Some Expansion of Fuzzy Paranormal Operators

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Abstract: Let $\mathbb{H}$ be a Fuzzy Hilbert space over the fields of $\mathbb{R}/\mathbb{C}$ and $FB(\mathbb{H})$ is the set of all fuzzy continuous linear operator on $\mathbb{H}$. In this paper we introduce the expansion of different fuzzy paranormal operators like $n$-fuzzy paranormal operator, $*$-fuzzy paranormal operator and $n^k$-fuzzy paranormal operator, which are developed from paranormal operators and their characteristics. The study resulted the properties of an $n$-fuzzy paranormal operator, $*$-fuzzy paranormal operator and $n^k$-fuzzy paranormal operator and their relationship between them. To investigate the nature of these operators, all it needs the nature of the $n$-fuzzy paranormal operator. It is emphasized on the characteristics and relationships of the $n^k$-Fuzzy paranormal operator and $*$ - fuzzy paranormal operator on FH-space. Therefore it necessary to study more about the properties of the $n^k$-Fuzzy paranormal operator and $*$ - fuzzy paranormal operator on FH-space.

Keywords: Adjoint Fuzzy operator, Fuzzy Hilbert space (FH-space), Self adjoint fuzzy operator, Fuzzy paranormal operator, $n$-fuzzy paranormal operator, $*$-fuzzy paranormal operator and $n^k$-fuzzy paranormal operator.

I. INTRODUCTION

The notion of fuzzy norm on a linear space is first introduced by Katsaras[5]. In 1991, the definition of fuzzy inner product space is first introduced by Biswas[7]. Riesz theorem was given by Youngst[9] using fuzzy concept in 2007. In 2009 Goudarzi and Vaeezpour [6] have been introduced the definition of fuzzy Hilbert space. They introduced triplets $(\mathbb{H}, F, \cdot)^*$, where $\mathbb{H}$ is the Fuzzy Hilbert space, F is the Fuzzy set on $\mathbb{H} \times \mathbb{R}$, and $\cdot^*$ is continuous t-norm. Sudad M. Rasheed [4] introduced the concept of adjoint fuzzy operator. That is $\langle u, a \cdot^*b \rangle = < a, u^*b \rangle$. Also self adjoint fuzzy operator if $u = u^*$ where $u^*$ is adjoint fuzzy operator of $u$. Radharamani et al.[1] introduced fuzzy normal operator if $u^* = u$ and let $u = u^*$. Then $u^* = u = u$. Also fuzzy unitary operator and fuzzy hyponormal operator was introduced. An operator $u \in FB(\mathbb{H})$ has the property of $\|u^2a\| = \|u^3a\| = \|u^4a\|$ for every unit vector $a \in \mathbb{H}$ is named as fuzzy paranormal operator and introduced by Radharamani et al.[3] in 2019. Then the fuzzy paranormal operator definition is expanded into positive integer $n$, $\|u^2a\| \geq \|u^n\cdot a\|^2$ for some unit vector $a \in \mathbb{H}$.

In this paper, we introduce an operator $u \in FB(\mathbb{H})$ with the property of $\|u^2a\| \geq \|u^n\cdot a\|^2$ is called $n$-fuzzy paranormal operator and we explained some characteristics of $n$-fuzzy paranormal operator. Before that requires to defining and explaining the theory of $n$-fuzzy paranormal operator. We give different properties of $n$-fuzzy paranormal operator and also give important result about it. An operator $u \in FB(\mathbb{H})$ with the characteristics of $\|u^2a\| \|a\| \geq \|u^a\|^2$ is said to be $*$-fuzzy paranormal operator with $u^*$ is Adjoint fuzzy operator of $u^*$.

In this study, the consolation of the $n$th-Fuzzy paranormal operator and $*$-fuzzy paranormal operator and is more emphasized on the characteristics and relationships of the $n$th-Fuzzy paranormal operator and $*$-fuzzy paranormal operator on FH-space. Therefore it necessary to study more about the properties of the $n$th-Fuzzy paranormal operator and $*$-fuzzy paranormal operator on FH-space.

II. PRELIMINARIES

Definition 2.1: [4] Fuzzy Hilbert space (FH-space)
Let $(\mathbb{H}, F, \cdot^*)$ be a fuzzy Hilbert space with IP: $\langle a, b \rangle = sup\{ u \in \mathbb{H} : F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$. Then $\mathbb{H}$ is complete in the $\| \cdot \|$ then $\mathbb{H}$ is called Fuzzy Hilbert space (FH-space).

