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Ranking and Similarity Measures of Interval-Valued Pentagonal Fuzzy Numbers

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Abstract*: We define generalized interval-valued pentagonal fuzzy numbers. Based on the height of the lower and upper pentagonal fuzzy numbers we propose to categorize interval-valued pentagonal fuzzy numbers into three different categories. Using signed distance concept de-fuzzification of interval-valued pentagonal fuzzy numbers is proposed and mathematical formulas are derived using $\alpha - cut$ representations. We also introduce two similarity measures for interval-valued pentagonal fuzzy numbers based on geometric distance, area and height of pentagonal fuzzy numbers and geometric distance, perimeter and height of pentagonal fuzzy numbers respectively. Algorithms for finding de-fuzzification value and similarity measures are proposed with flow-chart illustration.

Keywords*: De-fuzzification, Interval-valued Pentagonal Fuzzy Numbers, Ranking, Signed Distance, Similarity Measures.

I. INTRODUCTION

Fuzzy numbers are most used fuzzy tools in decision making. They are used to capture and interpret uncertain, vague and imprecise information. While dealing with uncertain, vague and imprecise information it is very important that the final outcome is interpreted and the alternatives are ranked in a proper manner. For ranking fuzzy numbers we require de-fuzzified value of fuzzy numbers. There have been many methods suggested by various researchers for ranking ordinary fuzzy numbers. In 1985, Chen proposed maximizing sets and minimizing sets ranking method [6]. D. Dubois and H. Prade, suggested ranking method based on the possibility theory in 1983 [3] and Liou and Wang proposed ranking method based on integral value in 1992 [9]. There are various ranking methods based on centroid, center of maxima, center of gravity, fuzzy preference relation and hamming distance etc. Different ranking methods have been proposed for different types of fuzzy numbers like intuitionistic fuzzy numbers, interval-valued fuzzy numbers [10], interval type-2 fuzzy numbers and type-2 fuzzy numbers. In this paper, we define generalized interval-valued pentagonal fuzzy numbers and propose ranking of Interval-Valued Pentagonal Fuzzy Numbers (IVPFNs) [7] using signed distance. We also propose two similarity measures for interval-valued pentagonal fuzzy numbers.

The rest of the paper is arranged in the following manner. In the second section we discuss preliminary concepts and definitions used in this paper. In section three we define generalized IVPFNs and categorize them into three different categories. In the fourth section, de-fuzzification methodology is proposed and mathematical formulas are derived to find signed distance of an IVPFN with reference to null fuzzy number $\underset{\sim}{0}$ using $\alpha - cut$ representations of lower and upper membership functions of IVPFN. We further propose ranking of IVPFNs using the signed distance. In section five, methodologies for two similarity measures are proposed. We discuss the results in section six. Finally, we conclude the paper.

II. PRELIMINARY CONCEPTS

A. Interval-valued Fuzzy Numbers

An interval-valued fuzzy number[10] $\underset{\sim}{A}_{IV} = \left[\underset{\sim}{A}^L_{IV}, \underset{\sim}{A}^U_{IV}\right]$ is an interval-valued fuzzy set on real line $\Re$, where $\underset{\sim}{A}^L$ and $\underset{\sim}{A}^U$ are two fuzzy numbers known as lower membership function and upper membership function respectively and $\underset{\sim}{A}^L_{IV} \subseteq \underset{\sim}{A}^U_{IV}$. If $\underset{\sim}{A}^L$ and $\underset{\sim}{A}^U$ are two triangular fuzzy numbers we call it interval-valued triangular fuzzy numbers and if they are trapezoidal fuzzy numbers they are called interval-valued trapezoidal fuzzy numbers.

B. Pentagonal Fuzzy Number

Linear pentagonal fuzzy number [8] $\underset{\sim}{A}_P$ is defined as $\underset{\sim}{A}_P = (a_1, a_2, a_3, a_4, a_5)$. The membership function is given as follows:

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$$\underset{\sim P}{A} = \begin{cases} \frac{x-a_1}{a_2-a_1} & ;a_1 \le x \le a_2 \\ \frac{x-a_2}{a_3-a_2} & ;a_2 \le x \le a_3 \\ 1 & ;x = a_3 \\ \frac{a_4-x}{a_4-a_3} & ;a_3 \le x \le a_4 \\ \frac{a_5-x}{a_5-a_4} & ;a_4 \le x \le a_5 \\ 0 & ;otherwise \end{cases} \quad (1)$$

C. Signed Distance

The signed distance [1] function of a set A in a metric space determines the distance of a given point x from the boundary of A, with the sign determined by whether x is in A. It takes negative values outside of A at the boundary the signed distance function is zero. If zero is taken as boundary point the signed distance of a point 'x' is defined as

$d(x,0) = x$ and $d(-x,0) = -x$

III. GENERALIZED INTERVAL-VALUED PENTAGONAL FUZZY NUMBERS

We extend interval-valued pentagonal fuzzy numbers (IVPFNs) [7] and propose the definition of generalized interval-valued pentagonal fuzzy numbers.

