Couple Stress Impact on S-Waves in an Anisotropic Pre-Stressed Medium under Gravity and Magnetic Field

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Abstract—The gravity field and magnetic field impact with the inclusion of couple stress on shear waves in a transversely isotropic elastic solid medium is studied. The dispersion equations for the particular problem have been obtained. Several particular cases are also examined. The numerical calculations are carried out for specific rigidity parameter $A$, density parameter $C$, anisotropic factor $N$, initial stress parameter $P$, gravity parameter $G$, magnetic field parameter $H$, couple stress parameter $R_s$ as well as wave number $k$ and the outcomes are visually represented by graph.

Keywords: incompressible, anisotropic, gravity field, pre-stressed, couple stress, magnetic field.

I. INTRODUCTION

In 2004, A. M. Abd-Alla., et.al. [1] examined the Rayleigh waves in anelasto-magnetic pre-stressed orthotropic material of half space under gravity field. Abd-Alla, A.M., et.al. [2] discussed the gravity field impact on S-waves in a non-homogeneous, incompressible and initially stressed anisotropic solid medium. In 1965, Biot [3] observed that the initial stresses in the solid medium have a significant effect on the reproduction of elastic waves and a comparative interpretation of a stratified medium by an extended structure with anisotropic features. Das., et.al. [6] have investigated the effect of initial stresses in the propagation of edge waves in homogeneous isotropic plates. In his study, compound structures of thinly strained materials were deliberated and used in the study of twisting and oscillations. The spreading of "Edge Waves" in an anisotropic pre-stressed plate of limited thickness and absolute length was investigated and the dispersion equations for the edge waves were derived by Dey, S., P. K. De., [7]. The propagation nature of the elastic waves in a layered stratified medium under both internal and couple stresses was discussed by Pal Roy [8]. Also P. Pal Roy examined the dynamics of a stratified medium of Maxwell type solids and weakness of the surface of a pre-stressed laminated material. Pijush Pal Roy., and Lokenath Debnath., [9] considered about the development of edge waves in a stratified pre-stressed medium with stress couples. Mainly the work was based on the comparative interpretation of a stratified medium by an equivalent pre-stressed anisotropic continuation with couple stresses and concluded that for a particular compression, the couple stress increases the speed of wave propagation with the increment in wave numbers. But in general the increment in wave numbers diminishes the speed of the wave. Selim [11] inspected the thermal stress impact on development of transverse waves and debilitation of seismic body waves in pre-stressed dissipative medium undercouple stresses.

In this work, an attempt has been made to find the velocities of propagation, damping of $S$-waves in a non-homogeneous, anisotropic, incompressible and initially stressed medium under the presence of magneto-gravity field with couple stress $R_s$. Also the dispersion equations for this particular problem have been obtained to determine the $S$-wave velocity using linearinhomogeneities and outcomes are visually represented by graph.

II. FORMULATION OF THE PROBLEM

A stratified medium of $n$ thin layers and of total thickness $H$ under the common initial compressive stress $P$ on the $x^-$ direction is considered. Suppose that the $y^-$ axis is normal to the plane of the laminations, the relationship between stress-strain in the composite medium are given by [3]

$$s_{11} = 2k_1P + s$$
$$s_{22} = 2k_2P + s$$
$$s_{33} = 2k_3P$$

where $s_{11}$ and $s_{22}$ are the components of principal stress along $x$ and $y$ directions respectively.

$s_{12}$ is the shear stress component in the $xy^-$ plane and

$$s_{12} = -s_{21} = 0$$

The bending rigidity of the medium taken into accounts by introducing couple stresses $R_s$ [4], the total bending moment at the medium is

$$Dz = \frac{\partial^2 w}{\partial x^2}$$
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where \( \vec{H} \) is the magnetic field intensity, \( \vec{B} \) is the magnetic induction, \( \vec{E} \) is the electric field intensity, \( \vec{J} \) is the current density vector, \( \vec{D} \) is the displacement current vector, \( c \) is the speed of the light in vacuum, \( \mu_r \) is the magnetic permeability and \( \varepsilon_r \) is the permittivity respectively.

Limiting the analysis to the plane strain only. The modified dynamical equations of equilibrium in \( x-y \) plane of the composite anisotropic medium under the impact of couple stress are given by [5] and [9] for the present problem

\[
\begin{align*}
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \mu_r \frac{\partial^2 v}{\partial z^2} &= 0 \\
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \mu_r \frac{\partial^2 w}{\partial z^2} &= 0
\end{align*}
\]

where \( \rho \) is the density, \( g \) is the acceleration due to gravity and \( \omega \) is the component of the local rotation perpendicular to \( x-y \) plane

\[
\omega = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y} \right)
\]

The components of incremental strain are given by [1]

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)
\]

The condition of incompressibility \( \varepsilon_{xx} + \varepsilon_{yy} = 0 \) is satisfied by

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}
\]

where \( \mathcal{S} = \mathcal{S}(x, y, t) \).

