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A Numerical Method for Solving Nonlinear Equations Arising in Astrophysics

Z. A. Zaki, Naeem Ahmad, Marwa I. Ghonamy, F. A. Abd El-Salam , A. M. Azam

Abstract*: In this paper, a parametric cubic spline function was used to get the solution of a non-linear problem for an isothermal gas sphere. The quasi-linearization procedure was used to reduce the given problem to a sequence of linear problems, the resulting equations are modified at the singular point and are handled by using parametric cubic spline for determining the numerical results. All computations have been carried out by the Mathematica software program package. The findings of computational outcomes on those astrophysics problems confirmed that the technique is legitimate for the solution of these kinds of equations.

***Keywords*: Non-linear Differential Equations, Parametric Cubic Spline, Lane–Emden Equations.**

I. INTRODUCTION

Several problems in the applied mathematics lead to a class of non-linear singular boundary value problem such as Lane–Emden differential equation, which is provided in equations 1 to 3.

$$y''(\mathrm{x}) + \frac{\alpha}{x} y'(\mathrm{x}) = \mathrm{f}(\mathrm{x}, \mathrm{y}) \quad (1)$$

$$y'(0) = 0 \quad (2)$$

$$y(1) = \beta \quad (3)$$

where: α and β are constants. The function f(x, y),$0 \leq x \leq 1$,is assumed to be a continuous,$\frac{\partial f}{\partial y}$ exists, continuous and $\frac{\partial f}{\partial y} \geq 0$ .Many ways were conducted by researchers to overcome the singularity problems that based on applying the procedure of series expansion in the neighborhood (0,ε) of singularity. The parameter ε is defined as tiny number and has value bigger than zero and less than one (vicinity of singularity). After that, the problem becomes regular boundary value problem in the interval(ε,1), the solution can be produced by any numerical method. the numerical solution of (1) to (3) are introduced in [1,2,3 and 4]. Thus, the objective of this paper is to apply a numerical method based on parametric cubic spline approximation [5] for the solution of Lane–Emden differential equation (equations 1 to 3).

II. METHODOLOGY

Firstly, the formulation of spline function is proposed and then the method was applied to the problem (1) to (3) and in the same time, Thomas Algorithm was used. Finally, the efficiency of that numerical method is evaluated by considering two Physical Model Problems. More details are introduced in the following section.

A. Parametric Spline Function

Let $x_i = a + ih$,$i = 0, 1, \ldots, n$; $x_0 = a$,$x_n = b$ and $h = \frac{b-a}{n}$,a function $S(x, \tau)$ of class $C^2[a, b]$ which approximates $y(x)$ at the grid point x_i depend on a parameter τ, reduces to cubic spline in $[a, b]$ when $\tau \to 0$, is termed as parametric cubic spline function. The spline function $S(x, \tau) = S(x)$ that satisfying for x in$[x_i, x_{i+1}]$ the differential equation:

$$S''(x) + \tau S(x) = [S''(x_i) + \tau S(x_i)] \frac{(x_{i+1} - x)}{h} + [S''(x_{i+1}) + \tau S(x_{i+1})] \frac{(x - x_i)}{h} \quad (4)$$

Where: $S(x_i) = y_i$ and $\tau > 0$ is named as cubic spline in compression.by solving (4) and obtaining the arbitrary constants from the following interpolator conditions: $S(x_{i+1}) = y_{i+1}$ and $S(x_i) = y_i$, the solution is displayed in (5) after assuming $\rho = h\sqrt{\tau}$

$$S(x) = \frac{-h^2}{\rho^2 \sin \rho} \left[M_{i+1} \sin \frac{\rho(x - x_i)}{h} + M_i \sin \frac{\rho(x_{i+1} - x)}{h} \right] + \frac{h^2}{\rho^2} \left[\frac{(x - x_i)}{h} \left(M_{i+1} + \frac{\rho^2}{h^2} y_{i+1} \right) + \frac{(x_{i+1} - x)}{h} \left(M_i + \frac{\rho^2}{h^2} y_i \right) \right] \quad (5)$$