A. Definition

Generalized interval-valued pentagonal fuzzy number is an interval-valued fuzzy set of $\Re$. It is defined as

$\underset{\sim IVP}{A} = \left[\underset{\sim IVP}{A^L}, \underset{\sim IVP}{A^U}\right]$ where $\underset{\sim IVP}{A^U} = (a_1, a_2, a_3, a_4, a_5; \lambda_1)$ and $\underset{\sim IVP}{A^L} = (a_1', a_2', a_3', a_4', a_5'; \lambda_2)$ are called lower pentagonal fuzzy number and upper pentagonal fuzzy numbers respectively such

that $\underset{\sim IVP}{A^L} \subseteq \underset{\sim IVP}{A^U}$; $a_1 \le a_2 \le a_3 \le a_4 \le a_5$ and $a_1' \le a_2' \le a_3' \le a_4' \le a_5'$. Also

(i) $a_3 = a_3'$

(ii) $\frac{1}{2} < \lambda_2 \le \lambda_1 \le 1$; where λ_1 is height of upper PFN and λ_2 is height of lower PFN

(iii) Variation alpha level cut = 0.5

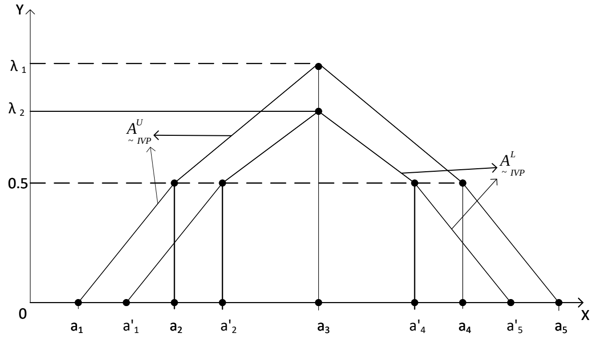

Fig. 1. Generalized IVPFN

The membership functions of IVPFN are given as follows:

$$\mu_{\underset{\sim IVP}{A^U}}(x) = \begin{cases} \frac{1}{2}\frac{x-a_1}{a_2-a_1} & ;a_1 \le x \le a_2 \\ \frac{1}{2} + \frac{\lambda_1 - \frac{1}{2}}{a_3-a_2}(x-a_2) & ;a_2 \le x \le a_3 \\ \lambda_1 & ;x = a_3 \\ \frac{1}{2} + \frac{\lambda_1 - \frac{1}{2}}{a_4-a_3}(a_4-x) & ;a_3 \le x \le a_4 \\ \frac{1}{2}\frac{a_5-x}{a_5-a_4} & ;a_4 \le x \le a_5 \end{cases} \quad (2)$$

$$\mu_{\underset{\sim IVP}{A^L}}(x) = \begin{cases} \frac{1}{2}\frac{x-a_1'}{a_2'-a_1'} & ;a_1' \le x \le a_2' \\ \frac{1}{2} + \frac{\lambda_2 - \frac{1}{2}}{a_3-a_2'}(x-a_2') & ;a_2' \le x \le a_3 \\ \lambda_2 & ;x = a_3 \\ \frac{1}{2} + \frac{\lambda_2 - \frac{1}{2}}{a_4'-a_3}(a_4'-x) & ;a_3 \le x \le a_4' \\ \frac{1}{2}\frac{a_5'-x}{a_5'-a_4'} & ;a_4' \le x \le a_5' \end{cases} \quad (3)$$

We propose to categorize IVPFNs into three following types:

B. Normal IVPFNs

If the upper as well as the lower pentagonal fuzzy numbers constituting an IVPFN are normal i.e. $\lambda_2 = \lambda_1 = 1$ then such fuzzy number is called normal IVPFN.

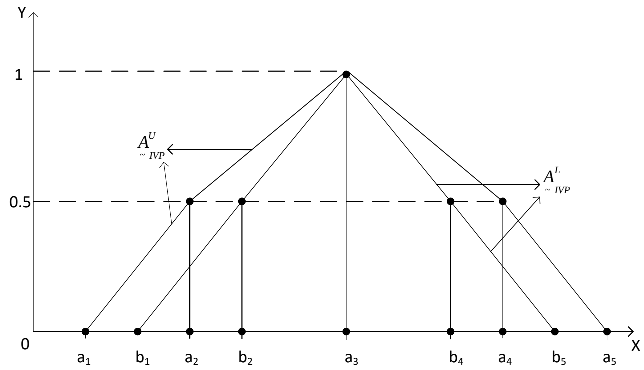

Fig. 2. Normal IVPFN

C. Partially Normal IVPFNs

If the upper pentagonal fuzzy number of an IVPFN is normal and the lower pentagonal fuzzy number is generalized one i.e. $\lambda_2 < \lambda_1 = 1$ then such a fuzzy number is called a partially normal IVPFN.

