Domination Integrity of Some Special Graphs

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Abstract: A major importance is given to the stability of a network by its users and designers. Domination Integrity is one of the parameter used to determine the network’s vulnerability and its defined for a connected graph G as “ DI(G) = min{ |X| + m(G - X) : X is a dominating set }” where m(G-X) is the order of maximum component of G-X”, by Sundaraeswaran and Swaminathan. Here we investigate the domination integrity of Tadpole graph, Lollipop graph and a line graph of composition of $P_n$ and $P_2$.

Keywords : Domination Integrity, Line graph, Lollipop graph, Tadpole graph.

I. INTRODUCTION

A communication network composed of processor (or nodes) and connection between the processor as a links. Unfortunately, the destruction of links, disturbance in nodes, any fault in software, hardware and failure of transmission occurs at various terminals (starting and ending points) will result in a disturbance of service for a prolonged duration may affect its effectiveness which is described as a Vulnerability of a network. While constructing the communication network, it must be more stable or vulnerability must be less. On modeling the network as a graph, we number of graph theoretic parameters such as binding number, edge tenacity, integrity, edge integrity, vertex-neighbor integrity, Domination integrity etc. and these are used to estimate the resistance level of a network after the failure of certain elements which happened in the case of interruptions.

For any modeled graph structure, the vulnerability means that ‘any small damage cause large consequences’. In a graph structure, the vulnerability is nothing but the inadequacy of a resistance due to the deletion of vertices or edges or both. On analyzing the vulnerability of a network after any interruption, the two quantities are necessary which are (i) number of non-working elements(edges or nodes), (ii) order or the biggest remaining component or sub network within which mutual communication can exists. Here the first one represents the imperfection (most targeted) nodes for more interruption and the second one provides the consequences after interruption.

Swaminathan and Sundaraeswaran[2]-[5] introduced the concept of domination integrity and investigated the domination integrity of middle graphs, trees and power of cycles.

A. Definition[1]

A subset D of V(G) is said to be a dominating set of a graph G if every vertex $u \in V$ is either in D or adjacent to atleast one element in D.

B. Definition[2]

The Domination integrity of a connected graph G is written as DI(G) and defined as DI(G) = min{ |X| + m(G - X) : X is a dominating set } where m(G - X) is the order(number of vertices) of a largest component of G - X.

The Kothari and Vaidya[7]-[9] studied the domination integrity of duplication of a vertex(or an edge) by an edge(or vertex), Splitting graph of path and cycle, and Splitting and degree splitting graph of some graphs. The domination integrity of Shadow graphs of bistar, path, complete bipartite and cycle, some path related graphs and total graphs of star, path and cycle graphs were analyzed by Shah and Vaidya[10]-[12]. Veena and Sultthan seena [6] have investigated the domination integrity of Line Splitting and central graph of some standard graphs. Some basic definitions used here are:

C. Definition

The Line graph L(G) of a graph G(V,E) is also a graph with vertex set as $E_0$(that is V(L(G)) is a edge set of a graph G), and the two vertices $e_1$, $e_2$ are adjacent in L(G) if and only if the edges $e_1$ and $e_2$ are adjacent in a graph G.

D. Definition

The (m,n) – Tadpole graph is denoted by $T_{mn}$ and is obtained by connecting any vertex of cycle $C_m$ to the end vertex of the path $P_n$ with the bridge and is also named as Kite graph or Dragon graph.

E. Definition

The Lollipop graph is a graph obtained by linking any mutual communication can exists. Here the first one represents the imperfection (most targeted) nodes for more interruption and the second one provides the consequences after interruption.

Swaminathan and Sundaraeswaran[2]-[5] introduced the concept of domination integrity and investigated the domination integrity of middle graphs, trees and power of cycles.
are adjacent in \( G_1 \) or \( u_1 = u_2 \) and \( v_1 \) is adjacent to \( v_2 \) in \( G_1 \) and is denoted as \( G_1[G_2] \).

