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On the Open Packing Number of a Graph

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***Abstract:** A non-empty set $S \subseteq V(G)$ of a graph G is an open packing set of G if no two vertices of S have a common neighbor in G. The maximum cardinality of an open packing set is the open packing number of G and is denoted by $\rho^o(G)$. An open packing set of cardinality $\rho^o(G)$ is a $\rho^o(G)$-set of G. In this paper, the classes of trees and unicyclic graphs for which the value of ρ^c is either 2 or 3 are characterized. Moreover, the exact values of the open packing number for some special classes of graphs have been found.*

***Keywords :** open packing number, trees, unicyclic graphs*

I. INTRODUCTION

Graphs are considered in this article is finite, simple and undirected graph. For graph theoretic terminology, we refer the book by Chartrand and Lesniak [1].

A set S of vertices of G is an *open packing set* of G if no two vertices of S have a common neighbor in G. The *lower open packing number* of G, denoted $\rho^o{}_L(G)$, is the minimum cardinality of a maximal open packing set of G while the *open packing number* of G, denoted $\rho^o(G)$, is the maximum cardinality among all open packing sets of G. An open packing set of G with cardinality $\rho^o{}_L(G)$ is called a $\rho^o{}_L(G)$ -set and similarly a ρ^c -set is defined. These parameters were introduced in [2] and further studied in [3], [4] and [5]. Before we enter into the main section, first we need the following result.

Theorem 1.1. Let C_n be a cycle on $n \geq 3$ vertices. Then $\rho^o(C_n) = \begin{cases} \frac{n}{2} - 1 & if\ n \equiv 2(mod\ 4) \\ \left\lceil \left\lfloor \frac{n}{2} \right\rfloor \right\rceil & otherwise. \end{cases}$

II. MAIN RESULTS

In this section we characterize the connected graphs for which the value of the open packing number is 2. Moreover, we obtain the structural characterization of trees and unicyclic graphs with the open packing number 2 or 3. A pair of vertices *(u, v)* is an *injective pair* in *G* if the vertices *u* and *v* have no common neighbor in *G*.

Theorem 2.1. Let G be a connected graph of order at least 3. Then $\rho^o(G) = 2$ if and only if the following are hold in G.

(i) G has at least one injective pair.
(ii) for each $x \in V(G) - \{u, v\}$ either (*u*,*x*) or (*v*,*x*) is not an injective pair in *G*, where (*u*,*v*) is an injective pair in *G*.

Proof: Assume that G satisfying the conditions (i) and (ii). Let (*u*,*v*) be an injective pair in G. Then the set {*u*,*v*} is an open packing set of G and so $\rho^o(G) \geq |S| = 2$. On the other hand, if there is an open packing set *D* of *G* such that |*D*|>2 then *D* contains an injective pair, say (*u*,*v*). Now, let $x \in D$ be such that $x \notin \{u, v\}$. Then both (*x*,*u*) and (*x*,*v*) are the injective pairs in G. This violates the condition (ii) and hence $\rho^o(G) = 2$.

Conversely, suppose $\rho^o(G) = 2$. Let S={u,v} be a $\rho^o - set$ of *G*. Then (*u*,*v*) is an injective pair in G and so the condition (i) follows. Further, if there is a vertex $x \in V(G) - \{u, v\}$ such that both (*x*,*u*) and (*x*,*v*) are injective pairs in *G*, then the set {*u*,*v*,*x*} will become an open packing set of *G* and so $\rho^o(G) \geq 3$. This contradiction yields *(ii).*

The following theorems characterize the trees and unicyclic graphs for which the value of the open packing number is 2.

Theorem 2.2. Let T be a tree of order at least 2. Then $\rho^o(T) = 2$ if and only if T is either a star or a double star.

Proof: Assume that $\rho^o(T) = 2$. If $diam(T) \geq 4$, then there is a path, say $P = (v_1, v_2, v_3, \ldots, v_k)$ such that $k \geq 5$ in *T*. Certainly, the set $\{v_1, v_2, v_k\}$ is an open packing set of *T* and so $\rho^o(T) \geq |S| = 3$, produces a contradiction. Thus $diam(T) \leq 3$ and hence *T* is either a star or a double star. Converse is just a verification.