Definition 2.2: [4] Adjoint Fuzzy operator
Let $(\mathbb{H}, F, \cdot^*)$ be a Fuzzy Hilbert space and let $u \in FB(\mathbb{H})$ be $T$ continuous linear functional. Then $3$ unique $u^* \in FB(\mathbb{H})$ such that $\langle u, b \rangle = (u, b)^*v$ $a \in \mathbb{H}$.

Note: Let $FB(\mathbb{H})$ be the set of all fuzzy continuous linear operator on $\mathbb{H}$.

Definition 2.3: [4] Self-Adjoint Fuzzy operator
Let $(\mathbb{H}, F, \cdot^*)$ be a FH-space with IP: $(a, b) = sup\{ u \in R, F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$ and let $u \in FB(\mathbb{H})$ then $u$ is self-adjoint Fuzzy operator, if $u = u^*$, where $u^*$ is adjoint Fuzzy operator of $u$.

Theorem 2.4: [4]
Let $(\mathbb{H}, F, \cdot^*)$ be a FH-space with IP: $(a, b) = sup\{ u \in R, F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$ and let $u \in FB(\mathbb{H})$. Then $u$ is a Fuzzy Unitary operator.

Theorem 2.5: [4]
Let $(\mathbb{H}, F, \cdot^*)$ be a FH-space with IP: $(a, b) = sup\{ u \in R, F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$ and let $u \in FB(\mathbb{H})$. Then $u$ is a Fuzzy Normal operator.

Definition 2.6: [4] Fuzzy Paranormal operator
Let $(\mathbb{H}, F, \cdot^*)$ be a Fuzzy Hilbert space with IP: $(a, b) = sup\{ u \in R, F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$ and let $u \in FB(\mathbb{H})$. Then $u$ is a Fuzzy Paranormal operator if $\|u^2a\| \geq \|u^3a\| \forall a \in \mathbb{H}$.

Definition 2.7: [4] Fuzzy Paranormal operator
Let $(\mathbb{H}, F, \cdot^*)$ be a Fuzzy Hilbert space with IP: $(a, b) = sup\{ u \in R, F(a, b, u) < 1 \} \forall a, b \in \mathbb{H}$ and let $u \in FB(\mathbb{H})$. Then $u$ is a Fuzzy Paranormal operator if $\|u^2a\| \geq \|u^3a\| \forall a \in \mathbb{H}$.

Note: Let $u \in FB(\mathbb{H})$. Then $u$ is a Fuzzy Paranormal operator if $\|u^2a\| \geq \|u^3a\| \forall a \in \mathbb{H}$.
Some Expansion of Fuzzy Paranormal Operators

Theorem 2.9:[4] Let $\langle\mathfrak{H}, \mathbb{F}^*, \alpha \rangle$ be a FH – space with IP: $(a, b) = \text{Sup}\{u \in \mathbb{R}; F(a, b, u) = \text{Sup}\{u \in \mathbb{R}; F(a, b, u) = 0\}$ and let $U'$ be the adjoint Fuzzy operator of $U \in FB(\mathfrak{H})$, then

i. $(U^*)' = U$

ii. $(\alpha U)' = \alpha U$

iii. $(\alpha U + \beta V)' = \alpha U' + \beta V' \text{ where } \alpha, \beta \text{ are scalars}$

and $U \in FB(\mathfrak{H})$.

\[\text{i.e. } (U^*)' = V^* \text{ where } V \in \mathbb{F}^*\]

III. MAIN RESULTS

Definition 3.1 Let $\langle\mathfrak{H}, \mathbb{F}^*, \alpha \rangle$ be a fuzzy Hilbert space with IP: $\langle a, b \rangle = \text{Sup}\{u \in \mathbb{R}; F(a, b, u) \leq 1\}$ and let $U \in FB(\mathfrak{H})$. The operator $U$ is called n- fuzzy paranormal operator if $\|U^{n+1}a\| \geq \|Ua\|^{n+1}$ for $n \in N$ and for some unit vector $a \in \mathbb{F}$.

Theorem 3.2 Let $U \in FB(\mathfrak{H})$. If $U$ is n-fuzzy paranormal operator then $U$ is fuzzy paranormal.

Proof: Given $U$ is n- fuzzy paranormal operator.

By the definition, $\|U^{n+1}a\| \geq \|Ua\|^{n+1}$

Put $n=1$, $\|U^2a\| \geq \|Ua\|^2$

Therefore $U$ is fuzzy paranormal.