The graphs considered here are undirected, loopless, finite and connected and the terminology used are in [1],[13].

II. MAIN RESULT

A. Theorem

\[
\text{DI}(T_{3,n}) = \begin{cases} 
3, & n = 1, \\
4 + i, & n = 2 + 3i + j, \\
\text{where } j = 0,1,2 \text{ and } i = \{0\} \cup \mathbb{N} 
\end{cases}
\]

Proof

Let \( |V(T_{3,n})| = 3 + n = p \) and \( |E(T_{3,n})| = 3 + n \) of the graph \( T_{3,n} \). We need two cases to prove the result.

Case (i)

For \( n = 1 \), the vertex of \( T_{3,1} \) be \( v_1, v_2, v_3, v_4 \). Consider \( S = \{v_3\} \) as a dominating set of \( T_{3,1} \) and we get \( m(T_{3,1} - S) = 2 \). Therefore, \( |S| + m(T_{3,1} - S) = 3 \). If we select a dominating set either \( S = \{v_1, v_2\} \) or \( S = \{v_2, v_4\} \), we get \( m(T_{3,1} - S) = 2 \) in both the cases we get \( |S| + m(T_{3,1} - S) = 4 \). Hence \( \text{DI}(T_{3,1}) = 3 \).

Case (ii)

- If \( n = 2 + 3i + j \) where \( j = 0 \) and \( i = \{0\} \cup \mathbb{N} \) (that is, for \( n = 2,5,8,11,\ldots \)), consider \( S = \{v_{3+3k} \mid k = 0 \text{ to } i\} \cup \{v_{3+n}\} \) and \( |S| = 2 + i \).
- If \( n = 2 + 3i + j \) where \( j = 1,2 \) and \( i = \{0\} \cup \mathbb{N} \) (i.e., for \( n = 3,4,6,7,9,10,12,13,\ldots \)), consider the set \( S = \{v_{3+3k} \mid k = 0 \text{ to } i + 1\} \) and \( |S| = 2 + i \).

In the above two cases, the set \( S \) is considered as a dominating set because \( v_1, v_2, v_4 \in N(v_3) \) and \( v_{3+3T}, v_{3+3T+2} \in N(V_{3+3T}) \) where \( t = 0,1,2,3,\ldots \), and also \( m(T_{3,n} - S) = 2 \).

Now we have to check the minimum of \( |S| + m(T_{3,n} - S) \).

Let us consider any other dominating set \( S_1 \) of \( T_{3,n} \) with \( m(T_{3,n} - S) > 2 \) because due to the nature of the construction of the graph \( T_{3,n} \), will have \( |S| > |S_1| \). Therefore,

\[
|S_1| + m(T_{3,n} - S_1) > |S| + m(T_{3,n} - S)
\]

Consider any other set \( S_2 \) as a dominating set of \( T_{3,n} \) with \( m(T_{3,n} - S) < 2 \), that is each component in \( T_{3,n} - S \) will be the isolated point, then automatically \( |S_2| > |S| \).

\[
|S| + m(T_{3,n} - S) \leq |S_2| + m(T_{3,n} - S_2)
\]

Therefore,
\[
|S| + m(T_{3,n} - S) = \min \{ |X| + m(T_{3,n} - X) \} : X \text{ is a dominating set}
\]

\[
\text{DI}(T_{3,n}) = \begin{cases} 
3, & n = 1, \\
4 + i, & n = 2 + 3i + j, \\
\text{where } j = 0,1,2 \text{ and } i = \{0\} \cup \mathbb{N} 
\end{cases}
\]

B. Theorem

\[
\text{DI}(T_{4,n}) = \begin{cases} 
3, & n = 1, \\
4 + i, & n = 2 + 2i + j \text{ where } j = 0,1 \text{ and } i = \{0\} \cup \mathbb{N}, \\
7 + i, & n = 8 + 3i + j \text{ where } j = 0,1,2 \text{ and } i = \{0\} \cup \mathbb{N},
\end{cases}
\]

Proof

Let \( V(T_{4,n}) = \{v_1, v_2, v_3, \ldots, v_{4+n}\} \) and \( E(T_{4,n}) = |V(T_{4,n})| \). The proof is given by the following three cases.