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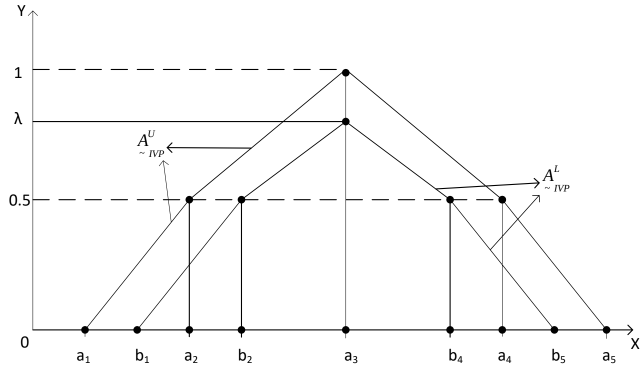

Fig. 3. Partially Normal IVPFN

D. Generalized IVPFNs

If the upper as well as the lower pentagonal fuzzy numbers constituting an IVPFN are generalized pentagonal fuzzy numbers i.e. $\lambda_2 < \lambda_1 < 1$ we call such fuzzy number generalized IVPFN.

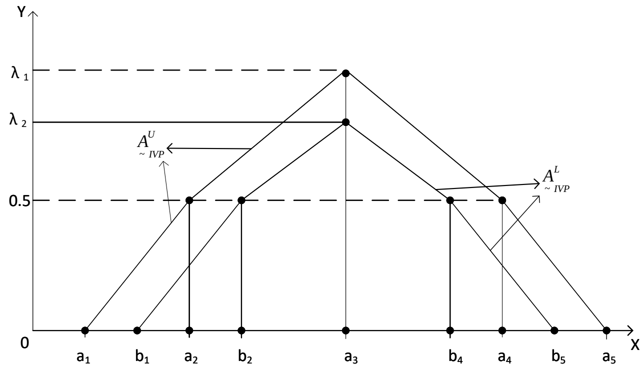

Fig. 4. Generalized IVPFN

E. $\alpha-cut$ Representation of IVPFNs

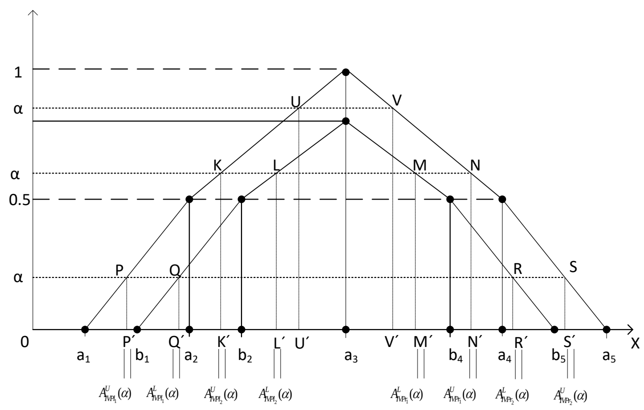

Fig. 5. $\alpha-cut$ representation for a IVPFN

The $\alpha-cuts$ representation of lower and upper membership functions are the following in the respective intervals:

Case 1. $0 \le \alpha < \frac{1}{2}$,

$$A^U_{IVPl_1}(\alpha) = 2\alpha(a_2 - a_1) + a_1,\ A^U_{IVPr_2}(\alpha) = a_5 - 2\alpha(a_5 - a_4)$$
$$A^L_{IVPl_1}(\alpha) = 2\alpha(b_2 - b_1) + b_1,\ A^L_{IVPr_2}(\alpha) = b_5 - 2\alpha(b_5 - b_4) \quad (4)$$

Case 2. $\frac{1}{2} \le \alpha \le \lambda_1$,

$$A^U_{IVPl_2}(\alpha) = \frac{\alpha - \frac{1}{2}}{\lambda_1 - \frac{1}{2}}(a_3 - a_2) + a_2,\ A^U_{IVPr_1}(\alpha) = a_4 - \frac{\alpha - \frac{1}{2}}{\lambda_1 - \frac{1}{2}}(a_4 - a_3) \quad (5)$$

Case 3. $\frac{1}{2} \le \alpha \le \lambda_2$,

$$A^L_{IVPl_2}(\alpha) = \frac{\alpha - \frac{1}{2}}{\lambda_2 - \frac{1}{2}}(b_3 - b_2) + b_2,\ A^L_{IVPr_1}(\alpha) = b_4 - \frac{\alpha - \frac{1}{2}}{\lambda_2 - \frac{1}{2}}(b_4 - b_3) \quad (6)$$

The alpha level set of $\underset{\sim}{A}_{IVP} = \left[\underset{\sim}{A}^L_{IVP}, \underset{\sim}{A}^U_{IVP}\right]$ is defined as:

$$\left\{x \setminus \mu_{\underset{\sim}{A}^U}(x) \ge \alpha\right\} - \left\{x \setminus \mu_{\underset{\sim}{A}^L}(x) > \alpha\right\}.$$

By Decomposition theorem an IVPFN can be represented as follows:

$$\underset{\sim}{A}_{IVP} = \bigcup_{0 \le \alpha < \frac{1}{2}} \left(\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha); \alpha\right] \cup \left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha); \alpha\right]\right)$$
$$\cup \left(\bigcup_{\frac{1}{2} \le \alpha < \lambda_2} \left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha); \alpha\right] \cup \left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha); \alpha\right]\right)\right) \quad (7)$$
$$\cup \left(\bigcup_{\lambda_2 \le \alpha \le \lambda_1} \left[A^U_{IVPl_2}(\alpha), A^U_{IVPr_2}(\alpha); \alpha\right]\right)$$

IV. DE-FUZZIFICATION OF INTERVAL-VALUED PENTAGONAL FUZZY NUMBERS

De-fuzzification of a fuzzy number is essential for ranking of two or more fuzzy numbers. In this section we propose methodology to de-fuzzify IVPFNs and to rank them.

