Fuzzy Eulerian and Fuzzy Hamiltonian Graphs with Their Applications

Abdul. Muneera, T. Nageswara Rao, R.V.N.Srinivasa Rao

Abstract: ***In this article we discussed prominence of Fuzzy Eulerian and Fuzzy Hamiltonian graphs. Fuzzy logic is introduced to study the uncertainty of the event. In Fuzzy set theory we assign a membership value to each element of the set which ranges from 0 to 1. The earnest efforts of the researchers are perceivable in the relevant establishment of the subject integrating coherent practicality and reality. Fuzzy graphs found an escalating number of applications in day to day life system where the information intrinsic in the system varies with different levels of accuracy. In this article we initiated the model of fuzzy Euler graphs (FEG) and also fuzzy Hamiltonian graphs (FHG). We explored about fuzzy walk, fuzzy path, fuzzy bridge, fuzzy cut node, fuzzy tree, fuzzy blocks, fuzzy Eulerian circuit and fuzzy Hamiltonian cycle. Here we studied some applications of Fuzzy Eulerian graphs and fuzzy Hamiltonian graphs in real life.***

Key Words: Fuzzy walk, Fuzzy path, Fuzzy Bridge, Fuzzy block, Fuzzy Euler graph, Fuzzy Eulerian circuit, Fuzzy Hamiltonian cycle.

I. INTRODUCTION

The river Pregl was crossed 7 bridges which connected 2 islands on the river with each other and with opposite banks. Euler thought that to have a walk along all the seven bridges exactly once by starting from any of the four land areas and return to the same starting point. While proving this problem Euler replaced each land area by nodes A, B, C, D and all the bridges by an edge joining the corresponding nodes. Using this Euler proved that such a route over the bridges of Konigsberg is impossible. This method used by Euler in solving this problem gave rise to the notion of graph theory. To study the relationship between objects, Graph is a convenient way. In this graph theory objects are represented by nodes and the line joining the nodes shows the relation between them known as edges. The idea of fuzziness is one of enrichment, not of replacement. To describe vagueness in the objects (or) relations (or) in both fuzzy graphs plays a vital role. Consider the set S = {Apple, Banana, Guava, Orange} to study the eating habits of fruits. One cannot decide that to like (or) dislike the set S because both things are there. To deal these types of uncertainty fuzzy set theory came into existence in 1965, which was introduced by [1] Lotfi A. Zadeh. In fuzzy sets and fuzzy relations elements are assigned by the membership values in the range from 0 to 1. i.e., the above set S defined in fuzzy set as S = {Apple/ {0.4}, Banana/ {0.6, Guava / {0.3}, Orange/ {0.9}}.

Kaufmann [3] introduced the definition of fuzzy graph; Rosenfield [2] defined elaborated definition of fuzzy node and fuzzy edge. H.J. Zimmermann thoroughly studied the conceptual phenomena of fuzzy set theory in 2010. Rosenfeld [3] has explained various concepts of fuzzy graph. An Eulerian trail of a certain fuzzy graph (FG) is open trail of fuzzy graph containing all the edges of fuzzy graph exactly once, which began and ends on different nodes. This kind of graph is said to be passable. We call Eulerian trail as an Eulerian circuit if it begins and ends on the same node. A fuzzy graph containing an Euler line is named as fuzzy Euler graph. If an edge set of connected fuzzy graph can be partitioned into cycles, then such a fuzzy graph is called Eulerian fuzzy graph. This condition is necessary and sufficient also. Hamiltonian graphs were introduced by great Irish mathematician Sir William Rowan Hamilton in nineteenth century (i.e, 1805-1865). A closed path that visits every node only once is a fuzzy Hamiltonian circuit.

If each node in a fuzzy graph visited exactly once excluding beginning vertex where as edges may repeat then such a circuit it called Hamiltonian circuit. A fuzzy graph which is having Hamiltonian circuit is called fuzzy Hamiltonian graph. In the same way a circuit which traverses every edge exactly once in a fuzzy graph where as nodes may repeat such a circuit is called fuzzy Eulerian circuit.