Where $M_i = S''(x_i)$, Then the continuity of the first derivative at $x_i$was used that generates the following useful spline relation.

$$y_{i+1} - 2y_i + y_{i-1} = h^2(\gamma_1 M_{i+1} + 2\gamma_2 M_i + \gamma_1 M_{i-1}) \quad (6)$$

Where:$\rho_1 = \frac{1}{\rho^2}\left(\frac{\rho}{\sin \rho} - 1\right), \rho_2 = \frac{1}{\rho^2}(1 - \rho \cot \rho)$,

$M_i = S''(x_i)$ and $i = 1, 2, \ldots \ldots n - 1$

The quasi-linearization approach is used to transform the non-linear problem as shown in Equation (1) to Equation (3) to a series of linear equations. An initial approximation was selected for the function $y(x)$ in $f(x, y)$, call it as $y_{(0)}(x)$ and expand $f(x, y)$ by Taylor series around the function $y_{(0)}(x)$, then (7) was

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* Correspondence Author
Z. A. Zaki *, Electrical Engineering Department, College of Engineering, Jouf University, Skaka, KSA, Zattwa@ju.edu.sa and also Engineering Mathematics and Physics Department, Shoubra Faculty of Engineering, Benha University, Egypt, zaki.atwa@feng.bu.edu.eg,
Naeem Ahmad, Department of Mathematics, College of Science, Jouf University, Skaka, KSA
Marwa I. Ghonamy, Department of Mathematics, Faculty of Science, Tabuk University, Tabuk, Saudi Arabia and also Mathematics Department, Faculty of Science, Benha University, Egypt
F. A. Abd El-Salam, Engineering Mathematics and physics Department, Shoubra Faculty of Engineering, Benha University, Egypt, vittous@yahoo.com
A. M. Azam, Civil Engineering Department, College of Engineering, Jouf University, KSA, Amazam@ju.edu.sa; and also Public Works Engineering Department, Mansoura University, Egypt, Abdelhalim.azam@mans.edu.eg

developed.

$$f(x,y_{(1)}) = f(x,y_{(0)}) + (y_{(1)} - y_{(0)})\left(\frac{\partial f}{\partial y}\right)_{(x,y_{(0)})} + \cdots \quad (7)$$

In general, (7) can be written as presented in (8),for iteration index k ,where $k \geq 0, k$ is an integer number

$$f(x,y_{(k+1)}) = f(x,y_{(k)}) + (y_{(k+1)} - y_{(k)})\left(\frac{\partial f}{\partial y}\right)_{(x,y_{(k)})} + \quad (8)$$

Equation (1) can be approximated to be in the form: $y''_{(k+1)}(\mathrm{x}) + \frac{\alpha}{x}\mathrm{y}'_{(k)}(\mathrm{x}) + a_{(k)}\mathrm{y}_{(k+1)}(\mathrm{x}) = b_{(k)}(\mathrm{x})$,

$\mathrm{k} \geq 0$, k is an integer number (9)

where:

$$a_{(k)}(x) = -\left(\frac{\partial f}{\partial y}\right)_{(x,y_{(k)})}$$

$$,\mathrm{b}_{(k)}(\mathrm{x}) = f(x,y_{(k)}) - y_{(k)}\left(\frac{\partial f}{\partial y}\right)_{(x,y_{(k)})}$$

Considering the conditions of:

$\mathrm{y}'_{(k+1)}(0) = 0$ (10)

$y_{(k+1)}(1) = \beta$ (11)

Equation (9) turned into changed at the singular point $x = 0$ after which the parametric spline technique become implemented for solving the set of linear singular Boundary value problems which given by Equation (9) considering the application of the boundary conditions (10) and (11). according to [8], L'Hospital's rule was applied to transform the given (9) to (11) to the following equations:

$y''_{(k+1)}(\mathrm{x}) + \mathrm{p}_{(k)}\mathrm{y}'_{(k+1)}(\mathrm{x}) + q^{(k)}\mathrm{y}_{(k+1)}(\mathrm{x}) = \mathrm{r}_{(k)}(\mathrm{x})$, (12)