Case (i)

If \( n = 1 \), then the vertex set of \( T_{4,1} \) is \( \{v_1, v_2, v_3, v_4, v_5\} \). Now, consider the set \( S = \{v_2, v_4\} \) as a dominating set of \( T_{4,1} \) then we have \( m(T_{4,1} - S) = 1 \) So \( |S| + m(T_{4,1} - S) = 3 \). The other possible dominating set of \( T_{4,1} \) are \( S = \{v_1, v_4\} \) and \( S = \{v_1, v_3, v_5\} \), then we have \( |S| + m(T_{4,1} - S) = 4 \).

Among all these dominating set, \( |S| + m(T_{4,1} - S) = 2 + 1 + 3 = 3 \) is minimum.

Case (ii)

For \( 2 \leq n \leq 7 \)

- If \( n = 2,3 \) and \( 5 \), consider the set \( S = \{v_2, v_4 \mid k = 0 \text{ to } i + 2\} \) and \( |S| = 3 + i \). Moreover we have \( m(T_{4,n} - S) = 1 \).
- If \( n = 4,6,7 \), then choose the set \( S = \{v_2, v_4 \mid \cup \{v_{7+3k} \mid k = 0 \text{ to } i - 1\} \) and \( |S| = 2 + i \). Further, we have \( m(T_{4,n} - S) = 2 \). Then for also \( S = \{v_2, v_4 \mid k = 0 \text{ to } i + 2\} \) and \( |S| = 3 + i \) we have \( m(T_{4,n} - S) = 1 \). So for both the set \( S \) we get \( |S| + m(T_{4,n} - S) = 4 + i \).

The above mentioned set \( S \) will be a dominating set as \( v_1, v_2, v_5 \in N(v_3) \) and \( v_{6+3}, v_{9+3} \in N(v_{7+3}) \) for \( t=0,1,2,\ldots \).

Now to analysis the minimality of \( |S| + m(T_{4,n} - S) \):

Let \( S_1 \) be any other dominating set with \( m(T_{4,n} - S_1) > 2 \), then it is easily verified that,

\[
|S| + m(T_{4,n} - S) \leq |S_1| + m(T_{4,n} - S_1)
\]

Now for another dominating set \( S_2 \) with \( m(T_{4,n} - S_2) = 1 \), then each component in \( T_{4,n} - S_2 \) is a isolated vertex. So,

\[
|S| + m(T_{4,n} - S) \leq |S_2| + m(T_{4,n} - S_2)
\]

Hence from (3) and (4), \( \text{DI}(T_{4,n}) = 4 + i \).

Case (iii)

Let \( S_1 = \{v_2, v_4\} \).

- If \( n = 8 + 3i + j \) where \( j = 0 \) and \( i = N \cup \{0\} \) (i.e. for \( n = 8,11,14,17,\ldots \)), then...
For the above mentioned set $S$ in $T_{n,n}$ will be a dominating set as $v_1,v_2,v_3,v_5 \in N(v_4)$ and $v_6,v_7,v_8,v_{10} \in N(v_4)$ for $t = 0,1,2,...$ and we have $m(T_{n,n} - S) = 2$.