II. PRELIMINARIES

The definitions and theorems in the preliminaries are in the Rosenfeld [2]

Definition 2.1: Let V be a finite non empty set and E be the collection of two element subset of V. A fuzzy graph $G(\sigma, \mu)$ is a pair of functions, where $\sigma : V \rightarrow [0, 1]$ and $\mu: E \rightarrow [0, 1]$, such that $\mu (xy) \leq (\sigma(x) \wedge \sigma(y))$ for all x, y ϵ V.

Example: Let a, b, c be the vertices. i.e., V = {a, b, c}. Consider the fuzzy set σ on V by membership values as $\sigma(a) = 0.7$, $\sigma(b) = 0.2$, $\sigma(c) = 0.4$ .Define a fuzzy set μ of E such that $\mu(ab) = 0.2$, $\mu(bc) = 0.2$, $\mu(ac) = 0.4$. Thus the graph $G(\sigma, \mu)$ is a fuzzy graph (FG).

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Definition 2.2: Let G (σ, μ) be a fuzzy graph. Sum of the membership values of the edges incident on the node v is called the degree of v, labeled as $d(v) = \sum_{u \neq v} \mu\,(u, v)$.
$\delta(G) = \wedge \{ d(v) / v \in V \}$ is the minimum degree of fuzzy graph G and
$\Delta(G) = \vee \{ d(v) / v \in V \}$ is the maximum degree of fuzzy graph G.

Definition 2.3: A Path P in a fuzzy graph (FG) G (σ, μ) is a sequence of different nodes $x_0, x_1, \ldots, x_n$ such that $\mu(x_{i-1}, x_i) > 0$, i = 1,2 ,…n. Here the length of the path is n. The alternate pairs are called the edges (or) arcs of the path. Weight of the weak edge is the Strength of the path .Suppose if we want to check the strength of chain which is having some links then its strength depends upon the weakest connection of that chain. The strength of connectedness $\mu^{\infty}(x, y)$ is denoted by $CONN_G(x, y)$. A fuzzy graph is connected if $CONN_G(x, y) > 0$ for every pair of nodes $x, y \in \sigma^*$.

Definition 2.4: A fuzzy walk W of the fuzzy graph (FG) in fig 1. a μ(a, b) b μ(b, c) c μ(c, d) d μ(d, e) e which started with the initial node a and the terminal node is e. The length of the fuzzy walk W is min $\mu(x_i, x_{i+1})$ for all nodes contained in the fuzzy walk. Length of the fuzzy walk in fuzzy graph (FG) in fig1 is 0.3.

Definition 2.5: If initial and terminal nodes of a fuzzy walk W with $\mu(x_i, x_{i+1}) \geq 0$ are distinct then that walk is called open fuzzy walk otherwise it is called a closed fuzzy walk.

Definition 2.6: Trail is a fuzzy walk in a fuzzy graph (FG) where the edge $\mu(x_i, x_{i+1}) \geq 0$ is not repeated. A fuzzy walk with distinct initial and terminal nodes is called open fuzzy walk otherwise it is closed fuzzy walk.

Example: (i). a μ(a, b) b μ(b, d) d μ(d, a) a μ(a, e) e is a trail in fuzzy graph (FG) in fig 1. (ii). b μ(b, c) c μ(c, d) d μ(d, a) a μ(a, e) e is a fuzzy path in the fuzzy graph (FG) in fig 1.

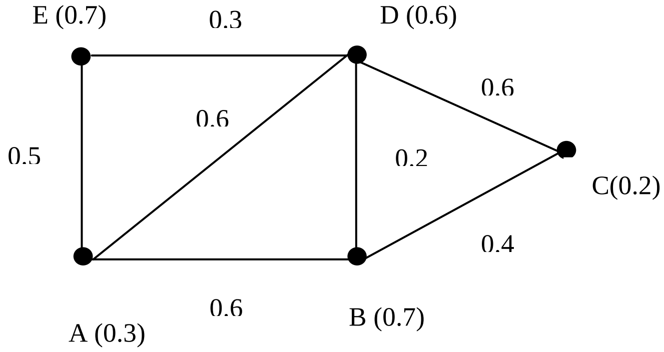

Fig.1. Fuzzy Graph (FG)

Definition 2.7: min $\mu(x_i, x_{i+1})$ for all nodes contained in the fuzzy walk is called the strength of the fuzzy path and the length of a fuzzy path n > 0 is the number of nodes contained in the fuzzy path and is a sum of the membership values.

