Group $S_3$ Cordial Remainder Labeling

A. Lourdusamy, S. Jenifer Wency, F. Patrick

Abstract: Let $G = (V(G), E(G))$ be a graph and let $g : V(G) \rightarrow S_3$ be a function. For each edge $xy$, assign the label $r$ where $r$ is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $o(g(x)) \geq o(g(y))$ or $o(g(y)) \geq o(g(x))$. The function $g$ is called a group $S_3$ cordial remainder labeling of $G$ if $|v_g(x) - v_g(y)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$, where $v_g(x)$ denotes the number of vertices labeled with $x$ and $e_g(i)$ denotes the number of edges labeled with $i$ ($i = 0, 1$). A graph $G$ which admits a group $S_3$ cordial remainder labeling is called a group $S_3$ cordial remainder graph. In this paper, we introduce the concept of group $S_3$ cordial remainder labeling. We prove that some standard graphs admit a group $S_3$ cordial remainder labeling.

Keywords: Group $S_3$ cordial remainder labeling, Group $S_3$ cordial remainder graph, path, cycle.

I. INTRODUCTION

By a graph we mean a finite, simple and undirected one. The vertex set and the edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ so that the order and size of $G$ are $|V(G)|$ and $|E(G)|$ respectively. Terms are not defined here follows from Harary [4]. Graph labeling was first introduced in 1960’s. Most of the graph labeling trace their origins in the paper presented by Alex Rosa in 1967 [6]. The complete survey of graph labeling is in [3]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by I. Cahit in [1]. Ponraj et al. introduced the concept of remainder cordial labeling in [5]. Chandra et al. introduced the concept of group $S_3$ cordial prime labeling in [2]. Motivated by these concepts, we introduce the concept of group $S_3$ cordial remainder labeling.

II. PRELIMINARIES

Definition 1.1. The join of two graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$, and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set is $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 1.2. The wheel $W_n$ is defined as the join $C_n + K_1$.

Definition 1.3. The graph $F_n = P_n + K_1$ is called a fan.

Definition 1.4. The bistar $B_{m,n}$ is the graph obtained by attaching the apex vertices of two copies of star $K_{1,m}$ by an edge.

Definition 1.5. The corona $G_1 \circ G_2$ of two graphs $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ (with $p_i$ vertices) and $p_i$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ with an edge to every vertex in the $i^{th}$ copy of $G_2$. The graph $P_n \circ K_1$ is called a Comb. The graph $C_n \circ K_1$ is called a Crown.

Definition 1.6. Let $A$ be a group. The order of $a \in A$ is the least positive integer $n$ such that $a^n = e$. We denote the order of $a$ by $o(a)$.

In this paper, we prove that path, cycle, star, bistar, complete bipartite, wheel, fan, comb and crown graphs admit a group $S_3$ cordial remainder labeling.

III. MAIN RESULTS

Definition 2.1. Consider the symmetric group $S_3$. Let the elements of $S_3$ be $e, a, b, c, d, f$ where

$e = (1 2 3) \quad a = (1 2 3) \quad b = (1 2 3)
(1 2 3) \quad c = (1 2 3) \quad d = (1 2 3) \quad f = (1 2 3)$

We have $o(e) = 1, o(a) = o(b) = o(c) = 2, o(d) = o(f) = 3$.

Definition 2.2. Let $G = (V(G), E(G))$ be a graph and let $g : V(G) \rightarrow S_3$ be a function. For each edge $xy$ assign the label $r$ where $r$ is the remainder when $o(g(x))$ is divided by $o(g(y))$ or $o(g(y))$ is divided by $o(g(x))$ according as $o(g(x)) \geq o(g(y))$ or $o(g(y)) \geq o(g(x))$. The function $g$ is called a group $S_3$ cordial remainder labeling of $G$ if $|v_g(x) - v_g(y)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$, where $v_g(x)$ denotes the number of vertices labeled with $x$ and $e_g(i)$ denotes the number of edges labeled with $i$ ($i = 0, 1$). A graph $G$ which admits a group $S_3$ cordial remainder labeling is called a group $S_3$ cordial remainder graph.

