)\le\gamma_s^{cs}(G)$.

C. Observation

For a Path P_m, $\gamma_s^{cs}(P_m)=\begin{cases} m-1, & \text{for m} = 2,3,\\ m-2, & \text{for } m=4,5,6,\\ \text{does not exist}, & \text{for m}\ge 7\end{cases}$

D. Observation

If G is a cycle graph of m vertices, then $\gamma_s^{cs}(C_m)=\begin{cases} m-2, & \text{for m} = 3,4,\\ m-3, & \text{for } 5\le \text{m}\le 9,\\ \text{does not exist}, & \text{for m}\ge 10\end{cases}$

E. Theorem

If G is a complete graph with m vertices, then $\gamma_s^{cs}(K_m)=1$ for $m\ge 3$.

Proof

Let $G=K_m$ be a graph with m vertices each of degree m-1, then we have to prove that $|S|=1$ is a minimum co-secure set dominating set. Instead of this, it is enough to show that S is a minimum dominating set of K_m because every vertex is of degree m-1. Thus every vertex of K_m is connected to every other vertex of that graph. So minimum dominating set of K_m is 1. Therefore, $\gamma_s^{cs}(K_m)=1$.

F. Theorem

If W_m is a wheel graph with m vertices then $\gamma_s^{cs}(W_m)=\left\lceil\frac{m-1}{3}\right\rceil$ for $m\ge 4$.

Proof

Let $W_m=C_{m-1}+\{v\}$, where C_{m-1} is a cycle of order m-1. Let S be any CSSDS. If $\{v\}$ is not in S, then S is a co-secure set dominating set of C_{m-1} and hence $|S|\ge\left\lceil\frac{m-1}{3}\right\rceil$ for $6\le m\le 10$ and no such S exists for $m\ge 11$ because the set S is having the occurrence of consecutive vertices continuously 3 or more than so S will be a set dominating set but that S will not be co-secure dominating set. If $\{v\}\in S$, then S will be considered as a dominating set because v is connected with all the vertices of C_{m-1} and there exist $u\in V(C_{m-1})$ such that u replaces v. Since $C_{m-1}-N[u]=P_{n-4}$ it follows from the corollary 1.1 that $|S|\ge\left\lceil\frac{m-4}{3}\right\rceil+1=\left\lceil\frac{m-1}{3}\right\rceil$. Thus $\gamma_s^{cs}(W_m)\ge\left\lceil\frac{m-1}{3}\right\rceil$. Further any $\gamma-set$ of C_{m-1} is an CSSDS of W_m (i.e. $\gamma-$set of $(C_{m-1}\setminus\{u\})\cup\{v\}$), gives that $\gamma_s^{cs}(W_m)\le\left\lceil\frac{m-1}{3}\right\rceil$. Hence, $\gamma_s^{cs}(W_m)=\left\lceil\frac{m-1}{3}\right\rceil$ for $m\ge 4$.

Remark

Also for $m\ge 4$, $\gamma^{cs}(W_m)=\left\lceil\frac{m-1}{3}\right\rceil$ is already found in [1].

Thus we have $\gamma^{cs}(W_m)=\gamma_s^{cs}(W_m)=\left\lceil\frac{m-1}{3}\right\rceil$.

G. Theorem

For a graph $G=K_{1,m-1}$, then $\gamma_s^{cs}(K_{1,m-1})=m-1$.

Proof

Let $G=K_{1,m-1}$, then we say that S, the set of all pendant vertex of $K_{1,m-1}$ is a minimum co-secure set dominating set because the central vertex must always be in V\S, then only for every vertex v_i in S there exists a vertex v in V\S such that vv_i is an edge of $K_{1,m-1}$ and $((S\setminus\{v_i\})\cup\{v\})$ will form a dominating set of $K_{1,m-1}$ and therefore $|S|=m-1$. Hence $\gamma_s^{cs}(K_{1,m-1})=m-1$.

H. Theorem

For a Complete Bipartite graph $K_{r,s}$ with $r\le s$, then

$$\gamma_s^{cs}(K_{r,s})=\begin{cases} s, & \text{if } r=1,\\ 2, & \text{if } r=2,\\ 3, & \text{if } r=3,\\ 4, & \text{if } r\ge 4,\end{cases}.$$

Proof

Let $R=\{u_1,u_2,u_3,...u_r\}$ and $S=\{u_1',u_2',u_3',...u_s'\}$ be the bipartition of $G=K_{r,s}$. If r = 1 then $G=K_{1,s}$ is a star and the proof is given in the theorem 2.7. If r = 2, then $\gamma_s^{cs}(K_{2,s})=2$. If r = 3 and let S_1 be a CSSDS of $K_{3,s}$. If $S_1\cap R=\{u_1\}$ and $S_1\cap S=\{u_1'\}$, then no vertex in R or S can replace u_1 or u_1'. Therefore $\gamma_s^{cs}(K_{3,s})\ge 3$. Also R is a CSSDS of $K_{3,s}$ and so $\gamma_s^{cs}(K_{3,s})=3$. Now, if $r\ge 4$ and let S_1 be a CSSDS of $K_{r,s}$. We know that if $S_1\cap R=\varnothing$, then $S_1=S$ and vice versa. Further if $S_1\cap R\ne\varnothing$, then $|S_1\cap R|\ge 2$ and if $S_1\cap S\ne\varnothing$, then $|S_1\cap S|\ge 2$. So in this case $|S_1|\ge 4$. Also if $\{u_1,u_2,u_1',u_2'\}$ is a CSSDS of

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$K_{r,s}$ and hence we conclude that $|S_1| = 4$. Therefore $\gamma_s^{cs}(K_{r,s}) = 4$ for $r \geq 4$.