Theorem 2.3. Let *G* be a connected unicyclic graph of order at least 4. Then $\rho^o(G) = 2$ if and only if G is either C_4 or, C_5 or, C_6 or, isomorphic to one the graphs G_i $(1 \leq i \leq 6)$ *as* shown in Figure 1.

Proof: Let C be the cycle in G. If G=C, then *G* is C_4 or, C_5 or, C_6 follows from Theorem 1.1. Assume that $G \neq C$. If G contains the cycle of length 5 or more, then we can find a vertex *v* from the outside of C such that there exist two adjacent vertices on *C* which have distance at least 3 apart from *v* in G.

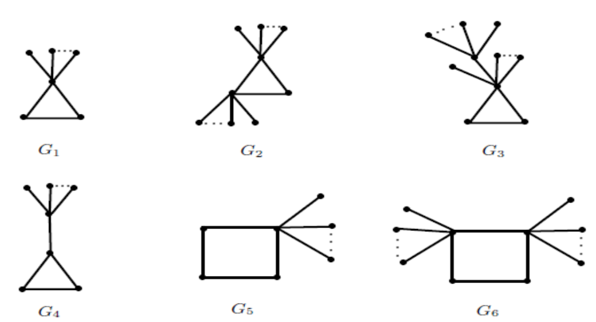

Figure 1: Unicyclic graphs G with $\rho^o(G) = 2$

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Certainly, we have $\rho^o(G) \geq 3$ and thus the length of C is either 3 or 4. Now, we prove the result in the following cases.

Case i. C=C$_3$

Let $V(C)$ ={x,y,z} and let $G_1 = \langle V(G) - V(C) \rangle$. We first claim that each component of G$_1$ is either K$_1$ or a star graph *H*. Suppose G_1 has a path of length 3, say $P = (v_1, v_2, v_3, v_4)$. Then exactly one vertex of *P* is adjacent with the vertex of *C* as *G* is unicyclic. If either v_1 or v_2 is adjacent with *x*, then the set {v_3,v_4,y} will forms an open packing set of G and this is a contradiction to $\rho^o(G) = 2$. On the other hand, if either v_3 or v_4 is adjacent with x, then the set {v_1 ,v_2, z} will become an open packing set of G, again we get a contradiction to $\rho^o(G) = 2$. Thus every component of G$_1$ is isomorphic to either K$_1$ or a star graph *H*. Also, it is clear that G$_1$ has at most one star graph H as its component. If G_1 is the star graph *H*, then the center vertex of *H* is adjacent with exactly one of the vertices of *C* as *G* is unicyclic and consequently G=G_4. Suppose every component of G_1 is K_1. Then either one vertex or two vertices on *C* will become the support vertices of *G*. Otherwise, each vertex of *C* is a support vertex of *G* and so one pendant neighbor of each support vertex of *G* together forms an open packing set of G which in turn implies that $\rho^o(G) \geq 3$. This is a contradiction and hence *G* is isomorphic to either G_1 or G_2. Finally, assume that exactly one component of G_1 is the star graph *H* and all the remaining components are K_1. If the center vertex *u* of *H* is adjacent with the vertex *x*, then *deg y* = *deg z* =2; for otherwise, there is a vertex *w* is adjacent with exactly one of the vertices *y* and *z* in G and so the set {*u*,*v*,*w*} is an open packing set of *G*, where $v \in N(u)$ in *H*, which is a contradiction to $\rho^o(G) = 2$. Thus, in this case G=G_3.