A. Methodology

Signed distance concept is used to de-fuzzify an IVPFN. Using $\alpha-cut$ representations of IVPFNs we find the formula for finding signed distance of an IVPFN with reference to $\underset{\sim}{0}$ fuzzy number.

B. Algorithm

We propose the following algorithm for de-fuzzification:

Step 1. Consider $\alpha-cut$ representations of $\underset{\sim}{A}_{IVP} = \left[\underset{\sim}{A}^L_{IVP}, \underset{\sim}{A}^U_{IVP}\right]$.

Step 2. Mapping the α - level intervals between $\underset{\sim}{A}^L_{IVP}$ and $\underset{\sim}{A}^U_{IVP}$ to the equivalent intervals in the universe of discourse.

Step 3. Finding signed distance of all α - level intervals between $\underset{\sim}{A}^L_{IVP}$ and $\underset{\sim}{A}^U_{IVP}$.

Step 4. Finding signed distance combining left and right sides of $\underset{\sim}{A}^L_{IVP}$ and $\underset{\sim}{A}^U_{IVP}$.

Step 5. Using decomposition theorem to find signed distance of an IVPFN.

C. Flow Chart

We can represent the above algorithm as the following flow chart:

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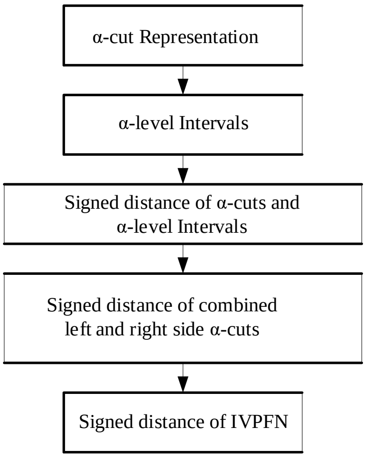

Fig. 6. Flow Chart for de-fuzzification

D. Derivation of Mathematical Results

Step 1. Consider all the $\alpha-cuts$ as in (4), (5) and (6).

Step 2. As in fig.5, we have following one-one onto mapping for each α

Case 1. $0 \le \alpha < \frac{1}{2}$;

$\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha);\alpha\right]$ (Corresponding PQ) $\leftrightarrow$ $\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right] = [P'Q']$,

$\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha);\alpha\right]$ (Corresponding RS) $\leftrightarrow$ $\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right] = [R'S']$ and

$\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right] \cap \left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right] = \varphi$

Case 2. $\frac{1}{2} \le \alpha < \lambda_2$;

$\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha);\alpha\right]$ (Corresponding KL) $\leftrightarrow$ $\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right] = [K'L']$,

$\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha);\alpha\right]$ (Corresponding MN) $\leftrightarrow$ $\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right] = [M'N']$ and

$\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right] \cap \left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right] = \varphi$

Case 3. $\lambda_2 \le \alpha \le \lambda_1$,

$\left[A^U_{IVPl_2}(\alpha), A^U_{IVPr_2}(\alpha);\alpha\right]$ (Corresponding UV) $\leftrightarrow$ $\left[A^U_{IVPl_2}(\alpha), A^U_{IVPr_2}(\alpha)\right] = [U'V']$

Step 3. Using the definition of signed distance we have:

$$d(A^U_{IVPl_1}(\alpha),0) = A^U_{IVPl_1}(\alpha), d(A^U_{IVPl_2}(\alpha),0) = A^U_{IVPl_2}(\alpha)$$

$$d(A^U_{IVPr_1}(\alpha),0) = A^U_{IVPr_1}(\alpha), d(A^U_{IVPr_2}(\alpha),0) = A^U_{IVPr_2}(\alpha)$$

$$d(A^L_{IVPl_1}(\alpha),0) = A^L_{IVPl_1}(\alpha), d(A^L_{IVPl_2}(\alpha),0) = A^L_{IVPl_2}(\alpha)$$

$$d(A^L_{IVPr_1}(\alpha),0) = A^L_{IVPr_1}(\alpha), d(A^L_{IVPr_2}(\alpha),0) = A^L_{IVPr_2}(\alpha)$$

Signed distance of $\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right]$ from 0 is $d\left(\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right],0\right)$ and it is defined as:

$$\frac{1}{2}\left[d(A^U_{IVPl_1}(\alpha),0) + d(A^L_{IVPl_1}(\alpha),0)\right] = \frac{1}{2}\left[A^U_{IVPl_1}(\alpha) + A^L_{IVPl_1}(\alpha)\right]$$

$$= \frac{1}{2}\left[2\alpha\left(a_2 + b_2 - a_1 - b_1\right) + \left(a_1 + b_1\right)\right] \quad (12)$$

