On the Open Packing Number of a Graph

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Abstract: A non-empty set $S \subseteq V(G)$ of a graph $G$ is an open packing set of $G$ if no two vertices of $S$ have a common neighbor in $G$. The maximum cardinality of an open packing set is the open packing number of $G$ and is denoted by $\rho^0(G)$. An open packing set of cardinality $\rho^0(G)$ is a $\rho^0(G)$-set of $G$. In this paper, the classes of trees and unicyclic graphs for which the value of $\rho^0$ is either 2 or 3 are characterized. Moreover, the exact values of the open packing number for some special classes of graphs have been found.

Keywords: open packing number, trees, unicyclic graphs

I. INTRODUCTION

Graphs are considered in this article, finite, simple and undirected graph. For graph theoretic terminology, we refer the book by Chartrand and Lesniak [1].

A set $S$ of vertices of $G$ is an open packing set of $G$ if no two vertices of $S$ have a common neighbor in $G$. The lower open packing number of $G$, denoted $\rho^0_1(G)$, is the minimum cardinality of a maximal open packing set of $G$ while the open packing number of $G$, denoted $\rho^0(G)$, is the maximum cardinality among all open packing sets of $G$. An open packing set of $G$ with cardinality $\rho^0_1(G)$ is called a $\rho^0_1(G)$-set and similarly a $\rho^0$-set is defined. These parameters were introduced in [2] and further studied in [3], [4] and [5]. Before we enter into the main section, first we need the following result.

Theorem 1.1. Let $C_n$ be a cycle on $n \geq 3$ vertices.

Then $\rho^n(C_n) = \begin{cases} \frac{n}{2} - 1 & \text{if } n \equiv 2(\mod 4) \\ \frac{n}{2} & \text{otherwise.} \end{cases}$

II. MAIN RESULTS

In this section we characterize the connected graphs for which the value of the open packing number is 2. Moreover, we obtain the structural characterization of trees and unicyclic graphs with the open packing number 2 or 3. A pair of vertices $(u, v)$ is an injective pair in $G$ if the vertices $u$ and $v$ have no common neighbor in $G$.

Theorem 2.1. Let $G$ be a connected graph of order at least 3. Then $\rho^0(G) = 2$ if and only if the following are hold in $G$.

(i) $G$ has at least one injective pair.

(ii) for each $x \in V(G) - \{u, v\}$ either $(u, x)$ or $(v, x)$ is not an injective pair in $G$, where $(u, v)$ is an injective pair in $G$.

Proof: Assume that $G$ satisfying the conditions (i) and (ii). Let $(u, v)$ be an injective pair in $G$. Then the set $\{u, v\}$ is an open packing set of $G$ and so $\rho^0(G) \geq |S| = 2$. On the other hand, if there is an open packing set $D$ of $G$ such that $|D| \geq 2$ then $D$ contains an injective pair, say $(u, v)$. Now, let $x \in D$ be such that $x \notin \{u, v\}$. Then both $(x, u)$ and $(x, v)$ are the injective pairs in $G$. This violates the condition (ii) and hence $\rho^0(G) = 2$.

Conversely, suppose $\rho^0(G) = 2$. Let $S = \{u, v\}$ be a $\rho^0$-set of $G$. Then $(u, v)$ is an injective pair in $G$ and so the condition (i) follows. Further, if there is a vertex $x \in V(G) - \{u, v\}$ such that both $(x, u)$ and $(x, v)$ are injective pairs in $G$, then the set $\{u, v, x\}$ will become an open packing set of $G$ and so $\rho^0(G) \geq |S| = 3$. This contradiction yields (ii).

The following theorems characterize the trees and unicyclic graphs for which the value of the open packing number is 2.

Theorem 2.2. Let $T$ be a tree of order at least 2. Then $\rho^0(T) = 2$ if and only if $T$ is either a star or a double star.

Proof: Assume that $\rho^0(T) = 2$. If $\text{diam}(T) \geq 4$, then there is a path, say $P = (v_1, v_2, \ldots, v_k)$ such that $k \geq 5$ in $T$. Certainly, the set $\{v_1, v_2, v_k\}$ is an open packing set of $T$ and so $\rho^0(T) \geq |S| = 3$, produces a contradiction. Thus $\text{diam}(T) \leq 3$ and hence $T$ is either a star or a double star. Converse is just a verification.

Theorem 2.3. Let $G$ be a connected unicyclic graph of order at least 4. Then $\rho^0(G) = 2$ if and only if $G$ is either $C_4$ or, $C_5$ or, $C_6$ or, isomorphic to one the graphs $G$, $(1 \leq i \leq 3)$ as shown in Figure 1.