Case ii. C=C$_4$

Let $V(C)$ = {w, x ,y, z} and let $G_2 = \langle V(G) - V(C) \rangle$. In this case, we prove that G$_2$ has no edges and at most two vertices of *C* are support vertices in *G*; this shows that G is isomorphic to either G$_4$ or G$_5$. On the contrary, let $u, v \in V(G_2)$ be such that u and v are adjacent in G$_2$. Since G is unicyclic, exactly one of the vertices *u* and *v* is adjacent with the vertex on the cycle C. Without loss of generality, assume that u is adjacent with the vertex x in G. Certainly, v is adjacent with none of the vertices *w*,*x*,*y* and *z* and so (*v*,*w*) and (*v*,*z*) are the injective pairs in G. This implies that the set {v,w,z} is an open packing set of G and so $\rho^o(G) \geq 3$, producing a contradiction. Thus every vertex lying outside of C in G is a pendant vertex. Further, if more than two vertices on *C* become the support vertices of G, then one pendant neighbor of each support vertex of G together forms an open packing set of G and hence the value of $\rho^o(G)$ exceeds 2. This is a contradiction and hence G is isomorphic to either G$_4$ or G$_5$.

In view of Theorem 2.2 that the value of $\rho^o(T)$ for a tree T with diameter is less than or equal to 3 is two, whereas trees with diameter 4 can assume arbitrarily large value of ρ^o as shown in the following proposition.

Proposition 2.4. If T is a tree with *diam(T)* =4, then $\rho^o(T) \geq 3$. Further, for given any integer $k \geq 3$, there is a tree T with *diam(T)*=4 and $\rho^o(T) = k$.

Proof: Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a diametral path in T. Then {v$_1$,v$_2$,v$_5$} becomes an open packing set of T and thus $\rho^o(T) \geq 3$.

Suppose $k \geq 3$ is a given integer. Now, we construct a tree T with *diam(T)*=4 and $\rho^o(T) = k$ as follows. Consider a star K$_{1,k-1}$ with the center vertex *v* and $V(K_{1,k-1}) = \{v, v_1, v_2, v_3, \dots, v_{k-1}\}$. Subdivide each edge of K$_{1,k-1}$ exactly once and let T be the resultant tree. For *k=6*, the tree T is shown in Figure 2.

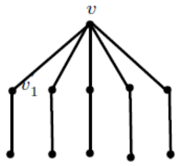

Figure 2: A tree T with $\rho^o(T) = 6$

Obviously, *diam(T)*=4 and the set S={ $v_1', v_1, v_2, v_3, \dots, v_{k-1}$} is an open packing set of T. Thus $\rho^o(T) \geq |S| = k$. On the other hand, let D be a maximal open packing set of T. Then D contains at most *k-1* pendant vertices of T and exactly one support vertex form T and so $\rho^o(T) \leq |D| = (k-1) + 1 = k$.

Next we proceed to characterize the trees T for which $\rho^o(T) = 3$. A *caterpillar* T is a tree whose removal of pendant vertices results in a path and this resulting path is called *the spine* of the caterpillar, denoted by SP(T).

Theorem 2.5. Let *T* be a tree of order at least 5. Then $\rho^o(T) = 3$ if and only if *diam(T)*=4 and *T* is a caterpillar with length of the spine of *T* is 2.

Proof: Suppose $\rho^o(T) = 3$. If $diam(T) \geq 5$, then we can find a path $P = (v_1, v_2, v_3, \dots, v_k)$ in T, where $k \geq 6$. Certainly, the set S={v_1, v_2, v_5, v_6} forms an open packing set of T and so $\rho^o(T) \geq 4$. This is a contradiction and hence $diam(T) \leq 4$. Now, Theorem 2.2 implies that $diam(T) = 4$.

Let S={x,y,z} be a $\rho^o - set$ of *T*. Since no two vertices of S have a common neighbor in T, it follows that at most one pair of vertices in S are adjacent in T. Without loss of generality, assume that x and y are adjacent in T. Since T is a tree, there is a unique path P' from y to z. Suppose P' contains the vertex x (See Figure 3).

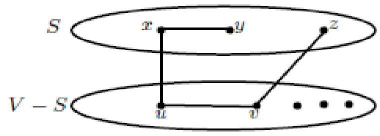

Figure 3: A step in the proof of Theorem 2.5.