Now to verify the minimality of $\lvert S \rvert + m(T_{n,n} - S)$:

Let us consider any other dominating set as $S_i$ with $m(T_{n,n} - S_i) > 2$, then due to the nature of the structure of $T_{n,n}$, we get $\lvert S_i \rvert > \lvert S \rvert$ such that

$$\lvert S_i \rvert + m(T_{n,n} - S_i) > \lvert S \rvert + m(T_{n,n} - S) \quad (5)$$

Now, $S_2$ is any other dominating set with $m(T_{n,n} - S) < 2$, (i.e. $m(T_{n,n} - S) = 1$), then obviously $\lvert S_2 \rvert > \lvert S \rvert$. So

$$\lvert S_2 \rvert + m(T_{n,n} - S_2) > \lvert S \rvert + m(T_{n,n} - S) \quad (6)$$

Therefore by (5) and (6),

$$\lvert S \rvert + m(T_{n,n} - S) = \min \{ X \mid m(T_{n,n} - X) \} : X \text{ is a dominating set}$$

$$= \text{Di}(T_{n,n})$$

Hence

$$\text{Di}(T_{n,n}) = \begin{cases} 3, & \text{if } n = 1, \\ 4 + i, & \text{if } n = 2 + 2i + j \text{ where } j = 0,1 \text{ and } i = 0 \cup N, \\ 7 + i, & \text{if } n = 8 + 3i + j \text{ where } j = 0,1,2 \text{ and } i = 0 \cup N. \end{cases}$$

C. Theorem

For $m \geq 5$

$$\text{Di}(T_{m,n}) = \begin{cases} 4, & \text{if } m = 5, \\ 5 + i, & \text{if } m = 7 + 3i + j \text{ where } j = 0,1,2 \text{ and } i = 0 \cup N. \end{cases}$$

Proof

Let the \( \text{Di}(T_{m,n}) = m + n \). The proof is given by the following two cases.

Case (i)

For $m = 5$

Let us consider $S = \{ v_2, v_3 \}$ as a dominating set of $T_{5,1}$, then $m(T_{5,1} - S) = 2$. So $\lvert S \rvert + m(T_{5,1} - S) = 4$. For any other choice of $S = \{ v_1, v_2, v_3 \}$ or $\{ v_2, v_3, v_6 \}$ we have $m(T_{5,1} - S)$ as 5 and 6 respectively. So $\lvert S \rvert + m(T_{5,1} - S) = 2 + 2 = 4$ is minimum. Hence $\text{Di}(T_{5,1}) = 4$.

If $m = 6$, consider $S = \{ v_2, v_6 \}$ as a dominating set of $T_{6,1}$, then $m(T_{6,1} - S) = 2$. So $\lvert S \rvert + m(T_{6,1} - S) = 4$. For any other choice of $S = \{ v_2, v_3, v_5 \}$ or $\{ v_1, v_2, v_3, v_6 \}$ we have $m(T_{6,1} - S)$ as 2 and 1 respectively.

So $\lvert S \rvert + m(T_{6,1} - S) = 2 + 2 = 4$ is minimum. Therefore, $\text{Di}(T_{6,1}) = 4$.

Case (ii)

If $m = 7 + 3i + j$, where $j = 0,1,2$ and $i = 0 \cup N$ (i.e., for $m = 7,8,9,10,11,12,...$), then consider the dominating set $S = \{ v_3, v_{1+k} \}$, where $k = 0 \cup i+1 \cup \{ v_m \}$ and $\lvert S \rvert = 3 + i$. Use the nature of the structure of $T_{m,n}$, we get $\lvert S \rvert + m(T_{m,n} - S) = 2$. So $\lvert S \rvert + m(T_{m,n} - S) = 3 + 2$. If for any other dominating set $S_i$ with $m(T_{m,n} - S_i) > 2$, then due to the nature of the structure of $T_{m,n}$, we get $\lvert S_i \rvert > \lvert S \rvert$ such that

$$\lvert S_i \rvert + m(T_{m,n} - S_i) > \lvert S \rvert + m(T_{m,n} - S) \quad (7)$$

Let $S_2$ be any other dominating set of $T_{m,n}$ with $m(T_{m,n} - S_2) < 2$, (i.e., $m(T_{m,n} - S_2) = 1$), then obviously we say that $\lvert S_2 \rvert > \lvert S \rvert$. So

$$\lvert S_2 \rvert + m(T_{m,n} - S_2) > \lvert S \rvert + m(T_{m,n} - S) \quad (8)$$