Theorem 2.3. Path $P_n$ is a group $S_3$ cordial remainder graph.
Group $S_3$ Cordial Remainder Labeling

**Proof.** Let $v_1,v_2,...,v_n$ denote the vertices of $P_n$. Fig. 1 gives a group $S_3$ cordial remainder labeling of $P_n$ for $n \leq 5$.

![Diagram of path $P_n$ for $n \leq 5$.](image)

Clearly, $|v(i)-v(j)| \leq 1$ for $i,j \in S_3$ and $|e_g(0)-e_g(1)| \leq 1$. Hence $g$ is a group $S_3$ cordial remainder labeling.

**Case 4.** $n=3$ (mod 6).

Let $n=6k+3$ and $k \geq 1$. Assign the label to the vertices $v_i$ for $1 \leq i \leq 6k$ as in Case (1) and for the remaining vertices assign the following labels:

$$g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ b & \text{if } i = 6k+2 \\ f & \text{if } i = 6k+3 \end{cases}$$

Here $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = k$ and $e_g(0) = e_g(1) = 3k+2$. Clearly $|v_g(i)-v_g(j)| \leq 1$ for $i,j \in S_3$ and $|e_g(0)-e_g(1)| \leq 1$. Hence $g$ is a group $S_3$ cordial remainder labeling.

**Case 5.** $n=2$ (mod 6).

Let $n=6k+2$ and $k \geq 1$. Assign the label to the vertices $v_i$ for $1 \leq i \leq 6k$ as in the Case (1) and for the remaining vertices assign the following labels:

$$g(v_i) = \begin{cases} a & \text{if } i = 6k+1 \\ f & \text{if } i = 6k+2 \end{cases}$$

Here $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = k$ and $e_g(0) = e_g(1) = 3k+3$. Therefore $|v_g(i)-v_g(j)| \leq 1$ for $i,j \in S_3$ and $|e_g(0)-e_g(1)| \leq 1$. Hence $g$ is a group $S_3$ cordial remainder labeling.

**Case 6.** $n=1$ (mod 6).

Let $n=6k+1$ and $k \geq 1$. Assign the label to the vertices $v_i$ for $1 \leq i \leq 6k$ as in the Case (1) and $g(v_{6k+1}) = a$. Here $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = k$ and $e_g(0) = e_g(1) = 3k$. Therefore $|v_g(i)-v_g(j)| \leq 1$ for $i,j \in S_3$ and $|e_g(0)-e_g(1)| \leq 1$. Hence $g$ is a group $S_3$ cordial remainder labeling.

Hence path $P_n$ is a group $S_3$ cordial remainder graph. □

**Theorem 2.4.** Cycle $C_n$ is a group $S_3$ cordial remainder graph.

**Proof.** Let $v_1,v_2,...,v_n$ denote the vertices of $C_n$. The same labeling pattern is followed as in Theorem 2.3, except for $n=2$ (mod 6).

Suppose $n=2$ (mod 6). Define a function $g : V(C_n) \rightarrow S_3$ as follows. Let $n=6k+2$ and $k \geq 1$.
\[ g(v_j) = \begin{cases} 
  a & \text{if } i = 1 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  b & \text{if } i = 2 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  c & \text{if } i = 3 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  d & \text{if } i = 4 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  e & \text{if } i = 5 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  f & \text{if } i = 6k + 1 \\
  a & \text{if } i = 6k + 2 
\end{cases} \]

Here \( v_j(a) = v_j(f) = k + 1 \), \( v_j(b) = v_j(c) = v_j(d) = v_j(e) = k \) and \( e_j(0) = e_j(1) = 3k + 1 \). Therefore \( |v_j(i) - v_j(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_j(0) - e_j(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