Similarly,

$$d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right],0\right) = \frac{1}{2}\left[A^U_{IVPl_2}(\alpha) + A^L_{IVPl_2}(\alpha)\right]$$

$$= \frac{1}{2}\left[\begin{array}{l} \frac{\alpha}{\lambda_1 - \frac{1}{2}}\left(a_3 - a_2\right) - \frac{1}{2\left(\lambda_1 - \frac{1}{2}\right)}\left(a_3 - a_2\right) \\ + \frac{\alpha}{\lambda_2 - \frac{1}{2}}\left(a_3 - b_2\right) - \frac{1}{2\left(\lambda_2 - \frac{1}{2}\right)}\left(a_3 - b_2\right) + \left(a_2 + b_2\right) \end{array}\right] \quad (8)$$

$$d\left(\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right) = \frac{1}{2}\left[A^L_{IVPr_1}(\alpha) + A^U_{IVPr_1}(\alpha)\right]$$

$$= \frac{1}{2}\left[\begin{array}{l} \frac{1}{2\left(\lambda_2 - \frac{1}{2}\right)}\left(b_4 - a_3\right) - \frac{\alpha}{\lambda_2 - \frac{1}{2}}\left(b_4 - a_3\right) \\ + \frac{1}{2\left(\lambda_1 - \frac{1}{2}\right)}\left(a_4 - a_3\right) - \frac{\alpha}{\lambda_1 - \frac{1}{2}}\left(a_4 - a_3\right) + \left(a_4 + b_4\right) \end{array}\right] \quad (9)$$

$$d\left(\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right) = \frac{1}{2}\left[A^L_{IVPr_2}(\alpha) + A^U_{IVPr_2}(\alpha)\right]$$

$$= \frac{1}{2}\left[\left(a_5 + b_5\right) - 2\alpha\left(a_5 + b_5 - a_4 - b_4\right)\right] \quad (10)$$

Step 4. As $[P',Q'] \cap [R',S'] = \varphi$ the signed distance of $[P',Q'] \cup [R',S']$ from 0; for $\alpha-cuts$ of $\underset{\sim}{A}_{IVP}$ on $0 \le \alpha < \lambda_2$ $(\lambda_2 > \frac{1}{2})$ is defined as:

$$d\left(\left[A^U_{l_1}(\alpha), A^L_{IVPl_1}(\alpha)\right] \cup \left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right)$$

$$= \frac{1}{2}\left[d\left(\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right],0\right) + d\left(\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right)\right]$$

$$= \frac{1}{4}\left[2\alpha\left(a_2 + b_2 + a_4 + b_4 - a_1 - b_1 - a_5 - b_5\right) + \left(a_1 + b_1 + a_5 + b_5\right)\right] \quad (11)$$

Similarly as $[K',L'] \cap [M',N'] = \varphi$

$$d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right] \cup \left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right)$$

$$= \frac{1}{2}\left[d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right],0\right) + d\left(\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right)\right]$$

$$= \frac{1}{4}\left[\begin{array}{l} \frac{\alpha}{\lambda_1 - \frac{1}{2}}\left(2a_3 - a_2 - a_4\right) - \frac{1}{2\left(\lambda_1 - \frac{1}{2}\right)}\left(2a_3 - a_2 - a_4\right) + \frac{\alpha}{\lambda_2 - \frac{1}{2}} \\ \left(2a_3 - b_2 - b_4\right) - \frac{1}{2\left(\lambda_2 - \frac{1}{2}\right)}\left(2a_3 - b_2 - b_4\right) + \left(a_2 + b_2 + a_4 + b_4\right) \end{array}\right] \quad (12)$$

$$d\left(\left[A^U_{IVPl_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right) = \frac{1}{2}\left[A^U_{IVPl_2}(\alpha) + A^U_{IVPr_1}(\alpha)\right]$$

$$= \frac{1}{4}\left[\left(a_2 + a_4\right) + \frac{\alpha - \frac{1}{2}}{\lambda_1 - \frac{1}{2}}\left(2a_3 - a_2 - a_4\right)\right] \quad (13)$$

The membership functions in (2) and (3) are continuous on $0 \le \alpha < \frac{1}{2}$ with respect to α. Therefore, we can find average value as the following:

$$\frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}} d\left(\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right]\cup\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right)d\alpha$$

$$=\frac{1}{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{1}{2}\left[d\left(\left[A^U_{l_1}(\alpha), A^L_{l_1}(\alpha)\right],0\right)+d\left(\left[A^L_{r_2}(\alpha), A^U_{r_2}(\alpha)\right],0\right)\right]$$

$$=\frac{1}{16}\left[a_1+b_1+a_5+b_5+a_2+b_2+a_4+b_4\right] \qquad (14)$$

Similarly on $\frac{1}{2}\le\alpha<\lambda_2$,

$$\frac{1}{\lambda_2-\frac{1}{2}}\int_{\frac{1}{2}}^{\lambda_2} d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right]\cup\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right)d\alpha$$

$$=\frac{1}{\lambda_2-\frac{1}{2}}\int_{\frac{1}{2}}^{\lambda_2}\frac{1}{2}\left[d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right],0\right)+d\left(\left[A^L_{IVPr_1}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right)\right]$$

$$=\frac{1}{4}\left[\frac{\lambda_2-\frac{1}{2}}{2\left(\lambda_1-\frac{1}{2}\right)}\left(2a_3-a_2-a_4\right)+\frac{1}{2}\left(2a_3-b_2-b_4\right)+\left(a_2+b_2+a_4+b_4\right)\right] \qquad (15)$$

