On Equitable Irregular graphs

P. Sanakara Narayanan, S. Saravanakumar

***Abstract**: An k−edge-weighting of a graph G = (V,E) is a map $\varphi: E(G) \to \{1,2,3,...k\}$, }where $k \geq 1$ is an integer. Denote $S_\varphi(v)$ is the sum of edge-weights appearing on the edges incident at the vertex v underφ . An k-edge -weighting of G is equitable irregular if $|S_\varphi(u) - S_\varphi(v)| \leq 1$, for every pair of adjacent vertices u and v in G. The equitable irregular strength $S_e(G)$ of an equitable irregular graph G is the smallest positive integer k such that there is a k-edge weighting of G. In this paper, we discuss the equitable irregular edge-weighting for some classes of graphs.*

Keywords **:Edge-weighting, equitable irregular graphs.**

I. INTRODUCTION

A graph labeling is an assignment of integers to the vertices or edges or both with respect to some conditions. A detailed survey of graph labeling is given by Gallian [4]. This paper considering an edge labeling of graphs. For our convenience, we call the term edge - weighting instead of edge - labeling. An k-edge weighting is a map $\varphi : E(G) \to \{1,2,\dots,k\}$, where $k \in N$N is equitable irregular if $|S_\varphi(u) - S_\varphi(v)| \leq 1$ for every pair of adjacent vertices u and v in G, where $S_\varphi(x)$is the sum of the edge - weights presenting on the edges incident at the vertex *x*. A graph admits such a labeling is called an equitable irregular. This notion was introduced by I. Sahul Hamid and S. Ashok Kumar in [3]. In that paper, the authors were discussed some properties of equitable irregular graphs and provided some classes of equitable irregular graphs along with its strength. In this paper, we extend the study of this parameter by proving closed helm graphs, windmill graphs, flower graphs, quadrilateral snake and double quadrilateral snake graphs are equitable irregular. Moreover, we determine the exact value of the strength for each of these classes of graphs. For this, we need the following theorem which is proved in [3].

Theorem 1.1. If G is equitable irregular graph, then$S_e(G) \geq \left\lceil \frac{\Delta(G)-2}{\mu(G)-1} \right\rceil$, where $\mu(G) = \min\{\mu_x : x \epsilon V(G)\}$, $deg\ x = \Delta(G)$and$\mu(x) = min\{\deg y : xy \epsilon E(G)\}$.

II. MAIN RESULTS

In this section, we prove that the closed helm graphs, windmill graphs, flower graphs, quadrilateral and double quadrilateral graphs are equitable irregular. Moreover, we compute the irregularity strength of these families.

Definition 2.1. A *closed helm*CH_n is the graph obtained by taking a helm H_n and adding edges between the pendant vertices to form a cycle.

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Theorem 2.2. For all *n*, the graph CH_n is equitable irregular and $S_e(CH_n) = \left\lceil \frac{n-2}{3} \right\rceil$.

Proof. Let $V(CH_n) = \{v_0, v_1, v_1', v_2, v_2', \dots, v_n, v_n'\}$ and $E(CH_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_iv_i' : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}' : 1 \leq i \leq n-1\} \cup \{v_1v_n, v_1'v_n'$. Define an edge-weighting φ of CH_n as follows.

Let $\varphi(v_0v_i) = 1$ for all $1 \leq i \leq n$ and assign $\left\lceil \frac{n-2}{3} \right\rceil$ forall the remaining edges of CH_n. (For the graph CH_n, the edge-weighting φ is illustrated in Figure 1).

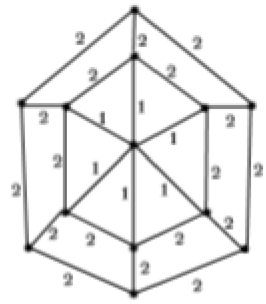

Figure 1: The graph CH_6 and its irregular edge-weighting

Then $S_\varphi(v_0) = n$ and for all $1 \leq i \leq n$, we have $S_\varphi(v_0) = \varphi(v_0v_i) + \varphi(v_iv_i') + \varphi(v_iv_{i+1}) + \varphi(v_iv_{i-1})$

$$= \begin{cases} n+1 & if\ n \equiv 0(mod3) \\ n & if\ n \equiv 1(mod3) \\ n-1 & if\ n \equiv 2(mod3) \end{cases}$$