I. Theorem

Let F_m be a Friendship graph having 2m+1 vertices, then $\gamma_s^{cs}(F_m) = m$ for all m.

Proof

Let F_m be a friendship graph with 2m+1 vertices such that F_m consists of m copies of K_3. Let the central vertex of F_m be u and its degree is 2m and $v_1, v_2, v_3 ......v_{2k}$ be the vertices labeled in clockwise direction (or anticlockwise) with each vertex v_i having degree 2.

If {u} is not in S, then every set in V\S is not connected to any set in S. Therefore, there is no such CSSDS for this case. If $\{u\} \in S$, and also choose one vertex from each m-1 copies of K_3 and form the CSSDS with m vertices. That is, $S = \{v_1, v_3, v_5, ...v_{2m-3}, u\}$ or $S_1 = \{v_2, v_4, v_6, ...v_{2m-2}, u\}$ be a CSSDS of F_m. For this S(or S_1), every set in V\S is connected to any set in S and also $(S \setminus \{v_i\}) \cup \{v_{i+1}\}$, $(S \setminus \{u\}) \cup \{v_{2m-1}\}$ for $i = 1, 3, 5, ...2m-3$ forms a dominating set (or $(S_1 \setminus \{v_i\}) \cup \{v_{i-1}\}$, $(S_1 \setminus \{u\}) \cup \{v_{2m}\}$ for all $i = 2, 4, 6, ...2m-2$ forms a dominating set). Thus $\gamma_s^{cs}(F_m) \leq m$.

Now we have to verify that S is the minimum CSSDS . That is, there exist no CSSDS with less than m vertices. Let us assume that $\gamma_s^{cs}(F_m) < m$. But at most one vertex from each m-1 copies of K_3 and central vertex u must be in the CSSDS (otherwise it's not a CSSDS), which is a contradiction.

Therefore, $\gamma_s^{cs}(F_m) = m$, for all m.

Remark

Also $\gamma^s(F_m) = \gamma^{cs}(F_m) = m$ is already found in [2]. Thus we have $\gamma^s(F_m) = \gamma^{cs}(F_m) = \gamma_s^{cs}(F_m) = m$.

J. Theorem

For any graph G of order m without isolated vertex, then $1 \leq \gamma_s^{cs}(G) \leq m-1$. Furthermore, the $\gamma_s^{cs}(G) = 1$ if and only if $G = K_m$ and the equality holds in the upper bound, $\gamma_s^{cs}(G) = m-1$ if and only if $G = K_{1,m-1}$.

Proof

The bounds are trivial. Now consider the lower bound. The result is obvious if $G = K_m$, then $\gamma_s^{cs}(G) = 1$. Conversely, if there is a co-secure set dominating set, S with $|S| = 1$, i.e., $S = \{u_1\}$ and that u_1 dominates every other vertex in V\S, therefore $\deg u_1 = m-1$. Also every set in V\S is connected with any set in S. Since $S = \{u_1\}$ is also a co-secure dominating set, so if we replace u_1 by any other vertex $\{u_i\}$ in V\S for $i = 2, 3, 4, ...m$ is also a dominating set. From this we conclude that every vertex in V\S is also adjacent to every other vertex of G. Therefore $\deg u_i = m-1$ for $i = 2, 3, 4, ...m$. Hence $G = K_m$.

Now consider the upper bound.

Suppose $G = K_{1,m-1}$, then $\gamma_s^{cs}(G) = m-1$ is given in the theorem 2.7. Conversely, suppose $\gamma_s^{cs}(G) = m-1$. Let S be a minimum co-secure set dominating set with $|S| = m-1$ and let $V \setminus S = \{v\}$. Then all the vertices in S are adjacent only to v. If not, then there exists a set $\{v_1, v_2\} \subseteq S$ such that $v_1 v_2$ is an edge of $K_{1,m-1}$, and also $S \setminus \{v_1\}$ is a co-secure set dominating set of cardinality less than m-1, a contradiction. Therefore, all the vertices in S are leaves of G (i.e., vertices in S are independent). Hence we conclude that $G = K_{1,m-1}$.

III. CONCLUSION

We have a new variant of the co-secure domination namely co-secure set domination set and co-secure domination number of various graph such as path, cycle, complete graph, wheel graph, star graph, complete bipartite graph and friendship graph and also its bounds. We observed for the wheel, complete, complete bipartite, star and friendship graph have $\gamma_s^{cs}(G) = \gamma^{cs}(G)$. In this paper we get the co-secure set domination number is always less than its order. What will happen to $\gamma_s^{cs}(G)$ if we remove or add a vertex to a graph G is an open area for investigation.

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