As *diam(T)*=4, the vertex z has no neighbor in V-S except v and the vertex y has no neighbor in V-S; and therefore every vertex in $(V - S) - \{u, v\}$ is adjacent with exactly one of the vertices x, u and v. Thus T is a caterpillar with spine $\langle x, u, v \rangle$.

If P' does not contain the vertex *x*, then the length of P' should be 3 as diam(T)=4.

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Now, let us take $P' = \langle y, u, v, z \rangle$. (see Figure 4). Since diam(T)=4, every vertex in $(V - S) - \{u, v\}$ is adjacent with exactly one of the vertices y, u and v. Hence T is a caterpillar and in this case $\langle y, u, v \rangle$ is the spine of T.

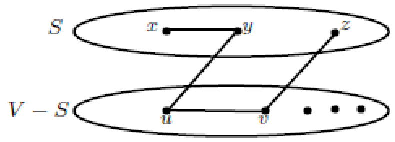

Figure 4: A step in the proof of Theorem 2.5.

Conversely, suppose T is a caterpillar with the spine $SP(T) = \langle x, y, z \rangle$. Since *diam*(*T*)=4, both the vertices *x* and *z* have at least one pendant neighbor in T. Let x' and z' be the pendant neighbors of x and z respectively in T. Certainly, the set $\{x', u, z'\}$ forms an open packing set of T and so $\rho^o(T) \geq 3$. Also, since $V(T) = N[x] \cup N[y] \cup N[z]$ and any maximal open packing set D of T can contains at most one vertex from each of the closed neighborhoods of the vertices on the spine of T, it follows that $\rho^o(T) \leq |D| \leq 3$ and hence $\rho^o(T) = 3$.

III. SPECIAL CLASSES OF GRAPHS

In this section we determine the exact value of the open packing number for some special classes of graphs such as triangular snakes, quadrilateral snakes, helm and closed helm graphs.

Definition 3.1. A triangular snake TS_n is obtained from the path $P = (v_1, v_2, v_3, \ldots, v_{n+1})$ by joining each v_i and v_{i+1} to a new vertex u_i for each $1 \leq i \leq n$. A quadrilateral snake QS_n is obtained from the path $P = (v_1, v_2, v_3, \ldots, v_{n+1})$ by introducing n copies of K_2, say $u_1u_1', u_2u_2', \ldots, u_nu_n'$ and joining u_i to v_i and u_i' to v_{i+1}, for all *i*=1,2,3,…,*n*.

Theorem 3.2. For the triangular snake TS_n, we have $\rho^o(TS_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof: Consider the set $S = \{u_1, u_3, u_5 \ldots, u_k\}$, where k is an odd integer and $1 \leq k \leq n$. It is obvious that the distance between any two vertices of S is at least 3 and thus no two vertices of S have a common neighbor in TS_n. For example, the graph TS_5 and the vertices (blue colored) forms an open packing set of TS_5 illustrated in Figure 5. Thus S is an open packing set of TS_n and hence have $\rho^o(TS_n) \geq |S| = \left\lceil \frac{n}{2} \right\rceil$.

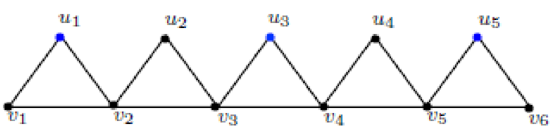

Figure 5: The graph TS_5

On the other hand, let D be any maximal open packing set of TS_n. Then D contains at most one vertex from any two adjacent blocks of TS_n, it follows that have $|D| \leq \frac{n}{2}$ or $\frac{n+1}{2}$ according as *n* is even or odd. Hence $\rho^o(TS_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.3. For the quadrilateral snake QS_n, we have $\rho^o(QS_n) = n + 1$.