On $\lambda_2\le\alpha\le\lambda_1$,

$$\frac{1}{\lambda_1-\lambda_2}\int_{\lambda_2}^{\lambda_1} d\left(\left[A^U_{IVPl_2}(\alpha), A^U_{IVPr_1}(\alpha)\right],0\right)=\frac{1}{\lambda_1-\lambda_2}\int_{\lambda_2}^{\lambda_1}\frac{1}{2}\left[A^U_{IVPl_2}(\alpha)+A^U_{IVPr_1}(\alpha)\right]$$

$$=\frac{1}{4}\left[\left(a_2+a_4\right)+\frac{\lambda_1+\lambda_2-1}{2\left(\lambda_1-\frac{1}{2}\right)}\left(2a_3-a_2-a_4\right)\right] \qquad (16)$$

Step 5. The sign distance of Generalized IVPFN $\underset{\sim IVP}{A}=[\underset{\sim IVP}{A^L},\underset{\sim IVP}{A^U}]$ is defined using decomposition theorem in (7) as follows:

$$d(\underset{\sim IVP}{A},\underset{\sim}{0})=\frac{1}{1-\frac{1}{2}}\int_0^{\frac{1}{2}} d\left(\left[A^U_{IVPl_1}(\alpha), A^L_{IVPl_1}(\alpha)\right]\cup\left[A^L_{IVPr_2}(\alpha), A^U_{IVPr_2}(\alpha)\right],0\right)d\alpha$$

$$+\frac{1}{\lambda_2-\frac{1}{2}}\int_{\frac{1}{2}}^{\lambda_2} d\left(\left[A^U_{IVPl_2}(\alpha), A^L_{IVPl_2}(\alpha)\right]\cup\left[A^L_{IVPr_1}(\alpha)+A^U_{IVPr_1}(\alpha)\right],0\right)d\alpha$$

$$+\frac{1}{\lambda_1-\lambda_2}\int_{\lambda_2}^{\lambda_1} d\left(\left[A^U_{l_2}(\alpha), A^U_{r_1}(\alpha)\right],0\right)d\alpha$$

$$d(\underset{\sim IVP}{A},\underset{\sim}{0})=\frac{1}{8}\left[\begin{array}{l}\left(2a_1+2b_1+6a_2+3b_2+2a_3+6a_4+3b_4+2a_5+2b_5\right)\\ +\frac{\left(\lambda_1+2\lambda_2-\frac{3}{2}\right)}{2\left(\lambda_1-\frac{1}{2}\right)}\left(2a_3-a_2-a_4\right)\end{array}\right] \qquad (17)$$

Case 1. $\frac{1}{2}<\lambda_2<\lambda_1=1$,

$$d(\underset{\sim IVP}{A},\underset{\sim}{0})=\frac{1}{8}\left(2a_1+2b_1+6a_2+3b_2+2a_3+6a_4+3b_4+2a_5+2b_5\right)$$
$$+\left(2\lambda-\frac{1}{2}\right)\left(2a_3-a_2-a_4\right) \qquad (18)$$

Case 2. $\lambda_2=\lambda_1=1$,

$$d(\underset{\sim IVP}{A},\underset{\sim}{0})=\frac{1}{8}\left(2a_1+2b_1+6a_2+3b_2+2a_3+6a_4+3b_4+2a_5+2b_5\right)$$
$$+\frac{3}{2}\left(2a_3-a_2-a_4\right) \qquad (19)$$

V. SIGNED DISTANCE BASED RANKING OF IVPFNS

Ranking of two interval-valued pentagonal fuzzy numbers using signed distance is defined in the following manner:

Let $\underset{\sim IVP}{A}=\left[\underset{\sim IVP}{A^L},\underset{\sim IVP}{A^U}\right]$ and $\underset{\sim IVP}{B}=\left[\underset{\sim IVP}{B^L},\underset{\sim IVP}{B^U}\right]$ be two IVPFNs. Then

i) $\underset{\sim IVP}{A}\prec\underset{\sim IVP}{B}$ iff $d(\underset{\sim IVP}{B},\underset{\sim}{0})<d(\underset{\sim IVP}{A},\underset{\sim}{0})$

ii) $\underset{\sim IVP}{A}\succ\underset{\sim IVP}{B}$ iff $d(\underset{\sim IVP}{B},\underset{\sim}{0})>d(\underset{\sim IVP}{A},\underset{\sim}{0})$

iii) $\underset{\sim IVP}{A}\approx\underset{\sim IVP}{B}$ iff $d(\underset{\sim IVP}{B},\underset{\sim}{0})=d(\underset{\sim IVP}{A},\underset{\sim}{0})$