Hence cycle \( C_n \) is a group \( S_3 \) cordial remainder graph. □

**Theorem 2.5.** Star \( K_{1,n} \) is a group \( S_3 \) cordial remainder graph for every \( n \).

**Proof.** Let \( v \) be the apex vertex and \( v_1, v_2, \ldots, v_n \) be the pendant vertices of \( K_{1,n} \). Then \( K_{1,n} \) is of order \( n+1 \) and size \( n \). We define \( g : V(K_{1,n}) \rightarrow S_3 \) as follows:

\[ g(v) = d ; \]

\[ g(v_j) = \begin{cases} 
  a & \text{if } i = 1 \pmod{6} \text{ and } 1 \leq i \leq n \\
  e & \text{if } i = 2 \pmod{6} \text{ and } 1 \leq i \leq n \\
  b & \text{if } i = 3 \pmod{6} \text{ and } 1 \leq i \leq n \\
  f & \text{if } i = 4 \pmod{6} \text{ and } 1 \leq i \leq n \\
  c & \text{if } i = 5 \pmod{6} \text{ and } 1 \leq i \leq n \\
  d & \text{if } i = 0 \pmod{6} \text{ and } 1 \leq i \leq n . 
\end{cases} \]

It is easy to verify that \( |v_j(i) - v_j(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_j(0) - e_j(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling. □

**Theorem 2.6.** Bistar \( B_{n,n} \) is a group \( S_3 \) cordial remainder graph for every \( n \).

**Proof.** Let \( v \) and \( u \) be the apex vertices and \( v_1, v_2, \ldots, u_1, u_2, \ldots, u_n \) be the pendant vertices of \( K_{1,n} \). Then \( B_{n,n} \) is of order \( 2n+2 \) and size \( 2n+1 \). We define \( g : V(B_{n,n}) \rightarrow S_3 \) as follows:

\[ g(u) = a ; g(u_1) = b ; g(u_2) = f ; \]

\[ g(v) = d ; g(v_1) = c ; g(v_2) = e ; \]

\[ g(u_j) = \begin{cases} 
  e & \text{if } i = 0 \pmod{3} \text{ and } 3 \leq i \leq n \\
  d & \text{if } i = 1 \pmod{3} \text{ and } 3 \leq i \leq n \\
  b & \text{if } i = 2 \pmod{3} \text{ and } 3 \leq i \leq n ; \\
  a & \text{if } i = 0 \pmod{3} \text{ and } 3 \leq i \leq n \\
  f & \text{if } i = 1 \pmod{3} \text{ and } 3 \leq i \leq n \\
  c & \text{if } i = 2 \pmod{3} \text{ and } 3 \leq i \leq n . 
\end{cases} \]

From Table - I, it is clear that \( g \) is a group \( S_3 \) cordial remainder labeling.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_g(a) )</th>
<th>( v_g(b) )</th>
<th>( v_g(c) )</th>
<th>( v_g(d) )</th>
<th>( v_g(e) )</th>
<th>( v_g(f) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( 3k ) (( k \geq 1 ))</td>
<td>( k+1 )</td>
<td>( k )</td>
<td>( k )</td>
<td>( k )</td>
<td>( k+1 )</td>
<td>( k )</td>
</tr>
<tr>
<td>( 3k+1 ) (( k \geq 1 ))</td>
<td>( k+1 )</td>
<td>( k )</td>
<td>( k )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
</tr>
<tr>
<td>( 3k+2 ) (( k \geq 1 ))</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
<td>( k+1 )</td>
</tr>
</tbody>
</table>

Hence bistar \( B_{n,n} \) is a group \( S_3 \) cordial remainder graph. □

**Theorem 2.7.** Wheel \( W_n \) is a group \( S_3 \) cordial remainder graph for \( n \geq 3 \).

**Proof.** Let \( W_n \) be the wheel \( C_n + K_1 \). Let \( u \) be the apex of the wheel and \( u_1, u_2, \ldots, u_n \) be the vertices on the cycle \( C_n \).