Proof. It is obvious that the set $S = \{v_1, u_1, u_2', u_3', \ldots, u_n'\}$ is an open packing set of QS_n and hence $\rho^o(QS_n) \geq |S| = n + 1$. For instance, the graph QS_5 and the vertices (blue colored) forms an open packing set of QS_5 is given in Figure 6.

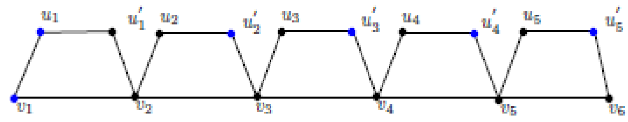

Figure 6: The graph QS_5

Let *D* be a maximal open packing set of QS_n. If *D* contains no vertex from the path *P*, then *D* has two vertices in exactly one edge u_iu_{i+1} for *i*=1,2,3,…,*n-1* and at most one vertex from the remaining edges of QS_n. This implies that $|D| \leq n + 1$. Similarly, one can easily get $|D| \leq n + 1$ when D contains some vertices of P. Hence $\rho^o(QS_n) = n + 1$.

Theorem 3.4. (i) If H_n is a helm graph, then $\rho^o(H_n) = n - 1$. (ii) If CH_n is a closed helm graph, then $\rho^o(CH_n) = \left\lceil \frac{n}{2} \right\rceil$.

***Proof:* (i)** It is clear that , the set S consisting of all pendant vertices of H_n will forms an open packing set of H_n and so we have $\rho^o(H_n) = n - 1$. Further, one can easily verify that *S* is the only open packing set of H_n with maximum cardinality and thus $\rho^o(H_n) = n - 1$. Hence (i) is proved.
(ii) It is observe that any open packing set of CH_n can have at most $\left\lceil \frac{n-1}{2} \right\rceil$ vertices from the outer cycle and at most one vertex from the rim of the wheel of CH_n, this implies that $\rho^o(CH_n) \leq \left\lceil \frac{n}{2} \right\rceil$.
For another inequality, let
$V(CH_n) = \{v_0, v_1, v_1', v_2, v_2', \ldots, v_n, v_n'\}$ and
$E(CH_n) = \{v_0v_i': 1 \leq i \leq n\}$
$\cup \{v_iv_i': 1 \leq i \leq n\} \cup \{v_iv_{i+1}': 1 \leq i \leq n-1\} \cup$
$\{v_iv_{i+1}: 1 \leq i \leq n-1\} \cup \{v_1v_n, v_1'v_n'\}$
. Now, we produce an open packing set of CH_n with cardinality $\left\lceil \frac{n}{2} \right\rceil$ in the following cases.

Case (i). $n \equiv 0 (mod\ 4)$
Consider the set $S = \{v_1, v_2, v_5, v_6, \ldots, v_{n-3}, v_{n-2}\}$. Obviously, no two vertices of S have a common neighbor in CH_n and so S is an open packing set of CH_n. Thus $\rho^o(CH_n) \geq |S| = \frac{n}{2}$.

Case (ii). $n \equiv 1\ or\ 2 (mod\ 4)$
Consider the set
$D = \{v_1, v_2, v_5, v_6, \ldots, v_{n-4}, v_{n-3}\} \cup \{v_{n-1}'\}$. One can easily verify that D forms an open packing set of CH_n and thus $\rho^o(CH_n) \geq |S| = \left\lceil \frac{n}{2} \right\rceil$.

Case (iii). $n \equiv 3 (mod\ 4)$
Consider the set $D = \{v_1, v_2, v_5, v_6, \ldots, v_{n-2}\} \cup \{v_{n-2}'\}$. It is observe that any two vertices of D have no common neighbor in CH_n and so D is an open packing set of CH_n. Therefore, $\rho^o(CH_n) \geq |S| = \left\lceil \frac{n}{2} \right\rceil$.

In each case we have produced an open packing set of CH_n with cardinality at least $\left\lceil \frac{n}{2} \right\rceil$ and hence $\rho^o(CH_n) = \left\lceil \frac{n}{2} \right\rceil$. Thus (ii) is proved.

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