Proof: Let $C$ be the cycle in $G$. If $G = C$, then $G$ is $C_4$ or, $C_5$ or, $C_6$ follows from Theorem 1.1. Assume that $G = C$. If $G$ contains the cycle of length 5 or more, then we can find a vertex $v$ from the outside of $C$ such that there exist two adjacent vertices on $C$ which have distance at least 3 apart from $v$ in $G$.

Figure 1: Unicyclic graphs $G$ with $\rho^0(G) = 2$
Certainly, we have $\rho^s(G) \geq 3$ and thus the length of $C$ is either 3 or 4. Now, we prove the result in the following cases.

Case i. $C=C_3$

Let $V(C)=\{x,y,z\}$ and let $G_3=(V(G)-V(C))$. We first claim that each component of $G_3$ is either $K_1$ or a star graph $H$. Suppose $G_3$ has a path of length 3, say $P=(v_1,v_2,v_3)$. Then exactly one vertex of $P$ is adjacent with the vertex of $C$ as $G$ is unicyclic. If either $v_1$ or $v_2$ is adjacent with $x$, then the set $(v_1,v_2,y)$ will form an open packing set of $G$ and this is a contradiction to $\rho^s(G) = 2$. On the other hand, if either $v_2$ or $v_3$ is adjacent with $x$, then the set $(v_2,v_3,z)$ will become an open packing set of $G$, again we get a contradiction to $\rho^s(G) = 2$. Thus every component of $G_3$ is isomorphic to either $K_1$ or a star graph $H$. Also, it is clear that $G_3$ has at most one star graph $H$ as its component. If $G_3$ is the star graph $H$, then the center vertex of $H$ is adjacent with exactly one of the vertices of $C$ as $G$ is unicyclic and consequently $G=G_3$. Suppose every component of $G_3$ is $K_1$. Then either one vertex or two vertices on $C$ will become the support vertices of $G$. Otherwise, each vertex of $C$ is a support vertex of $G$ and so one pendant neighbor of each support vertex of $G$ together forms an open packing set of $G$ which in turn implies that $\rho^s(G) \geq 3$. This is a contradiction and hence $G$ is isomorphic to either $G_1$ or $G_2$. Finally, assume that exactly one component of $G_3$ is the star graph $H$ and all the remaining components are $K_1$. If the center vertex $u$ of $H$ is adjacent with the vertex $x$, then $deg y = deg z = 2$; for otherwise, there is a vertex $w$ is adjacent with exactly one of the vertices $y$ and $z$ in $G$ and so the set $\{u,v,w\}$ is an open packing set of $G$, where $v \in N(u)$ in $H$, which is a contradiction to $\rho^s(G) = 2$. Thus, in this case $G=G_3$.

Case ii. $C=C_4$

Let $V(C)=\{w,x,y,z\}$ and let $G_4=(V(G)-V(C))$. In this case, we prove that $G_4$ has no edges and at most two vertices of $C$ are support vertices in $G$; this shows that $G$ is isomorphic to either $G_4$ or $G_5$. On the contrary, let $u,v \in V(G_4)$ be such that $u$ and $v$ are adjacent in $G_4$. Since $G$ is unicyclic, exactly one of the vertices $u$ and $v$ is adjacent with the vertex on the cycle $C$. Without loss of generality, assume that $u$ is adjacent with the vertex $x$ in $G$. Certainly, $v$ is adjacent with none of the vertices $w,x,y$ and $z$ and so $(v,w)$ and $(v,z)$ are the injective pairs in $G$. This implies that the set $(w,v,z)$ is an open packing set of $G$ so $\rho^s(G) \geq 3$, producing a contradiction. Thus every vertex lying outside of $C$ in $G$ is a pendant vertex. Further, if more than two vertices on $C$ become the support vertices of $G$, then one pendant neighbor of each support vertex of $G$ together forms an open packing set of $G$ and hence the value of $\rho^s(G)$ exceeds 2. This is a contradiction and hence $G$ is isomorphic to either $G_4$ or $G_5$.

In view of Theorem 2.2 that the value of $\rho^s(T)$ for a tree $T$ with diameter is less than or equal to 3 is two, whereas trees with diameter 4 can assume arbitrarily large value of $\rho^s$ as shown in the following proposition.

**Proposition 2.4.** If $T$ is a tree with $diam(T)=4$, then $\rho^s(T) \geq 3$. Further, for any integer $k \geq 3$, there is a tree $T$ with $diam(T)=4$ and $\rho^s(T) = k$.

**Proof:** Let $P=(v_1,v_2,v_3,v_4,v_5)$ be a diametral path in $T$. Then $(v_1,v_2,v_5)$ becomes an open packing set of $T$ and thus $\rho^s(T) \geq 3$.