A. Illustrative example

Let $\underset{\sim IVP}{A}=[\underset{\sim IVP}{A^L},\underset{\sim IVP}{A^U}]$ = [(2,4,5,6,8;0.8), (1,3,5,7,9;1)] and $\underset{\sim IVP}{B}=[\underset{\sim IVP}{B^L},\underset{\sim IVP}{B^U}]$ = [(2.5,4.5,5.5,6.5,8.5;0.6), (1.5,3.5,5.5,7.5,9.5;1)], then

$d(\underset{\sim IVP}{A},\underset{\sim}{0})=1.75$ and $d(\underset{\sim IVP}{B},\underset{\sim}{0})=1.888$

$d(\underset{\sim IVP}{B},\underset{\sim}{0})>d(\underset{\sim IVP}{A},\underset{\sim}{0})\rightarrow\underset{\sim IVP}{B}\succ\underset{\sim IVP}{A}$

VI. SIMILARITY MEASURES OF IVPFNS

Similarity measures are used to compare and contrast two fuzzy sets. We propose two similarity measure for IVPFNs.

A. Methodology

For the first similarity measure we use geometric distance, area under membership functions and height of pentagonal fuzzy number. For the second similarity measure we use geometric distance, perimeter and height of pentagonal fuzzy number.

B. Algorithm

We propose the following algorithm for computing similarity measures:

Step 1. Identify the height of lower as well as upper pentagonal fuzzy numbers.

Step 2. Find the area of pentagonal fuzzy numbers

Step 3. Find the geometric distance between two concerned pentagonal fuzzy numbers.

Step 4. Find the similarity measures between two lower pentagonal fuzzy numbers and two upper pentagonal fuzzy numbers.

Step 5. Find similarity measures between two IVPFNs combining similarity measures of lower as well as upper pentagonal fuzzy numbers.

C. Flow Chart

The above algorithm is represented as the following flow chart:

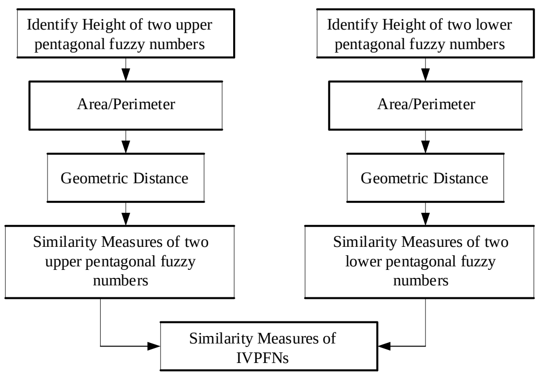

Fig. 7. Flow Chart to arrive at Similarity Measures

D. Mathematical Results

For similarity measures we propose the following formula:

$$sim\left(\underset{\sim IVP}{A},\underset{\sim IVP}{B}\right)=\left[\frac{sim\left(\underset{\sim IVP}{A^L},\underset{\sim IVP}{B^L}\right)+sim\left(\underset{\sim IVP}{A^U},\underset{\sim IVP}{B^U}\right)}{2}\right]^{1/2} \quad (20)$$

Case 1. Combining the formulas to find geometric distance, area and height we define the similarity measures of two lower and upper pentagonal fuzzy numbers in the following manner:

$$sim\left(\underset{\sim IVP}{A^L},\underset{\sim IVP}{B^L}\right)=\left(1-\frac{1}{5}\sum_{i=1}^{5}\left|a_i^L-b_i^L\right|\right) \times\left(1-\frac{1}{2}\left(\left|area\left(\underset{\sim IVP}{A^L}\right)-area\left(\underset{\sim IVP}{B^L}\right)\right|+\left|w_{\underset{\sim IVP}{A}}^L-w_{\underset{\sim IVP}{B}}^L\right|\right)\right) \quad (21)$$

$$sim\left(\underset{\sim IVP}{A^U},\underset{\sim IVP}{B^U}\right)=\left(1-\frac{1}{5}\sum_{i=1}^{5}\left|a_i^U-b_i^U\right|\right) \times\left(1-\left(\left|area\left(\underset{\sim IVP}{A^U}\right)-area\left(\underset{\sim IVP}{B^U}\right)\right|+\left|w_{\underset{\sim IVP}{A}}^L-w_{\underset{\sim IVP}{B}}^L\right|\right)\right) \quad (22)$$

Here $\frac{1}{5}\sum_{i=1}^{5}\left|a_i^L-b_i^L\right|$ is geometric distance, $w_{\underset{\sim IVP}{A}}^L$, $w_{\underset{\sim IVP}{A}}^U$ are height of lower and upper pentagonal fuzzy numbers respectively.