From (7) and (8), we get

$$\lvert S \rvert + m(T_{m,n} - S) = \min \{ X \mid m(T_{m,n} - X) \} : X \text{ is a dominating set}$$

$$= \text{Di}(T_{m,n})$$

Hence,

For $m \geq 5$

$$\text{Di}(T_{m,n}) = \begin{cases} 4, & \text{if } m = 5, \\ 5 + i, & \text{if } m = 7 + 3i + j \text{ where } j = 0,1,2 \text{ and } i = 0 \cup N. \end{cases}$$

D. Theorem

For $m = 5,6$ and $n \geq 2$

$$\text{Di}(T_{m,n}) = 5 \cup i, \text{if } n = 2 + 3i + j \text{ where } j = 0,1,2 \text{ and } i = 0 \cup N \}$$

Proof

Let $\lvert \text{Di}(T_{m,n}) \rvert = m + n$. To prove this we need the following two cases.

Case (i) For $m = 5$

If $n = 2 + 3i + j$, where $j = 0,1$ and $i = N \cup \{ 0 \}$, consider the set $S = \{ v_3, \} \cup \{ v_{m+n} \}$ of $T_{5,n}$ and $\lvert S \rvert = 3 + i$.

If $n = 2 + 3i + j$, where $j = 2$ and $i = N \cup \{ 0 \}$, consider the set $S = \{ v_3, \} \cup \{ v_{m+n+1} \}$ and $\lvert S \rvert = 3 + i$. The above defined set $S$ will be a dominating set as $\{ v_{3+k} \}$, where $k = 0 \cup i + 1 \cup \{ v_m \}$ for $t = 0,1,2$ etc. Moreover, we have $m(T_{5,n} - S) = 2$.

Let $S_2$ be any other dominating set of $T_{5,n}$ with $m(T_{5,n} - S_2) > 2$, then it is easily observed that $\lvert S_2 \rvert > \lvert S \rvert$. So we have that

$$\lvert S_2 \rvert + m(T_{5,n} - S_2) > \lvert S \rvert + m(T_{5,n} - S) \quad (9)$$

Let $S_2$ be any other dominating set of $T_{5,n}$ with $m(T_{5,n} - S_2) = 1$, then each component in $T_{5,n} - S_2$ is an isolated vertex and so we get $\lvert S_2 \rvert > \lvert S \rvert$. Therefore,

$$\lvert S \rvert + m(T_{5,n} - S_2) > \lvert S \rvert + m(T_{5,n} - S_2) \quad (10)$$

Hence from (9) and (10).
Let us consider any other dominating set $S_i$, with $m(T_{m,n} - S_i) < 2$, then all the components in $T_{m,n} - S_i$ are isolated. So we get $|S_i| > |S|$. Therefore,

$$|S_i| + m(T_{m,n} - S_i) > |S| + m(T_{m,n} - S)$$

(13)

Now consider other possible dominating set $S_2$ with $m(T_{m,n} - S_2) > 2$, then due to the structural nature of the graph we have $|S_2| \geq |S|$. So

$$|S_2| + m(T_{m,n} - S_2) > |S| + m(T_{m,n} - S)$$

(14)

Therefore from (13) and (14),

$$|S| + m(T_{m,n} - S) = \min \{X|X| + m(T_{m,n} - X): X \text{ is a dominating set}\} = DI(T_{m,n}).$$

Hence,

$$DI(T_{m,n}) = 6 + r + i, \text{ if } m = 7 + 3r + s, n = 2 + 3i + j \text{ where } r = N \cup \{0\}, i = N \cup \{0\} \text{ and } s = j = 0,1,2.$$  

F. Theorem

Let $|V(L_{r,s})| = r + s$. To prove this result we have to fix each value of $r$ and allow $s$ to vary from 0,1,2,....