Theorem 3.3 Let $U \in FB(\mathfrak{H})$. If $U$ is fuzzy paranormal then $U$ is n-fuzzy paranormal operator for $n \in N$.

Proof: Take $a \in \mathbb{F}$.

By using the mathematical induction, if $n=1$

$\|U^{n+1}a\| = \|U^{n+1}a\| \geq \|Ua\|^{n+1}$

i.e. $\|Ua\|^{n+1} \geq \|Ua\|^{n+1}$

if $n=2$,

$\|U^2a\| = \|U^2a\| \geq \|Ua\|^2$

i.e. $\|U^2a\|^2 \geq \|Ua\|^2$ etc.,

if $n=k$ is true

$\|U^k a\|^k \geq \|Ua\|^k$

Now we have to prove that is true for $n=k+1$.

Let $\|Ua\|^{k+1} \geq \|Ua\|^{k+1} + \|Ua\|$

$\leq \|U^k a\|^{k+1} + \|Ua\|$

$\|Ua\|^{k+1} \leq \|U^{k+1}a\|$

Hence $U$ is n-fuzzy paranormal operator.

Definition 3.4 Let $\langle\mathfrak{H}, \mathbb{F}^*, \alpha \rangle$ be a fuzzy Hilbert space with IP: $\langle a, b \rangle = \text{Sup}\{u \in \mathbb{R}; F(a, b, u) \leq 1\}$ and let $U \in FB(\mathfrak{H})$. The operator $U$ is called n- fuzzy paranormal operator if $\|U^{n+1}a\| \geq \|Ua\|^{n+1}$ for $n \in N$.

Note: Let $U \in FB(\mathfrak{H})$ be a n- fuzzy paranormal operator if $\|U^{n+1}a\| \geq \|Ua\|^{n+1}$ for $n \in N$ and $\forall$ unit vector $a \in \mathbb{F}$.

Theorem 3.5 Let $U, V \in FB(\mathfrak{H})$. If $U, V$ are Self-adjoint fuzzy operators then $U + V$ is n-fuzzy paranormal operator for $n \in N$.

Proof: Take $n \in N$ for a $a \in \mathbb{F}$ with $\|a\| = 1$. Let $\|(U + V)^{n+1}a\| \leq \|Ua\|^{n+1}$

\[\text{Sup}\{u \in \mathbb{R}; F(U + V, a, u) \leq 1\} \leq \|Ua\|^{n+1} \leq \|Ua\|^{n+1}\]

Therefore $U + V$ is n-fuzzy paranormal operator.

Theorem 3.6 Let $U, V \in FB(\mathfrak{H})$. If $U, V$ are Self-adjoint fuzzy operators then $UV$ is n-fuzzy paranormal operator for $n \in N$.

Proof: Take $n \in N$ for a $a \in \mathbb{F}$ with $\|a\| = 1$.

Since $U$ and $V$ are self-adjoint fuzzy operators.

We know that $U = UV = VU$.

To prove that $UV$ is an n-fuzzy paranormal operator.

Let $\|(UV)^{n+1}a\| \leq \|(U + V)^{2n}\|a\|a\|

$\text{Sup}\{u \in \mathbb{R}; F((UV)^{n+1}a, (U + V)^{2n}a, u) \leq 1\} \leq \|(U + V)^{2n}\|a\|a\|

Therefore $UV$ is n-fuzzy paranormal operator.

Theorem 3.7 Every fuzzy paranormal operator is 1-fuzzy paranormal operator.

Proof: Take $n \in N$ for a $a \in \mathbb{F}$ with $\|a\| = 1$.

Since $U \in FB(\mathfrak{H})$ is fuzzy paranormal operator

$\|U^2a\| \geq \|Ua\|^2$

$\|U^2a\| \geq \|Ua\|^2$

Hence $U$ is 1-fuzzy paranormal operator.

Theorem 3.8 Let $U \in FB(\mathfrak{H})$ is a Fuzzy Hilbert space and Self-adjoint fuzzy operator. If $U$ is a n-fuzzy paranormal operator then $U^n$ is n-fuzzy paranormal operator for $n \in N$.

Proof: Take $n \in N$ for a $a \in \mathbb{F}$ with $\|a\| = 1$.

We know that $U$ is n-fuzzy paranormal operator.