**Case 1.** \( n = 3 \).

Define \( g \) by \( g(u) = d, g(u_1) = a, g(u_2) = b \) and \( g(u_3) = c \).

It is easy to verify that \( |v_g(i) - v_g(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_g(0) - e_g(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Case 2.** \( n = 4 \).

Define \( g \) by \( g(u) = d, g(u_1) = a, g(u_2) = f, g(u_3) = b \) and \( g(u_4) = e \).

It is easy to verify that \( |v_g(i) - v_g(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_g(0) - e_g(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Case 3.** \( n = 5 \).

Define \( g \) by \( g(u) = d, g(u_1) = a, g(u_2) = b, g(u_3) = f, g(u_4) = c \) and \( g(u_5) = e \).

It is easy to verify that \( |v_g(i) - v_g(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_g(0) - e_g(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Case 4.** \( n \geq 6 \). Define \( g : V(W_n) \rightarrow S_3 \) as follows.

**Subcase (i).** \( n = 0 \pmod{6} \).

Let \( n = 6k \) and \( k \geq 1 \).

\[ g(u) = d ; \]

\[ g(v) = \begin{cases} 
  a & \text{if } i = 1 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  d & \text{if } i = 2 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  b & \text{if } i = 3 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  c & \text{if } i = 4 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  f & \text{if } i = 5 \pmod{6} \text{ and } 1 \leq i \leq 6k \\
  e & \text{if } i = 0 \pmod{6} \text{ and } 1 \leq i \leq 6k . 
\end{cases} \]

Here \( v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = k \), \( v_g(d) = k+1 \) and \( e_g(0) = e_g(1) = 6k \). Therefore \( |v_g(i) - v_g(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_g(0) - e_g(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Subcase (ii).** \( n = 5 \pmod{6} \).
Let \( n = 6k + 5 \) and \( k \geq 1 \). Assign the label to the vertices \( u \) and \( u_i \) for \( 1 \leq i \leq 6k \) as in Subcase (i) and for the remaining vertices assign the following labels:

\[
g(u_i) = \begin{cases} 
  a & \text{if } i = 6k+1 \\
  b & \text{if } i = 6k+2 \\
  f & \text{if } i = 6k+3 \\
  c & \text{if } i = 6k+4 \\
  e & \text{if } i = 6k+5. 
\end{cases}
\]

Here \( v_x(a) = v_x(b) = v_x(c) = v_x(d) = v_x(e) = v_x(f) = k+1 \) and \( e_x(0) = e_x(1) = 6k+5 \). Clearly \( |v_x(i)-v_x(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_x(0)-e_x(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Subcase (iii).** \( n = 4 \mod 6 \).

Let \( n = 6k + 4 \) and \( k \geq 1 \). Assign the label to the vertices \( u \) and \( u_i \) for \( 1 \leq i \leq 6k \) as in Subcase (i) and for the remaining vertices assign the following labels:

\[
g(u_i) = \begin{cases} 
  a & \text{if } i = 6k+1 \\
  b & \text{if } i = 6k+2 \\
  f & \text{if } i = 6k+3 \\
  c & \text{if } i = 6k+4. 
\end{cases}
\]

Here \( v_x(a) = v_x(b) = v_x(c) = v_x(d) = k+1, v_x(e) = v_x(f) = k \) and \( e_x(0) = e_x(1) = 6k+4 \). Clearly \( |v_x(i)-v_x(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_x(0)-e_x(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Subcase (iv).** \( n = 3 \mod 6 \).