Case 2. Combining the formulas to find geometric distance, perimeter and height we define the similarity measures of two lower and upper pentagonal fuzzy numbers in the following manner:

$$sim\left(\underset{\sim IVP}{A^L},\underset{\sim IVP}{B^L}\right)=\left(1-\frac{\sum_{i=1}^{5}\left|a_i^L-b_i^L\right|}{5}\right) \times\left[\frac{\min\left(P\left(\underset{\sim IVP}{A^L}\right),P\left(\underset{\sim IVP}{B^L}\right)\right)+\min\left(w_{\underset{\sim IVP}{A}}^L,w_{\underset{\sim IVP}{B}}^L\right)}{\max\left(P\left(\underset{\sim IVP}{A^L}\right),P\left(\underset{\sim IVP}{B^L}\right)\right)+\max\left(w_{\underset{\sim IVP}{A}}^L,w_{\underset{\sim IVP}{B}}^L\right)}\right] \quad (23)$$

$$sim\left(\underset{\sim IVP}{A^U},\underset{\sim IVP}{B^U}\right)=\left(1-\frac{\sum_{i=1}^{5}\left|a_i^U-b_i^U\right|}{5}\right) \times\left[\frac{\min\left(P\left(\underset{\sim IVP}{A^U}\right),P\left(\underset{\sim IVP}{B^U}\right)\right)+\min\left(w_{\underset{\sim IVP}{A}}^U,w_{\underset{\sim IVP}{B}}^U\right)}{\max\left(P\left(\underset{\sim IVP}{A^U}\right),P\left(\underset{\sim IVP}{B^U}\right)\right)+\max\left(w_{\underset{\sim IVP}{A}}^U,w_{\underset{\sim IVP}{B}}^U\right)}\right] \quad (24)$$

Here $P\left(\underset{\sim IVP}{A^L}\right)$ is perimeter of IVPFN which given as follows:

$$P\left(\underset{\sim IVP}{A^L}\right)=\sqrt{(a_2^L-a_1^L)^2+0.25}+\sqrt{(a_3^L-a_2^L)^2+(w_2-0.5)^2}+\sqrt{(a_4^L-a_3^L)^2+(0.5-w_2)^2}+\sqrt{(a_5^L-a_4^L)^2+0.25}+(a_5^L-a_1^L)$$

$$P\left(\underset{\sim IVP}{B^L}\right)=\sqrt{(b_2^L-b_1^L)^2+0.25}+\sqrt{(b_3^L-b_2^L)^2+(w_2'-0.5)^2}+\sqrt{(b_4^L-b_3^L)^2+(0.5-w_2')^2}+\sqrt{(b_5^L-b_4^L)^2+0.25}+(b_5^L-b_1^L)$$

$$P\left(\underset{\sim IVP}{A^U}\right)=\sqrt{(a_2^U-a_1^U)^2+0.25}+\sqrt{(a_3^U-a_2^U)^2+0.25}+\sqrt{(a_4^U-a_3^U)^2+0.25}+\sqrt{(a_5^U-a_4^U)^2+0.25}+(a_5^U-a_1^U)$$

$$P\left(\underset{\sim IVP}{B^U}\right)=\sqrt{(b_2^U-b_1^U)^2+0.25}+\sqrt{(b_3^U-b_2^U)^2+0.25}+\sqrt{(b_4^U-b_3^U)^2+0.25}+\sqrt{(b_5^U-b_4^U)^2+0.25}+(b_5^U-b_1^U)$$

E. Illustration for Similarity Measures:

Case 1. Using geometric distance, area and height:

Let $\underset{\sim IVP}{A}=[\underset{\sim IVP}{A^L},\underset{\sim IVP}{A^U}]$ = [(2,4,5,6,8;1), (1,3,5,7,9;1)] $\underset{\sim IVP}{B}=[\underset{\sim IVP}{B^L},\underset{\sim IVP}{B^U}]$ = [(2.5,4.5,5.5,6.5,8.5;1), (1.5,3.5,5.5,7.5,9.5;1)] be two IVPFNs. Then using the formula in (21) and (22) we have

$$sim\left(\underset{\sim IVP}{A^L},\underset{\sim IVP}{B^L}\right)=0.4$$

$$sim\left(\underset{\sim IVP}{A^U},\underset{\sim IVP}{B^U}\right)=0.5$$

Using the formula in (20) we have

$$sim\left(\underset{\sim IVP}{A},\underset{\sim IVP}{B}\right)=0.671$$

Case 2: Using geometric distance, perimeter and height:
We consider the two IVPFNs in the above example and using the formulas in (23) and (24) we get the following:

$$sim\left(\underset{\sim IVP}{A^L},\underset{\sim IVP}{B^L}\right)=0.492$$

$$sim\left(\underset{\sim IVP}{A^U},\underset{\sim IVP}{B^U}\right)=0.5$$

Using the formula in (20) we have

$$sim\left(\underset{\sim IVP}{A},\underset{\sim IVP}{B}\right)=0.704$$

VII. RESULTS

The result obtained in (17), (18) and (19) are used to rank general IVPFNs partially normal IVPFNs and normal IVPFNs respectively. We observe that if the IVPFNs are symmetric; heights of the fuzzy number does not have impact on ranking value.

Secondly, the formula in (20) gives us the similarity measures of IVPFNS. This value is an averaging value of similarity measures of lower and upper pentagonal fuzzy numbers. Raking and similarity measures of IVPFNs help us to draw conclusions particularly when dealing with decision problems under fuzzy environment.

VIII. CONCLUSION

Interval-valued pentagonal fuzzy numbers have two important characteristics namely interval and alpha level variation. They are used when vagueness and imprecision is expressed in intervals and there is inherent variation in the fuzzy concepts. These IVPFNs are applied in decision making when truth values are expressed in intervals.

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