Case (i)

For $r = 3,4,5,....$ and $s = 1$. Consider the set $S = \{v_r\}$ and $|S| = 1$, then this $S$ will be dominating set as $v_{r+1}, v_{r-1}, v_{r-2}, ..., v_1 \in N(v_r)$. Further we have $m(L_{r,s} - S) = r - 1$. So $|S| + m(L_{r,s} - S) = 1 + r - 1 = r$. Now to check the minimality of $|S| + m(L_{r,s} - S)$. Let us consider $S_1$ any other dominating set with $m(L_{r,s} - S_1) > m(L_{r,s} - S)$, then due to the nature of construction of the graph $L_{r,s}$, we have $|S_1| \geq |S|$. Therefore,

$$|S_1| + m(L_{r,s} - S_1) \geq |S| + m(L_{r,s} - S)$$

(15)

Now choose $S_2$ any other dominating set with $m(L_{r,s} - S_2) < m(L_{r,s} - S)$, then each component of $L_{r,s} - S_2$ will be isolated vertex. So $|S_2| > |S|$. Therefore,

$$|S_2| + m(L_{r,s} - S_2) > |S| + m(L_{r,s} - S)$$

(16)

Hence from (15) and (16), we have

$$|S| + m(L_{r,s} - S) = \min \{X|X| + m(L_{r,s} - X): X \text{ is a dominating set}\} = DI(L_{r,s}).$$

Case (ii)

For $r \geq 3$ and $s = 2 + 3i + j$ where $i = N \cup \{0\}$ and $j = 0,1,2$

- If $r \geq 3$ and $s = 2 + 3i + j$ where $i = N \cup \{0\}$ and $j = 0$ (i.e. for $s = 2,5,8,11,....$), consider the set
\[ S = \{ v_{r+3k} \mid k = 0, 1, \ldots, i \} \cup \{ v_r, v_s \} \] and also \[ |S| = 2 + i. \]

- If \( r \geq 3 \) and \( s = 2 + 3i + j \) where \( i = N \cup \{ 0 \} \) and \( j = 1, 2 \) (i.e. for \( s = 3, 4, 6, 7, 9, 10, \ldots \)), consider the set \[ S = \{ v_{r+3k} \mid k = 0, 1, \ldots, i \} \] and \[ |S| = 2 + i. \]

The above mentioned \( S \) will be a dominating set as \( v_{r+1}, v_{r+1}, v_{r+1}, \ldots, v_{r+1} \in N(v_r) \) and \( v_{r+3k-1}, v_{r+3k}, v_{r+3k+1} \in N(v_{r+3k}) \) where \( t = 1, 2, 3, 4, \ldots \). Moreover we have \( m(L_{r+1}, S) = r - 1 \).

Therefore, \[ |S| + m(L_{r+1}, S) = 2 + i + r - 1 = r + i + 1. \]

Now to check the minimum of \( |S| + m(L_{r+1}, S) \).

Let us consider any other dominating set \( S_1 \) with \( m(L_{r+1}, S_1) > m(L_{r+1}, S) \), then due to the nature of the structure we have \( |S_1| \geq |S| \). Therefore, \[ |S_1| + m(L_{r+1}, S_1) \geq |S| + m(L_{r+1}, S) \] \( (17) \)

Now consider another dominating set \( S_2 \) with \( m(L_{r+1}, S_2) < m(L_{r+1}, S) \), then each component in \( L_{r+1}, S_2 \) will be an isolated vertex and so \( |S_2| > |S| \). Therefore, \[ |S| + m(L_{r+1}, S) < |S_2| + m(L_{r+1}, S_2) \] \( (18) \)

From (17) and (18), we conclude that \[ |S| + m(L_{r+1}, S) = \min \{ |X| + m(L_{r+1}, S) : X \text{ is a dominating set} \} \]

\[ D_I(L_{r+1}) \]

Hence, \[ D_I(L_{r+1}) = \begin{cases} 5, & n = 2, \\ 9, & n = 3, \\ 11, & n = 4 \end{cases} \]

**G. Theorem**

\[ D_I(L(P_n[P_2])) = \begin{cases} 5, & n = 2, \\ 9, & n = 3, \\ 11, & n = 4 \end{cases} \]