Let \( n = 6k + 3 \) and \( k \geq 1 \). Assign the label to the vertices \( u \) and \( u_i \) for \( 1 \leq i \leq 6k \) as in Subcase (i) and for the remaining vertices assign the following labels:

\[
g(u_i) = \begin{cases} 
  a & \text{if } i = 6k+1 \\
  b & \text{if } i = 6k+2 \\
  c & \text{if } i = 6k+3. 
\end{cases}
\]

Here \( v_x(a) = v_x(b) = v_x(c) = v_x(d) = k+1 \), \( v_x(e) = v_x(f) = k \) and \( e_x(0) = e_x(1) = 6k+3 \). Clearly \( |v_x(i)-v_x(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_x(0)-e_x(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Case 5.** \( n = 2 \mod 6 \).

Let \( n = 6k + 2 \) and \( k \geq 1 \). Assign the label to the vertices \( u \) and \( u_i \) for \( 1 \leq i \leq 6k \) as in Subcase (i) and for the remaining vertices assign the following labels:

\[
g(u_i) = \begin{cases} 
  f & \text{if } i = 6k+1 \\
  a & \text{if } i = 6k+2. 
\end{cases}
\]

Here \( v_x(a) = v_x(b) = v_x(c) = v_x(d) = k+1 \), \( v_x(e) = v_x(f) = k \) and \( e_x(0) = e_x(1) = 6k+2 \). Therefore \( |v_x(i)-v_x(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_x(0)-e_x(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling.

**Case 6.** \( n = 1 \mod 6 \).

Let \( n = 6k + 1 \) and \( k \geq 1 \). Assign the label to the vertices \( u \) and \( u_i \) for \( 1 \leq i \leq 6k \) as in Subcase (i) and \( g(u_{6k+1}) = f \). Here \( v_x(a) = v_x(b) = v_x(c) = v_x(e) = k \), \( v_x(d) = v_x(f) = k+1 \) and \( e_x(0) = e_x(1) = 6k+1 \). Therefore \( |v_x(i)-v_x(j)| \leq 1 \) for \( i, j \in S_3 \) and \( |e_x(0)-e_x(1)| \leq 1 \). Hence \( g \) is a group \( S_3 \) cordial remainder labeling of \( K_{2,n} \), if \( n = 0 \mod 6 \).

Theorem 2.9. The complete bipartite graph \( K_{2,n} \) is a group \( S_3 \) cordial remainder graph if and only if \( n = 1, 2, 3, 4 \) or \( n = 0 \mod 6 \).

**Proof.** Let \( V_1, V_2 \) be the bipartition of \( K_{2,n} \) with \( V_1 = \{v_1, v_2\} \) and \( V_2 = \{u_1, u_2, ..., u_n\} \).

Suppose \( n = 1, 2, 3 \) or 4. Table - II gives a group \( S_3 \) cordial remainder labeling of \( K_{2,n} \) for \( 1 \leq i \leq 4 \).

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a )</td>
<td>( e )</td>
<td>( d )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( a )</td>
<td>( e )</td>
<td>( d )</td>
<td>( f )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( d )</td>
<td>( e )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td>( d )</td>
<td>( e )</td>
<td>( f )</td>
</tr>
</tbody>
</table>

Suppose \( n = 0 \mod 6 \). Let \( n = 6k \) and \( k \geq 1 \). Define \( g : V(K_{2,n}) \rightarrow S_3 \) as follows.

\[
g(v_i) = d; \quad g(v_1) = f;
\]

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a )</td>
<td>( i = 1 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( b )</td>
<td>( i = 2 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( c )</td>
<td>( i = 3 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( d )</td>
<td>( i = 4 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( e )</td>
<td>( i = 5 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( f )</td>
<td>( i = 0 \mod 6 )</td>
<td>( 1 \leq i \leq 6k )</td>
<td></td>
</tr>
</tbody>
</table>

Hence \( g \) is a group \( S_3 \) cordial remainder labeling of \( K_{2,n} \) if \( n = 0 \mod 6 \).
Conversely, assume that the complete bipartite graph $K_{2,n}$ is a group $S_3$ cordial remainder graph. Let $g$ be a group $S_3$ cordial remainder labeling of $K_{2,n}$.