III. SOLUTION OF THE PROBLEM

Applying the equations from (2.1a) to (2.1d) and (2.3) to (2.5) in (2.2a) and (2.2b), we obtain

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \mu_r \frac{\partial^2 u}{\partial z^2} &= 0 \\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \mu_r \frac{\partial^2 v}{\partial z^2} &= 0
\end{align*}
\]

Assuming the inhomogeneities

\[
Q_i = N_i = \rho = \lambda_i(1+\alpha)
\]

where \( Q_i \) and \( N_i \) are rigidities and \( \rho_i \) is the density in homogeneous isotropic generalized medium.

Applying the equation (3.3) in (3.1) and (3.2)

\[
\begin{align*}
\left[ (1+\alpha) + \mu_r \right] \varepsilon_{xx} + \left[ (1+\alpha) + \mu_r \right] \varepsilon_{yy} + \left[ (1+\alpha) + \mu_r \right] \varepsilon_{xy} &= 0 \\
\left[ (1+\alpha) + \mu_r \right] \varepsilon_{xx} + \left[ (1+\alpha) + \mu_r \right] \varepsilon_{yy} + \left[ (1+\alpha) + \mu_r \right] \varepsilon_{xy} &= 0
\end{align*}
\]

The solution of equation (3.4) for propagation of sinusoidal waves at any direction is

\[
\varepsilon(x, y, t) = M e^{(\omega+i\beta)x}
\]

where \( \phi \) is the angle of propagation with the \( x \)-axis, \( V_1 \) is the S-wave velocity and \( k \) is the wave number. Applying equation (3.5) in (3.4)

\[
\left( \frac{\omega}{\beta} \right)^2 = \frac{1}{1+\alpha} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2 \\
\varepsilon_{xx} + \varepsilon_{yy} = \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2
\]

where \( \beta \) is the S-wave velocity in generalized homogeneous isotropic medium.

The equation (3.6) represents the square of the phase velocity \( V_p \) and equation (3.7) represents the square of the damping velocity \( V_I \) of the shear waves.

IV INSPECTION OF PARTICULAR PROBLEM IN GENERAL HOMOGENEOUS MEDIUM:

(i) Investigation of equation (3.6):

Case A: When \( a \to 0 \), \( Q \) is homogeneous i.e., Constant rigidity along vertical direction.

\[
\left( \frac{\omega}{\beta} \right)^2 = \frac{1}{1+\alpha} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2
\]

Equation (4.2) depends on the couple stress coefficient.

When \( \cos \phi = 0, \sin \phi = 1 \), the velocity along (vertical) \( y \)-direction is

\[
V_y = \frac{\rho \beta^2}{1+\alpha} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2
\]

Case B: When \( b \to 0 \), \( N \) is homogeneous i.e., Constant rigidity along horizontal direction.

\[
\left( \frac{\omega}{\beta} \right)^2 = \frac{1}{1+\alpha} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2
\]

When \( \cos \phi = 1, \sin \phi = 0 \), the velocity along (horizontal) \( x \)-direction is

\[
V_x = \frac{\rho \beta^2}{1+\alpha} \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right]^2
\]
Equation (4.5) depends on the average couple stress coefficient.

When \( \cos \phi = 0, \sin \phi = 1 \), the velocity of propagation along (vertical) \( y \)-direction is

\[
\begin{align*}
\nu_1^2 &= \frac{2N_1}{Q_1} \cos \phi \\
\nu_2^2 &= \frac{2N_2}{Q_2} \cos \phi
\end{align*}
\] (4.6)

The velocity along vertical direction increases for \( P > 0 \) and the waves are dispersive.

**Case C:** When \( N(b \to 0), Q(a \to 0) \) and \( \rho(c \to 0) \) are homogeneous

\[
\begin{align*}
\left( \frac{\nu_1}{\beta} \right)^2 &= \frac{2N_1}{Q_1} \cos \phi \\
\left( \frac{\nu_2}{\beta} \right)^2 &= \frac{2N_2}{Q_2} \cos \phi
\end{align*}
\] (4.7)

The velocity equation when the initial stress is absent becomes

\[
\begin{align*}
\left( \frac{\nu_1}{\beta} \right)^2 &= \frac{2N_1}{Q_1} \cos \phi \\
\left( \frac{\nu_2}{\beta} \right)^2 &= \frac{2N_2}{Q_2} \cos \phi
\end{align*}
\] (4.8)

When \( \cos \phi = 1, \sin \phi = 0 \), the velocity along (horizontal) \( x \)-direction is

\[
\nu_2^2 = \beta^2
\] (4.9)

When \( \cos \phi = 0, \sin \phi = 1 \), the velocity along (vertical) \( y \)-direction is

\[
\nu_1^2 = \beta^2
\] (4.10)

In this case, the anisotropy does not affect the velocity.