**Proof**

Let \( v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_n} \) be the vertices of the path \( P_n \) and \( v_{v_1}, v_{v_2} \) be the vertices of the path \( P_2 \). Let \( L(P_n[P_2]) \) be the line graph of composition of \( P_n \) and \( P_2 \) with \( |V(L(P_n[P_2]))| = q \) (that is, \( e_1, e_2, e_3, \ldots, e_q \)) and the cardinality of edge set of \( L(P_n[P_2]) \) is \( |E(L(P_n[P_2]))| = q - 1/2 \sum (d(v_i))^2 \). We need the following three cases to prove the result.

**Case (i)**

For \( n = 2 \)

Let us consider the dominating set \( S = \{ e_1, e_2, e_3, e_4 \} \) of \( L(P_n[P_2]) \). Then \( |S| = 4 \) and \( m(L(P_n[P_2]) \ldots S) = 1 \). So \[ |S| + m(L(P_n[P_2]) \ldots S) = 5 \]. Also for \( S = \{ e_2, e_4, e_5, e_6 \} \) get \[ |S| + m(L(P_n[P_2]) \ldots S) = 5 \]. For other choice of \( S = \{ e_1, e_3 \} \) or \( S = \{ e_2, e_4 \} \), then \( m(L(P_n[P_2]) \ldots S) = 4 \) and \[ |S| + m(L(P_n[P_2]) \ldots S) = 6 \]. So \( |S| + m(L(P_n[P_2]) \ldots S) = 5 \) will be the minimum. Therefore, \( D_I(L(P_n[P_2])) = 5 \).

**Case (ii)**

For \( n = 3 \)

Consider the set \( S = \{ e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} \} \) as a dominating set of \( L(P_n[P_3]) \) and \( |S| = 7 \). Moreover \( m(L(P_n[P_3]) \ldots S) = 1 \). So \[ |S| + m(L(P_n[P_3]) \ldots S) = 9 \]. For other choice of \( S = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} \} \) and \( m(L(P_n[P_3]) \ldots S) = 5 \), then \[ |S| + m(L(P_n[P_3]) \ldots S) = 10 \]. Hence \[ |S| + m(L(P_n[P_3]) \ldots S) = 9 \] is minimum. Therefore, \( D_I(L(P_n[P_3])) = 9 \).

**Case (iii)**

For \( n = 4 \), Consider the set \( S = \{ e_2, e_4, e_6, e_8, e_{12}, e_{13} \} \) as a dominating set of \( L(P_n[P_2]) \) and \( |S| = 6 \). Then \( m(L(P_n[P_2]) \ldots S) = 5 \). So \[ |S| + m(L(P_n[P_2]) \ldots S) = 11 \]. If we consider the other dominating set \( S = \{ e_1, e_2, e_4, e_6, e_8, e_{10}, e_{12}, e_{13} \} \), then \( m(L(P_n[P_2]) \ldots S) = 6 \). So \[ |S| + m(L(P_n[P_2]) \ldots S) = 14 \]. Hence \[ |S| + m(L(P_n[P_2]) \ldots S) = 11 \] is minimum. Therefore \( D_I(L(P_n[P_2])) = 11 \).

Therefore, \[ D_I(L(P_n[P_2])) = \begin{cases} 5, & n = 2, \\ 9, & n = 3, \\ 11, & n = 4 \end{cases} \]
\[ |S_1| + m(L(P_n[P_2]) - S_2) > |S| + m(L(P_n[P_2]) - S) \quad (20) \]

From (19) and (20), we get
\[ |S| + m(L(P_n[P_2]) - S) = \min \{ |X| + m(L(P_n[P_2]) - X) : X \text{ is a dominating set} \} = D(L(P_n[P_2])) \]

Hence for \( 5 \leq n \leq 14 \),
\[ D(L(P_n[P_2])) = 15 + 4i + j, \text{ where } i = 0, 1, 2, 3, 4 \text{ and } j = 0, 1 \]

I. Theorem

For \( n \geq 15 \), \( D(L(P_n[P_2])) = n + 8i + 18 \), if \( n = 15 + 9i + 4 \sum_{p=0}^{i} p \), where \( i = N \cup \{0\} \) and also the Domination integrity for \( n = 15 + 9i + 4 \sum_{p=0}^{i} p \).