Suppose $k \geq 3$ is a given integer. Now, we construct a tree $T$ with $diam(T)=4$ and $\rho^s(T) = k$ as follows. Consider a star $K_{1,k-1}$ with the center vertex $v$ and $(V(K_{1,k-1}) = \{v,v_1,v_2,v_3,...,v_{k-1}\}$ Subdivide each edge of $K_{1,k-1}$ exactly once and let $T$ be the resultant tree. For $k=6$, the tree $T$ is shown in Figure 2.

![Figure 2: A tree $T$ with $\rho^s(T) = 6$](image)

Obviously, $diam(T)=4$ and the set $S=\{v_1,v_2,v_3,v_4,...,v_{k-1}\}$ is an open packing set of $T$. Thus $\rho^s(T) = k$. On the other hand, if $D$ be a maximal open packing set of $T$ and $D$ contains at most $k$-1 pendant vertices of $T$ and exactly one support vertex form $T$ and so $\rho^s(T) \leq |D| = (k-1)+1 = k$.

Next we proceed to characterize the trees $T$ for which $\rho^s(T) = 3$. A caterpillar $T$ is a tree whose removal of pendant vertices results in a path and this resulting path is called the spine of the caterpillar, denoted by $SP(T)$.

**Theorem 2.5.** Let $T$ be a tree of order at least 5. Then $\rho^s(T) = 3$ if and only if $diam(T)=4$ and $T$ is a caterpillar with length of the spine of $T$ is 2.

**Proof:** Suppose $\rho^s(T) = 3$. If $diam(T) \geq 5$, then we can find a path $P=(v_1,v_2,v_3,...,v_k)$ in $T$, where $k \geq 6$. Certainly, the set $S=\{v_1,v_2,v_3,v_k\}$ forms an open packing set of $T$ and so $\rho^s(T) \geq 4$. This is a contradiction and hence $diam(T) \leq 4$. Now, Theorem 2.2 implies that $diam(T) = 4$.

Let $S=\{x,y,z\}$ be a $\rho^s$ - set of $T$. Since no two vertices of $S$ have a common neighbor in $T$, it follows that at most one pair of vertices in $S$ are adjacent in $T$. Without loss of generality, assume that $x$ and $y$ are adjacent in $T$. Since $T$ is a tree, there is a unique path $P'$ from $y$ to $z$. Suppose $P'$ contains the vertex $x$ (See Figure 3).

![Figure 3: A step in the proof of Theorem 2.5](image)

As $diam(T)=4$, the vertex $z$ has no neighbor in $V-S$ except $v$ and the vertex $x$ has no neighbor in $V-S$; and therefore every vertex in $(V-S)-\{u,v\}$ is adjacent with exactly one of the vertices $x,u$ and $v$. Thus $T$ is a caterpillar with spine $(x,u,v)$.

If $P'$ does not contain the vertex $x$, then the length of $P'$ should be 3 as $diam(T)=4$.
Now, let us take $P' = (y, u, v, z)$. (see Figure 4). Since $\text{diam}(T)=4$, every vertex in $(V-S) = \{u, v\}$ is adjacent with exactly one of the vertices $y, u$ and $v$. Hence $T$ is a caterpillar and in this case $(y, u, v)$ is the spine of $T$.

Conversely, suppose $T$ is a caterpillar with the spine $SE(T) = (x, y, z)$. Since $\text{diam}(T)=4$, both the vertices $x$ and $z$ have at least one pendant neighbor in $T$. Let $x'$ and $z'$ be the pendant neighbors of $x$ and $z$ respectively in $T$. Certainly, the set $\{x', u, z'\}$ forms an open packing set of $T$ and so $\rho^O(T) \geq 3$. Also, since $V(T) = N[x] \cup N[y] \cup N[z]$ and any maximal open packing set $D$ of $T$ can contains at most one vertex from each of the closed neighborhoods of the vertices on the spine of $T$, it follows that $\rho^O(T) \leq |D| \leq 3$ and hence $\rho^O(T) = 3$.

III. SPECIAL CLASSES OF GRAPHS

In this section we determine the exact value of the open packing number for some special classes of graphs such as triangular snakes, quadrilateral snakes, helm and closed helm graphs.

**Definition 3.1.** A triangular snake $TS_n$ is obtained from the path $P = (v_1,v_2,v_3,...,v_{n+1})$ by joining each $v_i$ and $v_{i+1}$ to a new vertex $u_i$ for each $1 \leq i \leq n$. A quadrilateral snake $QS_n$ is obtained from the path $P = (v_1,v_2,v_3,...,v_{n+1})$ by introducing $n$ copies of $K_2$, say $u_i,u_1, u_2, ..., u_n u_n$, and joining $u_i$ to $v_i$ and $u_1$ to $v_1$ for all $i=1,2,3,...,n$.