Further, $S_\varphi(v_i') = \varphi(v_iv_i') + \varphi(v_i'v_{i+1}') + \varphi(v_i'v_{i-1}')$

$$= \begin{cases} n & if\ n \equiv 0(mod3) \\ n-1 & if\ n \equiv 1(mod3) \\ n-2 & if\ n \equiv 2(mod3) \end{cases}$$

It is clear that, the difference of the weights of any two adjacent vertices of CH_n under φ is at most 1. Hence CH_n is equitable irregular and so $S_e(CH_n) \leq \left\lceil \frac{n-2}{3} \right\rceil$. Sinceμ$(CH_n) = 4$ and $\Delta(CH_n) = n$, it follows by Theorem 1.1, we get $S_e(CH_n) \geq \left\lceil \frac{n-2}{3} \right\rceil$. Thus $S_e(CH_n) = \left\lceil \frac{n-2}{3} \right\rceil$.

Definition 2.3. The *windmill graph* $W_n^{(m)}$ is the graph obtained by taking *m*-copies of the complete graph $K_n$with a vertex in common.

Theorem 2.4. For all *n* and *m*, the windmill graph$W_n^{(m)}$ is equitable irregular.

Proof: Let $V(W_n^{(m)}) = \{v_0, v_{11}, v_{12,} v_{13}, \dots, v_{1(n-1)}, v_{21}, v_{22,} v_{23}, \dots, v_{2(n-1)} \dots, v_{m1}, v_{m2,} v_{m3}, \dots, v_{m(n-1)}\}$ and$E(W_n^{(m)}) = \{E(G_i) : 1 \leq i \leq m\} \cup \{v_0v_{ij} : 1 \leq i \leq n, 1 \leq j \leq n-1\}$,

$V(G_i) = \{v_{i1}, v_{i2,} v_{i3}, \dots, v_{i(n-1)} : 1 \leq i \leq m\}$.

We now assign the weights to the edges of ${W_n}^{(m)}$ as follows. Let $\varphi(v_0 v_{ij}) = 1$ for all $i = 1,2,3 \dots m$ and $j = 1,2,3, \dots, n-1$. Further assign the weights $\left\lceil \frac{mn-m-2}{n-2} \right\rceil$ to all the remaining edges of ${W_n}^{(m)}$.(For the graph ${W_4}^{(3)}$ the edges-weighting φ is illustrated in Figure 2). Then edge weighting of ${W_n}^{(m)}$ as follows.

For all $1 \le i \le m$and $1 \le i \le n-1$, we have $S_\varphi(v_{ij}) = 1 + (n-2) \left\lceil \frac{mn-m-2}{n-2} \right\rceil$ and $S_\varphi(v_0) = m(n-1)$,

$$S_\varphi(v_{nm}) = \begin{cases} m(n-1)+1 & if\ n,m\ are\ odd \\ m(n-1) & if m\ is\ odd, n\ is\ even \\ m(n-1)-1 & otherwise \end{cases}$$

Clearly for any two adjacent vertices of ${W_n}^{(m)}$, their weights are differ by at most one and hence ${W_n}^{(m)}$is equitable irregular. Thus $S_e({W_n}^{(m)}) \le \left\lceil \frac{mn-m-2}{n-2} \right\rceil$.

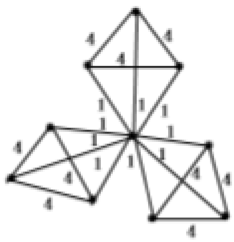

Figure 2: The graph $W_4^{(3)}$ and its irregualr edge weighting

Since $\Delta({W_n}^{(m)}) = (n-1)m$ and $\mu({W_n}^{(m)}) = n-1$, it follows from Theorem 1.1 we get $S_e({W_n}^{(m)}) \ge \left\lceil \frac{mn-m-2}{n-2} \right\rceil$. Thus $S_e({W_n}^{(m)}) = \left\lceil \frac{mn-m-2}{n-2} \right\rceil$.

Definition 2.5. A *flower graph*Fl_n is the graph obtained from a closed helm by joining each pendant vertex of the helm toits center vertex.

Theorem 2.6. The flower graph Fl_n is equitable irregular for all n and $S_e(Fl_n) = \left\lceil \frac{2n}{3} \right\rceil$.

Proof: Let $V(Fl_n) = \{v_0, v_1, v_1', v_2, v_2', \dots, v_n, v_n'\}$ and $E(Fl_n) = \{v_0 v_i : 1 \le i \le n\} \cup \{v_i v_i' : 1 \le i \le n\} \cup \{v_i v_{i+1} : 1 \le i \le n-1\} \cup \{v_i v_{i+1}' : 1 \le i \le n-1\} \cup \{v_0 v_i' : 1 \le i \le n\} \cup \{v_1 v_n, v_1' v_0'\}$.