(ii) Analysis of Equation (3.7):

**Case A:** When \( \alpha \to 0, Q \) is homogeneous i.e., Constant rigidity along vertical direction.

\[
\begin{align*}
\left( \frac{\nu_1}{\beta} \right)^2 &= \frac{2N_1}{Q_1} \cos \phi \\
\left( \frac{\nu_2}{\beta} \right)^2 &= \frac{2N_2}{Q_2} \cos \phi
\end{align*}
\]

i.e., the S-wave velocity is damped

When \( \cos \phi = 1, \sin \phi = 0 \), the velocity along (horizontal) \( x \)-direction is

\[
\left( \frac{\nu_1}{\beta} \right)^2 = 0
\] (4.11)

**Case B:** When \( b \to 0, N \) is homogeneous i.e., Constant rigidity along the horizontal direction.

\[
\begin{align*}
\left( \frac{\nu_1}{\beta} \right)^2 &= \frac{\rho \beta^2}{Q_1} \sin \phi \\
\left( \frac{\nu_2}{\beta} \right)^2 &= \frac{\rho \beta^2}{Q_2} \sin \phi
\end{align*}
\] (4.12)

When \( \cos \phi = 1, \sin \phi = 0 \), the velocity along (horizontal) \( x \)-direction is

\[
\left( \frac{\nu_1}{\beta} \right)^2 = \frac{\rho \beta^2}{Q_1}
\] (4.13)

The velocity in horizontal direction is damped by \( \frac{a}{c} \).

When \( \cos \phi = 0, \sin \phi = 1 \), the velocity along (vertical) \( y \)-direction is

\[
\left( \frac{\nu_1}{\beta} \right)^2 = \frac{\rho \beta^2}{Q_2}
\] (4.14)

**Case C:** When \( N(b \to 0), Q(a \to 0) \) both are homogeneous

\[
\left( \frac{\nu_1}{\beta} \right)^2 = 0
\] (4.15)

V MATHEMATICAL ANALYSIS AND DISCUSSION RESULTS

Consider the non-dimensional parameters to get numerical data on the phase velocity \( V_R \) of shear waves in the non-homogeneous pre-stressed medium:

\[
A = \frac{a}{b} \quad \text{(rigidity parameter)}, \quad C = \frac{c}{b} \quad \text{(density parameter)}, \quad \bar{N} = \frac{N_1}{Q_1} \quad \text{(anisotropy factor)}, \quad B = by \quad \text{(depth)}, \quad V_R = \frac{V_1}{\beta}
\]

(Shear wave velocity), \( \bar{P} = \frac{P}{2Q_1} \quad \text{(initial stress factor)}, \quad \bar{H} = \frac{\mu_eH_0^2}{Q_{11}} \quad \text{(magnetic field parameter)}, \quad G = \frac{gb}{k^2 \beta^2} \quad \text{(gravity parameter)}, \quad \bar{R} = \frac{R_s}{Q_{11}} \quad \text{(couple stress parameter)}.
\]

Using these parameters in the equation (3.8)

\[
\frac{\nu_1^2}{\beta^2} = \frac{1}{1+BC} \left[ \left( 1+AF \right) \left[ 1+AF \right] + \left[ 1+AF \right] \left[ 1+AF \right] - 2 \left( 1+AF \right) \left[ 1+AF \right] + \bar{H} \cos \phi \right] \cos \phi
\]

The outcomes are plotted for the parameters: \( A=4; B=0 \) to \( 6; C=0.8; \bar{P} =0.5; \bar{N} =0.5; \bar{R} =0.08; \bar{H} =0.5; \bar{G}=0.2 \) given by [11] and [9].
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Figure 1: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct density parameters $C = 0.5, 0.6, 0.7, 0.8, 0.9$.

Figure 2: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct rigidity parameter $A = 4, 5, 6, 7, 8$.

Figure 3: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct anisotropy parameter $\bar{N} = 0.5, 1, 1.5, 2, 2.5$.

Figure 4: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct initial stress parameter $\bar{P} = 0.5, 0.9, 1.3, 1.7, 2.1$.

Figure 5: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct gravity parameter $G = 0.2, 0.4, 0.6, 0.8, 1$.

Figure 6: Changes in wave velocity at $\phi = 30^\circ$ with depth $B$ and distinct magnetic field parameter $\bar{H} = 0.5, 0.6, 0.7, 0.8, 0.9$.
VI CONCLUSION

The velocity $V_R$ of the S-wave increases when the rigidity parameter $A$, magnetic field parameter $\overline{H}$, wave number $\kappa$ as well as anisotropy parameter $\mathcal{N}$ increases with the inclusion of couple stress parameter $\mathcal{R}$. Also the wave velocity $V_1$ decreases when the density parameter $C$, gravity parameter $G$ as well as the initial stress parameter $P$ increases with the inclusion of couple stress parameter $\mathcal{R}$. Finally, it is observed that the velocity decreases within the range $\phi = (0^\circ, 45^\circ)$, increases within the range $\phi = (46^\circ, 90^\circ)$ and also the increasing values of couple stress $r_z$ increases the S-wave velocity.

REFERENCES