$y'_{(k+1)}(0) = 0$ (13)

$y_{(k+1)}(1) = \beta$ (14)

Where:

$$\mathrm{p}_{(k)}(x) = \begin{cases} 0, x = 0 \\ \frac{\alpha}{x}, x \neq 0 \end{cases}, \quad \mathrm{q}_{(k)}(x) = \begin{cases} \frac{a_{(k)}(0)}{\alpha+1}, x = 0 \\ a_{(k)}(x), x \neq 0 \end{cases}$$

and: $\mathrm{r}_{(k)}(x) = \begin{cases} \frac{b_{(k)}(0)}{\alpha+1}, x = 0 \\ b_{(k)}(x), x \neq 0 \end{cases}$

The independent variable x is ranged from 0 to 1. At the grid points x_i, (12) may be discretized by:

$y''_{(k+1)}(\mathrm{x}_i) + \mathrm{p}_{i(k)}y'^{(k+1)}(\mathrm{x}_i) + \mathrm{q}_{i(k)}\mathrm{y}_{(k+1)}(\mathrm{x}_i) = \mathrm{r}_{i(k)}, \mathrm{k} \geq 0$ (15)

Where:

$\mathrm{p}_{i(k)}(x) = \mathrm{p}_{(k)}(x_i), \mathrm{q}_{i(k)} = \mathrm{q}_{(k)}(x_i)$ and $\mathrm{r}_{i(k)} = \mathrm{r}_{(k)}(x_i)$

Equation (15) is then rearranged and written in the form:

$M_{i(k+1)} + \mathrm{p}_{i(k)}\mathrm{y}'_{i(k+1)} + \mathrm{q}_{i(k)}\mathrm{y}_{i(k+1)} = r_i^{(k)}$ (16)

the first derivative of y is estimated by using the use of the subsequent approximations

$$y'_i \cong \frac{(y_{i+1} - y_{i-1})}{2h}$$

$$y'_{i+1} \cong \frac{(3y_{i+1} - 4y_i + y_{i-1})}{2h} \quad (17)$$

$$y'_{i-1} \cong \frac{(-y_{i+1} + 4y_i - 3y_{i-1})}{2h}$$

Substituting (17) into (16) and simplifying, the following equation produced:

$\left(-1 - \frac{1}{2}\rho_1 hp_{i+1(k)} + \rho_2 hp_{i(k)} + \frac{3}{2}\rho_1 hp_{i-1(k)} - \rho_1 h^2 q_{i-1(k)}\right) y_{i-1}, + \left(2 - 2\rho_1 hp_{i-1(k)} + 2\rho_1 hp_{i+1(k)} - 2\rho_2 h^2 q_{i(k)}\right) y_i + \left(-1 + \frac{1}{2}\rho_1 hp_{i-1(k)} - \rho_2 hp_{i(k)} - \frac{3}{2}\rho_1 hp_{i+1(k)} - \rho_1 h^2 q_{i+1(k)}\right) y_{i+1} = -h^2\left(\rho_1 r_{i-1(k)} + 2\rho_2 r_{i(k)} + \rho_1 r_{i+1(k)}\right), i = 1(1)n - 1$ (18)

The new forms of the boundary conditions (13) and (14) are displayed in (19) and (20):

$S_0 y_0 = S_1 y_1 + S_2$ (19)

$y_n = \beta$ (20)

Where:

$$S_0 = \frac{h}{a_1}\left[\frac{-\rho_1}{h}\mathrm{p}_{1(k)} - \mathrm{p}_{1(k)}\mathrm{q}_{0(k)}h({\rho_1}^2 - {\rho_2}^2) - \mathrm{p}_{0(k)}\mathrm{p}_{1(k)}({\rho_2}^2 - {\rho_1}^2) - \frac{\rho_2\mathrm{p}_{0(k)}}{h} + \rho_2\mathrm{q}_{0(k)} - \frac{a_1}{h}\right]$$