**Case 1.** $g(v_1) = e$ or $g(v_2) = e$.

Without loss of generality, let $g(v_1) = e$. The $n$ edges incident with $v_1$ get label 0. So the remaining $n$ edges should get label 1. If $g(v_2) = d$, then only $a, b$ and $c$ can be used to label the remaining vertices. Hence $n = 3$. If $g(v_2) = a$, then only $d$ and $f$ can be used to label the remaining vertices. Hence $n = 2$. If $g(v_2) = a$, then only $d$ can be used to label the remaining vertices. Hence $n = 1$.

**Case 2.** $g(v_i) ≠ e$ for $i = 1, 2$.

It is enough to consider the following subcases.

**Subcase (i).** $g(v_1) = d, g(v_2) = f$.

Assign labels $d, e, f$ to any three vertices. We get six edges with label 0. Next we assign labels $a, b, c$ to another set of three vertices and in this process we get six edges with label 1. Thus for $n = 0 \pmod{6}$, we assign the labels $d, e, f$ for every 3 vertices and we need 3 vertices with labels $a, b, c$. Hence $n = 0 \pmod{6}$.

**Subcase (ii).** $g(v_1) = a, g(v_2) = b$.

Assign labels $e, c$ to any two vertices. We get four edges with label 0. Next we assign labels $d, f$ to another set of two vertices and in this process we get four edges with label 1. Hence $n = 4$.

**Subcase (iii).** $g(v_1) = d, g(v_2) = a$.

Assign the label $e$ to a vertex. We get 2 edges with label 0. Vertices with every other label give one edge with label 0 and another edge with label 1. So, $|e_0(0) - e_1(1)| > 1$. Hence this is impossible. □

**Theorem 2.10.** The comb $P_n \circ K_1$ is a group $S_3$ cordial remainder graph for every $n$.

**Proof.** Let $V(P_n \circ K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(P_n \circ K_1) = \{u_{i+1}, v_i : 1 \leq i \leq n-1\} \cup \{u_1, v_{n} : 1 \leq i \leq n\}$. Then $P_n \circ K_1$ is of order $2n$ and size $2n - 1$.

Define $f : V(P_n \circ K_1) \to S_3$ as follows:

$g(u_i) = \begin{cases} 
  a & \text{if } i = 1 \pmod{6} \\
  b & \text{if } i = 2 \pmod{6} \\
  d & \text{if } i = 3 \pmod{6} \\
  c & \text{if } i = 4 \pmod{6} \\
  e & \text{if } i = 5 \pmod{6} \\
  f & \text{if } i = 0 \pmod{6} 
\end{cases}$

From Table III, it is clear that $g$ is a group $S_3$ cordial remainder labeling. Hence comb is a group $S_3$ cordial remainder graph. □

**Corollary 2.11.** The crown $C_n \circ K_1$ is a group $S_3$ cordial remainder graph for every $n$.

**Proof.** Let $V(C_n \circ K_1) = \{u_i, v_i : 1 \leq i \leq n\}$ where $u_1, u_2, ..., u_n$ are the vertices of the cycle and $v_1, v_2, ..., v_n$ are the pendant vertices adjacent to $u_1, u_2, ..., u_n$ respectively.

Assign the labels to the vertices $u_i$ and $v_i$ as in the Theorem 2.10, except by interchanging the labels of the vertices $v_i$ and $v_j$ for $n = 1, 2, 5 \pmod{6}$. It is easy to verify that $|v_g(i) - v_g(f)| \leq 1$ for $i, j \in S_1$ and $|e_g(0) - e_g(1)| \leq 1$.

Hence $g$ is a group $S_3$ cordial remainder labeling. □

**Theorem 2.12.** Let $g$ be a group $S_3$ cordial remainder labeling of a graph $G_1$ of order $6m$ and size $2n$ and let $h$ be any group $S_1$ cordial remainder labeling of a graph $G_2$. Then $G_1 \cup G_2$ is also a group $S_3$ cordial remainder graph.