Example: Strength of the above defined fuzzy path is 0.4 and the length is 2.3.

Definition 2.8: In a fuzzy graph G , if removal of a node reduces the strength of connectedness $\mu^{\infty}(x, y)$ between some other pair of nodes then such a node is called fuzzy cut vertex (or) fuzzy cut node of a fuzzy graph (FG) G(σ, μ).

Definition 2.9: In a fuzzy graph G , if removal of an edge (u, v) reduces the strength of connectedness $\mu^{\infty}(x, y)$ between some other pair of nodes then such an edge (or) arc (u, v) is called a fuzzy bridge of G(σ, μ).

Theorem 2.10: If (u, v) is a fuzzy bridge, then $\mu^{\infty}(u, v) = \mu(u, v)$.

Theorem 2.11: In a fuzzy graph G, every fuzzy bridge is strong.

Proof: Let xy be the fuzzy bridge of fuzzy graph (FG), G = (σ, μ).
Let us suppose that edge xy is not strong. Then $\mu(xy) < CONN_{G-xy}(x,y)$. The strongest path in fuzzy graph G from x to y is p. Then the strength of this path p is $CONN_{G-xy}(x,y)$. If we adjoin edge xy to p to obtain a cycle, then xy becomes the weakest edge of this cycle. By known property the edge xy is not a fuzzy bridge of G. Hence clearly it proves that a fuzzy bridge must be strong.

Theorem 2.12: An edge xy in a fuzzy graph G is strong if and only if $\mu(xy) = CONN_G(x, y)$.

Definition 2.13: A fuzzy graph is said to be a block if it is connected and has no fuzzy cut nodes. Note that in a graph, a block cannot have bridges. But in fuzzy graphs, a block may have fuzzy bridges.

Note: **A** complete fuzzy graph with n vertices is denoted by K_n

Example:

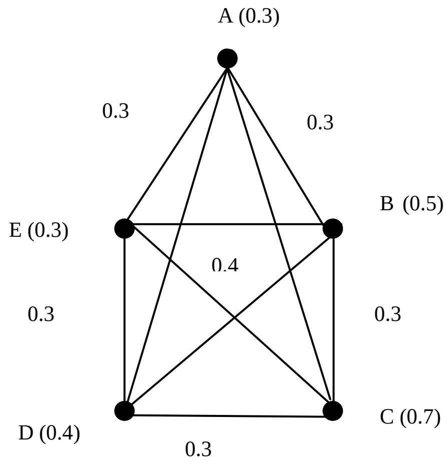

Figure 2 is the example of complete fuzzy graph with 5 vertices (K_5) which has two Hamiltonian fuzzy cycles 123451, 135241.All the membership values are greater than zero.

Definition 2.14: If all the arcs of a fuzzy cycle are strong then such a cycle is called strong fuzzy cycle.

Definition 2.15: The strength of a cycle C in an f-graph is defined as the weight of a weakest arc in C. In graphs, any two nodes of a block belong to a cycle and conversely.

Theorem 2.16: Let G be a connected fuzzy graph. If there is at most one strongest path between any two nodes of G, then G is a fuzzy tree.

Theorem 2.17:
G= (σ, μ) is a fuzzy tree if and only if the following are equivalent.
(1). (u, v) is a fuzzy bridge.
(2). $\mu^{\infty}(u,v) = \mu(u,v)$

Result 2.18: A connected fuzzy graph (FG), G (σ, μ) with n nodes has at most n-1 fuzzy bridges.

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III. FUZZY EULERIAN AND FUZZY HAMILTONIAN GRAPHS

In the Fig . 3 has an Euler circuit but Fig. 4 is not having Euler circuit. In Euler graph each node should have an even number of neighborhood nodes.