$\text{i.e. } \|U^n a\|^2 \leq \|U^2a\|^2$

To show that $\|(U^n)^{n+1}a\|^2 \leq \|(U^n)^{2n}\|a\|a\|

$\text{Sup}\{u \in \mathbb{R}; F((U^n)^{n+1}a, (U^n)^{2n}a, u) \leq 1\} \leq \|(U^n)^{2n}\|a\|a\|

Therefore $U^n$ is n-fuzzy paranormal operator.
An operator $U \in FB(\mathbb{I})$ is $n^0$-fuzzy paranormal operator if and only if $U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2 \geq 0$ for all $\lambda \geq 0, n \in N$.

**Proof:**

Take $n \in N$ for every $a \in \mathbb{I}$ with $\|a\| = 1$

$U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2 \geq 0 \iff$

$((U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2)a, a) \geq 0$

$U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2 \geq 0 \implies$

$\sup \{ u \in \mathbb{R} : F(U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2) a, a, u) < 1 \} \geq 0$

$U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2 \geq 0 \implies$

$\sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} U^{2n} a, a, u) < 1 \} \geq 0$

$-2\lambda \sup \{ u \in \mathbb{R} : F(U^{\ast n} U^{2n} a, a, u) \} < 1 \} \geq 0$

$U^{\ast n} U^{2n} - 2U^n U^n + \lambda^2 \geq 0 \implies$

$\sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} U^{2n} a, a, u) \} \geq 0$

Since if $a > 0$, $b$ and $c$ are real numbers then $a^2 + b + c \geq 0$ for every real $a$ and if only if $b^2 - 4ac \leq 0$ in an analogous manner, using elementary property of real quadratic forms.

$\implies 4\|U^n a\|^4 - 4\|a\|^2\|U^{\ast n} a \|^2 \leq 0$

$\implies \|U^n a\|^4 \leq \|a\|^2\|U^{\ast n} a \|^2$

$\implies \|U^n a\|^2 \leq \|U^{\ast n} a\|^2$

$\|U^n a\|^2 \geq \|U^{\ast n} a\|^2$

Hence $U$ is $n^0$-fuzzy paranormal operator.

**Theorem 3.10:**

Let $U \in FB(\mathbb{I})$. If $U$ is $n^0$-fuzzy paranormal operator and self-adjoint fuzzy operator then $U^\ast$ is $n^0$-fuzzy paranormal operator for $n \in N$.

**Proof:**

Take $n \in N$ for a $a \in \mathbb{I}$ with $\|a\| = 1$

Since $U$ is $n^0$-fuzzy paranormal operator, $i.e$. $\|U^{\ast n} a\|^2 \geq \|a\|^2$

Let $\|U^\ast n a\|^2 = (U^{\ast n} a, (U^\ast n) a)$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a) \} < 1 \}$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$\sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$\|U^\ast n a\|^2 \leq \|U^{\ast n} a\|^2$

Implies that $\|U^\ast n a\|^2 \geq \|U^{\ast n} a\|^2$

Therefore $U^\ast$ is $n^0$-fuzzy paranormal.

**Theorem 3.11:**

If $U \in FB(\mathbb{I})$ is $n^0$-fuzzy paranormal operator and self-adjoint fuzzy operator. Then $U^\ast$ is $n^0$-fuzzy paranormal operator for $n \in N$.

**Proof:**

Take $n \in N$ for a $a \in \mathbb{I}$ with $\|a\| = 1$

Since $U$ is $n^0$-fuzzy paranormal operator, $i.e$. $\|U^{\ast n} a\|^2 \geq \|a\|^2$

Let $\|U^\ast n a\|^2 = (U^{\ast n} a, (U^\ast n) a)$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a) \} < 1 \}$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$\sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$= \sup \{ u \in \mathbb{R} : \mathbb{F}(U^{\ast n} a, (U^\ast n) a, u) \} < 1 \}$

$\|U^\ast n a\|^2 \leq \|U^{\ast n} a\|^2$

Implies that $\|U^\ast n a\|^2 \geq \|U^{\ast n} a\|^2$

Therefore $U^\ast$ is $n^0$-fuzzy paranormal.
Some Expansion of Fuzzy Paranormal Operators

Let (I,I,F,*) be a fuzzy Hilbert space with IP: ⟨a,b⟩= sup{u∈ IR; F(a,b,u) < 1} ∀a,b ∈ I and self adjoint fuzzy operator.
Let U,V ∈ FB(I) be a Self adjoint fuzzy operators. Then UV is *-fuzzy paranormal operator for n ∈ N.