Now, let us define an edge-weighting φ of Fl_nas follows. For all $1 \le i \le n$, let $\varphi(v_0 v_i) = \varphi(v_0 v_i') = 1$ and assign $\left\lceil \frac{2n}{3} \right\rceil$ to all remaining edges of Fl_n. Then $S_\varphi(v_0) = 2n$ and for all $1 \le i \le n$, we have

$$S_\varphi(v_i) = S_\varphi(v_i') = \begin{cases} 2n & if\ n \equiv 2(mod3) \\ 2n+1 & if\ n \equiv 0(mod3) \\ 2n-1 & if\ n \equiv 1(mod3) \end{cases}$$

One can easily verify that the difference of the weights of any two adjacent vertices of Fl_n is at most 1 and hence Fl_n is equitable irregular. Thus $S_e(Fl_n) \le \left\lceil \frac{2n}{3} \right\rceil$. Here $\mu(Fl_n) = 4$and $\Delta(Fl_n)$ = 2n, we get $S_e(Fl_n) \ge \left\lceil \frac{2n}{3} \right\rceil$by Theorem1.1. Hence $S_e(Fl_n) = \left\lceil \frac{2n}{3} \right\rceil$.

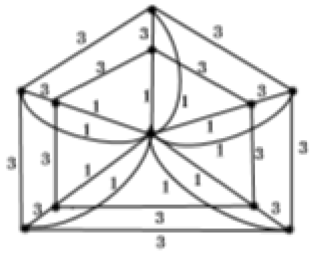

Figure 3: The graph Fl_5 and its irregular edge weighting

Definition 2.7. The *Lotus inside a circle*LC_n is a graph obtained from the cycle C_n:$(u_1, u_2, \dots, u_m, u_1)$and star $K_{1,n}$ with center vertex $v_0$and end vertices $v_1, v_2, \dots, v_n$ by joining each $v_i$to $u_i$and $u_{i+1}(mod n)$.

Theorem 2.8. The Lotus inside a circle graph LC_n is equitable irregular for all *n* andS$_e$((LC_n) =$\left\lceil \frac{n-1}{2} \right\rceil$.

Proof: Let $V(LC_n) = \{v_0, v_1, u_1, v_2, u_2, \dots, v_n, u_n\}$ and $E(LC_n) = \{v_0 v_i : 1 \le i \le n\} \cup \{v_i u_i : 1 \le i \le n\} \cup \{v_i u_{i+1} : 1 \le i \le n-1\} \cup \{v_n u_1\} \cup \{u_i u_{i+1}' : 1 \le i \le n-$ *1∪unu1.*

Now, let us define an edge-weighting φ of LC_n as follows. $\varphi(v_0 v_i) = \varphi(u_i u_{i+1}) = \varphi(u_1 u_n) = 1$ for all $1 \le i \le n$ and assign $\left\lceil \frac{n-1}{2} \right\rceil$ to all the remaining edges of LC_n. (Forthe graph LC_5 , the equitable irregular edge-weighting φ is illustrated in Figure 4.)

Clearly, $S_\varphi(v_0) = S_\varphi(u_n) = n$ and

$$S_\varphi(v_n) = \begin{cases} n & if\ n\ is\ odd \\ n-1 & if\ n\ is\ even \end{cases}$$

It is easy to observe that the weights of any two adjacent vertices of $LC_n$are differ by at most 1. Thus LC_n is equitable irregular.

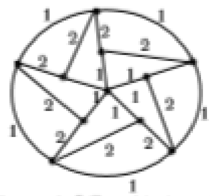

Figure 4: The graph LC_5 and its irregular edge weighting

Also,μ(LC_n) = 3 and Δ(LC_n) = *n*, by Theorem 1.1 that S$_e$(LC_n) ≥ $\left\lceil \frac{n-1}{2} \right\rceil$and the edge-weighting φ deduces that S$_e$(LC_n) ≤ $\left\lceil \frac{n-1}{2} \right\rceil$. Hence S$_e$ (LC_n) =$\left\lceil \frac{n-1}{2} \right\rceil$

Definition 2.9. A *quadrilateral snake*Q_n is obtained from a path $u_1, u_2, \dots, u_m$ by joining $u_i$and u_{i+1} to new vertices v_i and $w_i$respectively and joining the verticesv_i and $w_i$for $i = 1,2,\dots, n-1$.