But both the above figures are having Hamiltonian paths but not Hamiltonian circuits. In Fig. 3 Hamiltonian paths is A –F-E-D-B-C and in Fig 4. Hamiltonian path is A – E-D-C-B-F.

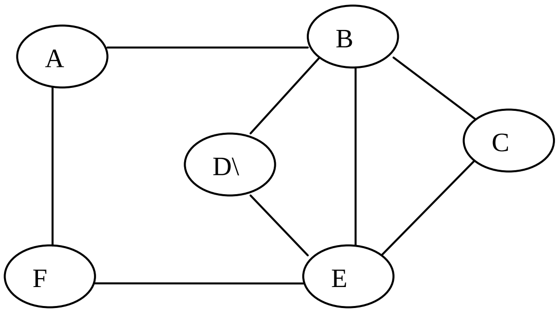

Fig. 3

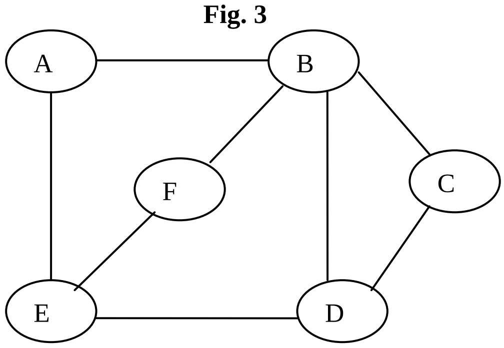

Fig. 4

Definition 3.1: A connected fuzzy graph G (σ, µ) is said to be a fuzzy Eulerian graph if there exists a circuit which includes every edge of G(σ, µ) exactly once which starts and ends with different nodes.

Note: The dual of dual fuzzy graph (FG) is the fuzzy graph itself, Eulerian fuzzy graph can be obtained by considering the dual of a bipartite fuzzy graph (FG).

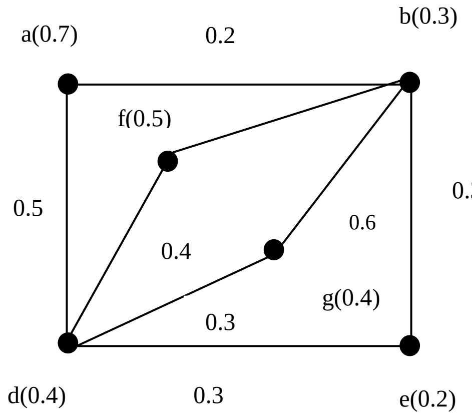

Fig. 5. Eulerian but Not Hamiltonian fuzzy graph

Definition 3.2: A connected fuzzy graph G (σ, µ) is said to be a fuzzy Hamiltonian graph if there exists a fuzzy cycle which covers all the nodes of fuzzy graph G (σ, µ) exactly once except the terminal nodes.

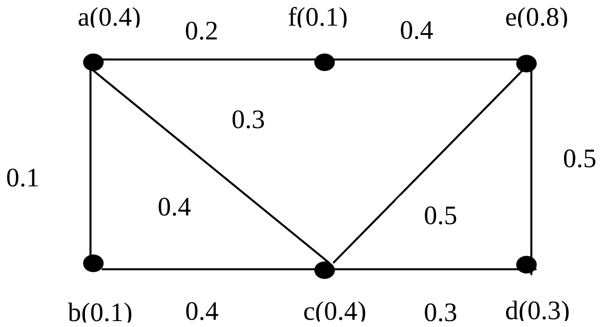

Fig. 6. Hamiltonian fuzzy graph but not Eulerian fuzzy graph.

Definition 3.3: A cycle C in an fuzzy graph (FG), G(σ, µ) is called a strongest strong cycle (SSC) if C is the union of two strongest strong u − v paths for every pair of nodes u and v in C except when (u, v) is an fuzzy bridge of G in C.

Definition 3.4: A disconnection of a fuzzy graph G: (σ, µ) is a node set D whose removal results in a disconnected or a single vertex graph. The weight of D is defined to be Σ v ∈ D {min µ (v, u) | µ (v, u) ≠ 0}.