**Proof.** Let $g, h$ be a group $S_3$ cordial remainder labeling of $G_1$ and $G_2$ respectively. Since $G_1$ has $6m$ vertices and $2n$ edges, we have $v_g(a) = v_g(b) = v_g(c) = v_g(d) = v_g(e) = v_g(f) = m$ and $e_g(0) = e_g(1) = n$. We define a vertex labeling $l$ of $G_1 \cup G_2$ such that

$l(u) = \begin{cases} 
  g(u) & \text{if } u \in V(G_1) \\
  h(u) & \text{if } u \in V(G_2).
\end{cases}$

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_g(a)$</th>
<th>$v_g(b)$</th>
<th>$v_g(c)$</th>
<th>$v_g(d)$</th>
<th>$v_g(e)$</th>
<th>$v_g(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6k - 5$ ($k \geq 1$)</td>
<td>$2k - 1$</td>
<td>$2k - 2$</td>
<td>$2k - 1$</td>
<td>$2k - 2$</td>
<td>$2k - 2$</td>
<td>$2k - 2$</td>
</tr>
<tr>
<td>$6k - 4$ ($k \geq 1$)</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 2$</td>
<td>$2k - 2$</td>
<td>$2k - 1$</td>
</tr>
<tr>
<td>$6k - 3$ ($k \geq 1$)</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
</tr>
<tr>
<td>$6k - 2$ ($k \geq 1$)</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
</tr>
<tr>
<td>$6k - 1$ ($k \geq 1$)</td>
<td>$2k - 1$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k - 1$</td>
<td>$2k - 1$</td>
</tr>
<tr>
<td>$6k$ ($k \geq 1$)</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k$</td>
<td>$2k$</td>
</tr>
</tbody>
</table>

\[\text{Table III: Comb graph}\]
Hence, \( v_i(i) = v_g(i) + v_h(i) \) for \( i \in S_3 \) and \\
\( e_i(i) = e_g(i) + e_h(i) \) for \( i = 0,1 \). Therefore \( |v_i(i) - v_j(j)| \leq 1 \) \\
for \( i, j \in S_3 \) and \( |e_i(0) - e_j(1)| \leq 1 \). Hence \( I \) is a group \( S_3 \) 

cordial remainder labeling. Thus \( G_1 \cup G_2 \) is also a group \( S_3 \) 
cordial remainder graph. □

IV. CONCLUSION

Labeled graphs serve as useful models for a broad range of 
applications such as: coding theory, radar, astronomy, circuit 
design and communication network. In this paper, we have 
introduced the concept of group \( S_3 \) cordial remainder 
labeling. Researchers can explore the possibility of applying 
this concept to the above mentioned areas. We have proved 
that path, cycle, star, bistar, complete bipartite, wheel, fan, 
comb and crown graphs are group \( S_3 \) cordial remainder 
graphs.

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AUTHORS PROFILE

A. Lourdusamy received M.Sc. from St.Joseph’s 
College, Trichy, India (affiliated to Bharathidasan 
University, Trichy) and Ph.D at Manonmaniam 
Sundaranar University, Tirunelveli in India. His Ph.D. 
was in Graph Theory. At present he is an Associate 
Professor and IQAC Coordinator of St.Xavier’s College, Palayamkottai. 
Since 1986 he has served many colleges in Tamil Nadu as assistant 
professor. He has published 80 publications in National/ International 
Journals so far. He is also reviewer for Math Review (American Mathematical 
Society) and ZtMath (European Mathematical Society).

S. Jenifer Wency, M.Sc., B.Ed., M.Phil., is pursuing 
Ph.D. at Manonmaniam Sundaranar University under the 
guidance of Dr. A. Lourdusamy. Her area of interest is 
Graph Theory.

Dr. F. Patrick, M.Sc., B.Ed., M.Phil., Ph.D., has 
published 19 publications in National and International 
Journals so far. His area of interest is Graph Theory.