Theorem 2.10. The Quadrilateral snake Q_n is equitable irregular for all *n* and $S_e(Q_n)$ = 2.

Proof : Let V (Q_n) = $\{u_1, u_2, u_3, \dots, u_{n+1}\}$ ∪ $\{v_i w_i :$ $1 \le i \le n-1\}$ and E(Q_n) = $\{u_i u_{i+1} : 1 \le i \le n\}$ ∪$\{u_i v_i, v_i w_i, w_i u_{i+1} : 1 \le i \le n-1\}$. Define an edge weighting φ of Q_n as follows.

Let $\varphi(u_i u_{i+1}) = \varphi(w_i u_{i+1}) = 1$ and $\varphi(u_i v_i) = 1$ for all $2 \le i \le n-1$ and assign the label 2 to all the remaining edges of Q_n.Then the weight of any vertex of Q_n is either 3 or 4.

Therefore, the weights of any two adjacent vertices of Q_n differ by at most 1 and so Q_n is equitable irregular. (For the graph Q_3 , the equitable irregular edge-weighting φ is illustrated in Figure 5). Hence $S_e(Q_n) \leq 2$.

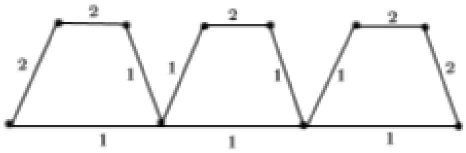

Figure 5: The graph Q_3 and its irregular edge weighting

Since $\mu(Q_n) = 2$ and $\Delta(Q_n) = 4$, it follows by Theorem 1.1, $S_e(Q_n) \geq 2$.Thus $S_e(Q_n) = 2$.

Definition 2.11. A *Double Quadrilateral snake* DQ_n is a graph consists of two quadrilateral snakes that have a common path.

Theorem 2.12. The Double Quadrilateral snake DQ_n is equitable irregular for all *n* and $S_e\ (DQ_n) = 4$.

Proof. Let V $(DQ_n) = \{u_1, u_2, u_3, \dots, u_{n+1}\} \cup \{v_i, w_i, x_i, y_i : 1 \leq i \leq n\}$and E $(DQ_n) = \{u_iu_{i+1}, u_iv_i, v_iw_i, w_iu_{i+1}, u_ix_i, x_iy_i, y_iu_{i+1} : 1 \leq i \leq n\}$.

Now, let us define an edge weighting φ of DQ_nas follows. Let $\varphi(u_1v_1) = \varphi(u_1x_1) = \varphi(u_{n+1}w_n) = \varphi(u_{n+1}y_n) = 2$ and $\varphi(v_iw_i) = \varphi(x_iy_i) = 4$ for all $1 \leq i \leq n$.

Further, assign the label 1 to all the remaining edges of DQ_n.

Then $S_\varphi(u_1) = S_\varphi\ (u_{n+1}) = 5$; $S_\varphi(v_1) = S_\varphi\ (x_1) = S_\varphi\ (w_n) = S_\varphi\ (y_n) = 6$ and for all $1 \leq i \leq n - 1$, we have $S_\varphi\ (y_i) = S_\varphi\ (w_i) = 5$ and for all $2 \leq i \leq n$, $S_\varphi\ (u_i) = 6$.

Clearly, the weights of any two adjacent vertices of $DQ_n$are differ by atmost 1. Therefore DQ_n is equitable irregular and hence $S_e\ (DQ_n) \leq 4$.

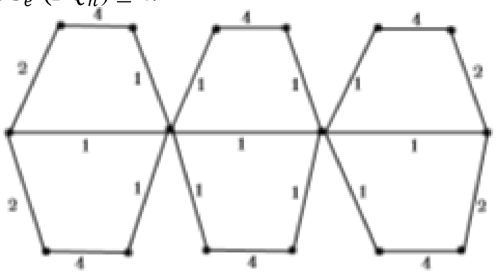

Figure 6: The graph DQ_3 and its irregular edge weighting

Since $\mu(DQ_n) = 2$ and $\Delta(DQ_n) = 6$, this implies $S_e\ (DQ_n) \geq 4$ by Theorem 1.1. and thus $S_e\ (DQ_n) = 4$.

III. CONCLUSION

In this paper, wehave proved some special classes of graphs such as closed helm graphs, windmill graphs, flower graphs, quadrilateral snakes and double quadrilateral snakes are equitable irregular.

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