Definition 3.5: In a fuzzy graph (FG), G: (σ, µ) the node connectivity is defined to be the minimum weight of a disconnection in G and is denoted by Ω (G).

Definition 3.6: The node set of a fuzzy graph (FG), be partitioned into two sets {V_1, V_2}. The set of edges joining nodes of V_1 and nodes of V_2 is called a cut- set of fuzzy graph G, denoted by (V1, V2) relative to the partition {V_1, V_2}. The weight of the cut-set (V_1, V_2) is defined as Σ µ (u, v) where u ∈ V_1 and v ∈ V_2.

Definition 3.7: Let G be a fuzzy graph. The edge connectivity of G denoted by λ (G) is defined to be the minimum weight of cut-sets of G.

Theorem 3.8: Let G (σ, µ) be a connected fuzzy graph (FG), The relations between node connectivity Ω (G), edge connectivity λ (G) and minimum degree is given as δ (G) ≥ λ (G) ≥ Ω (G).

Theorem 3.9: An Eulerian fuzzy graph (FG) is a graph obtained from the dual of a fuzzy bipartite graph.

Proof: Consider G (σ, µ), a maximal fuzzy bipartite graph (FBG),

Suppose the degree of the membership for the vertices is σ(v), degree of the membership for the edges is µ(e) and the degree of membership of faces is λ(f) respectively.

It is a maximal fuzzy bipartite planar graph so each cycle's length is even, such that every face f in FG has even lengths. So the dual of these faces f_i for all I =1,2,….k obtained a vertices v_i* of even degree. We know that "A given connected graph G is an Eulerian Graph if and only if all vertices of G are of even degree".

Hence the dual graph is Eulerian fuzzy graph with the degree of membership

σ(v*) = λ(f), µ(e*)= µ(e) and λ(f*)= σ(v).

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Theorem 3.10: If every node of a fuzzy graph (FG) has exactly two adjacent nodes, then there exists a fuzzy Hamiltonian cycle.

Proof: Let us Consider a fuzzy graph (FG), G(σ, μ) with n nodes $x_1, x_2, \ldots . x_n$.

Suppose that each node of fuzzy graph (FG) has exactly two adjacent nodes.

Suppose any edge between any two nodes is removed. Say $\mu(x_1, x_2)$.

Now if we begin to find a path starting with the vertex x_1 (or) x_2 and end with x_2 (or) x_1 by covering all the nodes of fuzzy graph (FG). In this case it is not possible to reach to the staring node, so can't get a Hamiltonian cycle.

Suppose any node x_i is removed, either it takes the path $x_{i-1}, x_{i-2}, \ldots$ and end with x_1 or end with x_2, excluding some nodes of fuzzy graph (FG), in this case also we won't get Hamiltonian cycle.

Hence it is necessary in fuzzy Hamiltonian cycle that each vertex of (FG) has exactly two adjacent nodes.

IV. APPLICATIONS

Fuzzy set theory is the modern mathematical apparatus that enables considering the initial information uncertainty. Eulerian graphs are used to resolve various practical problems. Many applications ask for a path or circuit that traverses each street in a neighborhood, each connection in a utility grid, or each link in a communications network exactly once. It is also used for ranking hyperlinks or by GPS to find shortest path home. The existence of the Hamilton's cycle will also allow the design of a programmed test so that once the data is entered and finally the test conditions are created. By using Hamiltonian circuit we can solve the practical problems such as road intersections, pipeline crossings etc.

V. CONCLUSION

The manuscript studied about the importance of fuzzy Eulerian and fuzzy Hamiltonian graphs. Also we established the model of fuzzy Eulerian and fuzzy Hamiltonian graphs. More over it deliberated the some applications of these graphs in real life problems. Fuzzy graph theory is becoming gradually more considerable as it is useful in computers, mathematics, science and technology. We can see the implementation of fuzzy logic in WI FI technology, IOT, GPS map , GSM mobile phone networks, data mining, electronic chip design, web designing, coding etc.

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