$$S_1 = \frac{h}{a_1}\left[\mathrm{p}_{0(k)}\mathrm{p}_{1(k)}h({\rho_1}^2 - {\rho_2}^2) - \rho_1\mathrm{q}_{1(k)} - \frac{\rho_1\mathrm{p}_{1(k)}}{h} - \frac{\rho_2\mathrm{p}_{0(k)}}{h} - \frac{a_1}{h^2}\right]$$

$$S_2 = \frac{h}{a_1}\left[h\mathrm{p}_{1(k)}\mathrm{r}_{0(k)}({\rho_2}^2 - {\rho_1}^2) + \rho_1\mathrm{r}_{1(k)} - \rho_2\mathrm{r}_{0(k)}\right]$$

$$a_1 = 1 - h\rho_2\mathrm{p}_{0(k)} + h\rho_2\mathrm{p}_{1(k)} - h^2{\rho_2}^2\mathrm{p}_{0(k)}\mathrm{p}_{1(k)} + h^2{\rho_1}^2\mathrm{p}_{0(k)}\mathrm{p}_{1(k)}$$

Remark

- If $\rho_1 = \frac{1}{6}$ and $\rho_2 = \frac{2}{6}$ the parametric spline method reduced to the method which reported by [8].
- Choosing different values of ρ_1 and ρ_2 provided that $\rho_1 + \rho_2 = \frac{1}{2}$ we are able to attain the classes of second order methods.

III. SOLUTION

Equation (18) can be put in discretized form:

$E_i\mathrm{y}_{i-1(k+1)} - F_i\mathrm{y}_{i(k+1)} + {G_i}^{(k)}\mathrm{y}_{i+1(k+1)} = \mathrm{H}_{i(k)}$

$, i = 1(1)n - 1$ (21)

Where:

$$E_i = \left(-1 - \frac{1}{2}\rho_1 h\mathrm{p}_{i+1(k)} + \rho_2 h\mathrm{p}_{i(k)} + \frac{3}{2}\rho_1 h\mathrm{p}_{i-1(k)} - \rho_1 h^2\mathrm{q}_{i-1(k)}\right)$$

$$F_i = \left(-2 - 2\rho_1 h\mathrm{p}_{i-1(k)} - 2\rho_1 h\mathrm{p}_{i+1(k)} + 2\rho_2 h^2\mathrm{q}_{i(k)}\right)$$

$$G_i = \left(-1 + \frac{1}{2}\rho_1 h\mathrm{p}_{i-1(k)} - \rho_2 h\mathrm{p}_{i(k)} - \frac{3}{2}\rho_1 h\mathrm{p}_{i+1(k)} - \rho_1 h^2\mathrm{q}_{i+1(k)}\right)$$

$$H_i = -h^2\left(\rho_1\,\mathrm{r}_{i-1(k)} + 2\rho_2\mathrm{r}_{i(k)} + \rho_1\mathrm{r}_{i+1(k)}\right)$$

Then (19), (20) and (21) form $n+1$ equations with $n+1$ unknowns$y_0, y_1, y_2, \ldots, y_n$.the matrix problem related here is a tridiagonal algebraic system whose answer can easily be obtained by means of an efficient set of rules referred to as Thomas algorithm. on this set of rules, we begin with a difference relation of the form

$y_i = W_i y_{i+1} + T_i, i = 0(1)n - 1$ (22)

where W_i and T_i correspond to $W(x_i)$ and $T(x_i)$ which are to be obtained.by using (22) in (21), the recurrence relations W_i and T_i for $i = 1(1)n - 1$ are obtained as:

$$W_i = \frac{G_i}{F_i - E_i W_{i-1}}$$

(23)

$$T_i = \frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \qquad (24)$$

to solve these recurrence relations W_i and T_i ,the initial conditions W_0 and T_0 must be required. these values can be easily verified from (19) as:

$$W_0 = \frac{S_1}{S_0} \qquad (25)$$

and

$$T_0 = \frac{S_2}{S_0} \qquad (26)$$

so the values of W_i and T_i for $i = 1(1)n-1$ can be evaluated in the forward process from (25) and (26). Then the solutions y_i from (20) and (22) can be determined in the backward process.