**Theorem 3.2.** For the triangular snake $TS_n$, we have $\rho^O(TS_n) = \left\lceil \frac{n}{2} \right\rceil$.

**Proof:** Consider the set $S = \{u_1, u_2, u_3, ..., u_n\}$, where $k$ is an odd integer and $1 \leq k \leq n$. It is obvious that the distance between any two vertices of $S$ is at least 3 and thus no two vertices of $S$ have a common neighbor in $TS_n$. For example, the graph $TS_5$ and the vertices (blue colored) forms an open packing set of $TS_5$ illustrated in Figure 5. Thus $S$ is an open packing set of $TS_n$ and hence have $\rho^O(TS_n) \geq |S| = \left\lceil \frac{n}{2} \right\rceil$.

![Figure 5: The graph $TS_5$.](image)

On the other hand, let $D$ be any maximal open packing set of $TS_n$. Then $D$ contains at most one vertex from any two adjacent blocks of $TS_n$, it follows that have $|D| \leq \frac{n}{2}$ or $\frac{n+1}{2}$ according as $n$ is even or odd. Hence $\rho^O(TS_n) = \left\lceil \frac{n}{2} \right\rceil$.

**Theorem 3.3.** For the quadrilateral snake $QS_n$, we have $\rho^O(QS_n) = n + 1$.

![Figure 6: The graph $QS_5$.](image)

Let $D$ be a maximal open packing set of $QS_n$. If $D$ contains no vertex from the path $P$, then $D$ has two vertices in exactly one edge $u_{i+1}$ for $i=1,2,3,...,n-1$ and at most one vertex from the remaining edges of $QS_n$. This implies that $|D| \leq n + 1$. Similarly, one can easily get $|D| \leq n + 1$ when $D$ contains some vertices of $P$. Hence $\rho^O(QS_n) = n + 1$.

**Theorem 3.4.** (i) If $H_n$ is a helm graph, then $\rho^O(H_n) = n - 1$. (ii) If $CH_n$ is a closed helm graph, then $\rho^O(CH_n) = \left\lceil \frac{n}{2} \right\rceil$.

**Proof:** (i) It is clear that, the set $S$ consisting of all pendant vertices of $H_n$ will forms an open packing set of $H_n$ and so we have $\rho^O(H_n) = n - 1$. Further, one can easily verify that $S$ is the only open packing set of $H_n$ with maximum cardinality and thus $\rho^O(H_n) = n - 1$. Hence (i) is proved.

(ii) It is observe that any open packing set of $CH_n$ can have at most $\left\lceil \frac{n-1}{2} \right\rceil$ vertices from the outer cycle and at most one vertex from the rim of the wheel of $CH_n$, this implies that $\rho^O(CH_n) \leq \left\lceil \frac{n}{2} \right\rceil$.

For another inequality, let $V(CH_n) = \{v_0, u_1, v_1, v_2, v_3, ..., v_n, u_n\}$ and $E(CH_n) = \{v_0 u_1, 1 \leq i \leq n\}$ and $u_i v_i: 1 \leq i \leq n\} \cup \{u_i u_{i+1}: 1 \leq i \leq n - 1\} \cup \{v_{n-1} v_0, v_{n-1} u_n\}$. Now, we produce an open packing set of $CH_n$ with cardinality $\left\lceil \frac{n}{2} \right\rceil$ in the following cases.

**Case (i).** $n \equiv 0 \pmod{4}$

Consider the set $S = \{v_1, v_2, v_3, ..., v_{n-3}, v_n\}$. Obviously, no two vertices of $S$ have a common neighbor in $CH_n$ and so $S$ is an open packing set of $CH_n$. Thus $\rho^O(CH_n) \geq |S| = \frac{n}{2}$.

**Case (ii).** $n \equiv 1 \pmod{4}$

Consider the set $D = \{v_1, v_2, v_3, v_4, v_5, v_6, ..., v_{n-1}, v_{n-2}\} \cup \{v_{n-1}\}$. One can easily verify that $D$ forms an open packing set of $CH_n$ and thus $\rho^O(CH_n) \geq |S| = \frac{n}{2}$.

**Case (iii).** $n \equiv 3 \pmod{4}$

Consider the set $D = \{v_1, v_2, v_5, v_6, v_7, ..., v_{n-3}, v_{n-2}\} \cup \{v_{n-1}\}$. It is observe that any two vertices of $D$ have no common neighbor in $CH_n$ and so $D$ is an open packing set of $CH_n$. Therefore, $\rho^O(CH_n) \geq |S| = \frac{n}{2}$.
In each case we have produced an open packing set of $CH_n$ with cardinality at least $\left\lceil \frac{n}{2} \right\rceil$ and hence $\rho^o(CH_n) = \left\lceil \frac{n}{2} \right\rceil$.

Thus (ii) is proved.

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REFERENCES


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