Proof:
Take n ∈ N for a ∈ I with ||a||=1.
Since U and V are self adjoint fuzzy operators.
we know that U = U*, V = V*.
To prove that UV is a *-fuzzy paranormal operator.
Let ||(UV)^* a||^2 ≤ ||(UV)^2 a||a||
= sup {u ∈ IR; F((UV)^* a, (UV)^* a, u) < 1} ≤ sup {u ∈ IR; F((UV)^* (UV)^* a, a, u) u < 1} ≤ sup {u ∈ IR; F((U^2)^* a, a, u) u < 1} ≤ sup {u ∈ IR; F(U^2 a, a, u) u < 1} ≤ ((U^2)^* a, a)
Therefore UV is *-fuzzy paranormal operator.

Theorem: 3.18
Let (I,I,F,*) be a fuzzy Hilbert space with IP: ⟨a,b⟩= sup{u∈ IR; F(a,b,u) < 1} ∀a,b ∈ I and self adjoint fuzzy operator.
Let U,V ∈ FB(I). If U and V are self adjoint fuzzy operators then U + V is *-fuzzy paranormal operator for n ∈ N.
Proof:
Take n ∈ N for a ∈ I with ||a||=1.
Let ||(U + V)^* a||^2 = ((U + V)^* a, (U + V)^* a) ≤ sup {u ∈ IR; F((U + V)^* (U + V)^* a, a, u) u < 1} ≤ sup {u ∈ IR; F(U^2 a, a, u) u < 1} ≤ ((U^2 a, a)
Therefore U + V is *-fuzzy paranormal operator.

Theorem: 3.19
If U ∈ FB(I) is fuzzy paranormal operator and self adjoint fuzzy operator then UV is *-fuzzy paranormal operator for.
Proof:
For a ∈ I with ||a||=1.
We know that U is fuzzy paranormal operator.
i.e ||U^2 a|| ≥ ||U a||^2 and U = U*.
Let ||U’a||^2 = (U’a, U’a) ≤ sup {u ∈ IR; F(U U*a, a, u) u < 1} ≤ sup {u ∈ IR; F(U^2 a, a, u) u < 1} ≤ ((U^2 a, a)
Therefore U is *-fuzzy paranormal operator.

Theorem: 3.20
If U ∈ FB(I) is *-fuzzy paranormal operator and self adjoint fuzzy operator then U^2 is *-fuzzy paranormal operator for n ∈ N.
Proof:
For a ∈ I with ||a||=1.
We know that U is fuzzy paranormal operator.
i.e ||U^2 a|| ≥ ||U a||^2 and U = U*.
Let ||U’a||^2 = (U’a, U’a) ≤ sup {u ∈ IR; F(U U*a, a, u) u < 1} ≤ sup {u ∈ IR; F(U^2 a, a, u) u < 1} ≤ ((U^2 a, a)
Therefore U is *-fuzzy paranormal operator.
Since $U$ is *- fuzzy paranormal operator i.e $\|U^* a\|^2 \leq \|U^2 a\|\|a\|$ and $U = U^*$. Let $\|(U^n)^* a\|^2 = (U^n)^* a, (U^n)^* a) = \text{Sup} \{u \in \mathbb{R} : F((U^n)^* a, (U^n)^* a) < 1\}$ = $\text{Sup} \{u \in \mathbb{R} : F(U^n (U^n)^* a, a, u) < 1\}$ $\text{Sup} \{u \in \mathbb{R} : F(U^n (U^n)^* a, a, u) < 1\}$ = $\text{Sup} \{u \in \mathbb{R} : F(U^n a, a + t) < 1\}$ $\|(U^n)^* a\|^2 \leq \|(U^n)^* a\|^2 \leq \|(U^n)^* a\|^2 \leq \|(U^n)^* a\|^2$ Implies that $\|(U^n)^* a\|^2 \leq \|(U^n)^* a\|^2$. Therefore $U^n$ is *- fuzzy paranormal operator for $n \in N$.

III. CONCLUSION

The conclusion that can be taken from a new idea of fuzzy paranormal operator in Fuzzy Hilbert space, example and characteristics of $n$- Fuzzy paranormal operator, ^*-Fuzzy paranormal operator and $n^*$-Fuzzy paranormal operator including addition and multiplication operators and its connection with self-adjoint fuzzy operator. In addition other characteristics are found and its connection between each of the operators of such other definition.

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REFERENCES


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