Proof

To give the result we need the following two cases.

Case (i)

If \( n = 15 + 9i + 4 \sum_{p=0}^{i} p \), where \( i = N \cup \{0\} \) (i.e., for \( n = 15, 28, 45, 66, \ldots \)), and let us consider the dominating set of \( L(P_n[P_2]) \) as
\[ S = \left\{ e_{(3+i)(i+1)}, e_{2+3(i+1)}(3+i), e_{2+3(i+1)} \right\} \quad \text{for } i = 0 \left( \frac{n}{3} + i \right) - 1 \text{ and} \]
\[ k = 0 \left( \frac{n}{3} + i \right) \left\{ e_{2n+3(i+2)}(3+i), e_{2n+3(i+2)}(3+i) \right\} \quad \text{for } i = 0 \left( \frac{n}{3} + i \right) - 2 \]
and \( |S| = n + 4i + 2 \).

Moreover \( m(L(P_n[P_2]) - S) = 10 + 4i \).

Therefore, \( |S| + m(L(P_n[P_2]) - S) = n + 8i + 18 \).

Now to analyze the minimum of \( |S| + m(L(P_n[P_2]) - S) \).

Let \( S_i \) be any other dominating set of \( L(P_n[P_2]) \) with \( |S_i| < |S| \), then due to the nature of the construction of the graph \( L(P_n[P_2]) \), we have
\[ m(L(P_n[P_2]) - S_i) > m(L(P_n[P_2]) - S) \]

such that
\[ |S| + m(L(P_n[P_2]) - S) \leq |S| + m(L(P_n[P_2]) - S_i) \quad (21) \]

Let \( S_i \) be any dominating set of \( L(P_n[P_2]) \) with \( |S_i| > |S| \), then
\[ m(L(P_n[P_2]) - S) < m(L(P_n[P_2]) - S_i) \]

So
\[ |S| + m(L(P_n[P_2]) - S) \leq |S| + m(L(P_n[P_2]) - S_i) \quad (22) \]

From (21) and (22), we get
\[ |S| + m(L(P_n[P_2]) - S) = \min \{ |X| + m(L(P_n[P_2]) - X) : X \text{ is a dominating set} \} = D(L(P_n[P_2])) \]

Hence, \( D(L(P_n[P_2])) = n + 8i + 18 \), if \( n = 15 + 9i + 4 \sum_{p=0}^{i} p \).

Case (ii)

For \( n \neq 15 + 9i + 4 \sum_{p=0}^{i} p \).

The number of terms in between \( n = 15 + 9i + 4 \sum_{p=0}^{i} p \), where \( i = N \cup \{0\} \) (i.e., \( n = 15, 28, 45, 66, \ldots \)) are \( 12 + 4i \). We can generate the four groups for the number of terms in between the \( n \). The number of members (or elements) in each group are \( \frac{12 + 4i}{4} = 3 + i \), for \( i = N \cup \{0\} \). Then the first member in each group have domination integrity value is computed by adding 3 to the previous domination integrity value and the other members of each groups have consecutive numbers as their domination integrity value.

III. CONCLUSION

The network users and designers give much more preference to its stability. Here we have analyzed the one of the parameter known as domination integrity and is used to find the vulnerability of a network. We investigated the domination integrity Tadpole, Lollipop graph and line graph formed from the composition of \( P_n \) and \( P_2 \) and we conclude that the order of the graph increases, then the domination integrity value also increases. The Domination integrity of some graph operations on tadpole, lollipop, snake and grid graphs are an open are for investigation.

REFERENCES