IV. COMPUTATIONAL RESULTS AND DISCUSSION

In this section, the introduced method is applied on two examples of physical model. After the implementation of the new proposed approach, the solution is excellent in comparison to the alternative methods and that approach is realistic and can without difficulty be carried out.

Example (1)

We keep in mind the problem arising in Astronomy; (the standard Lane-Emden equation) can be described via:

$$y''(\mathrm{x}) + \frac{2}{x} y'(\mathrm{x}) + y^5(\mathrm{x}) = 0$$

$$y'(0) = 0 \,, y(1) = \frac{\sqrt{3}}{2}$$

this example has been considered by [3,4,6 and 7]. the exact answer is $y(x) = (1 + \frac{x^2}{3})^{\frac{-1}{2}}$ .The maximum absolute errors of Example (1) for different values of ρ_1, ρ_2 and n is presented in Fig. 1. It can be seen that as known that the value of n increased, the maximum absolute error decreased. The exact solution compared with the spline solution at $\rho_1 = \frac{1}{6}$, $\rho_2 = \frac{2}{6}$ and $n = 100$ is displayed in Fig. 2. it is able to be mentioned that no difference among the two solutions. solution of example 1 at $\rho_1 = \frac{1}{6}$, $\rho_2 = \frac{2}{6}$ and $n = 64$ by different numerical methods is illustrated in Fig. 3. It can be noted that there is no difference between the techniques.

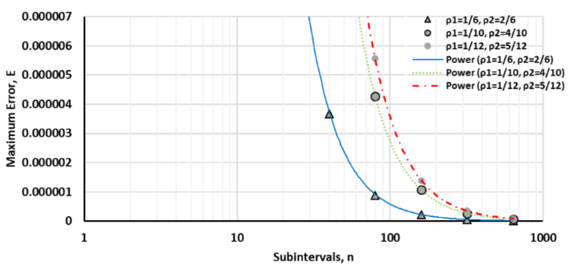

Fig. 1.Maximum absolute errors of Example (1) at different values of at ρ_1 , ρ_2 and n

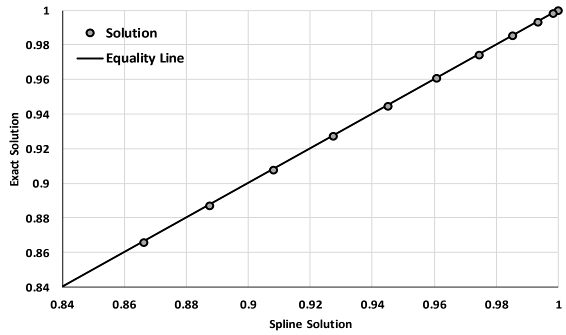

Fig. 2. Comparison between the exact solution and the spline method at different values of x

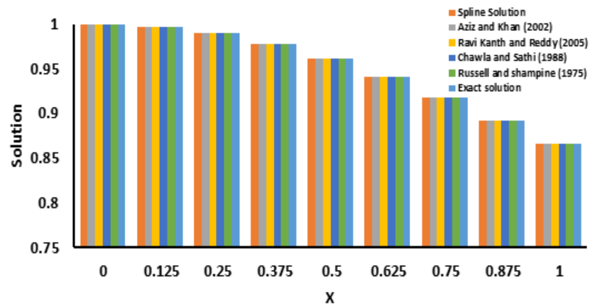

Fig. 3. Solution of example 1 at $\rho_1 = \frac{1}{6}$, $\rho_2 = \frac{2}{6}$ and $n = 64$ by different numerical methods

Example (2)

Consider the following problem (The isothermal gas spheres equation):

$$y''(\mathrm{x}) + \frac{1}{x} y'(\mathrm{x}) + \mathrm{e}^{y(x)} = 0$$

$$y'(0) = 0 \,, y(1) = 0$$

This example has been considered by [3,4 and 8]. The exact solution is $y(x) = 2\ln\frac{B+1}{Bx^2+1}$,where $B = 3 \pm 2\sqrt{2}$ the numerical solutions are tabulated in Tables I, II and III. The maximum absolute errors of Example (2) for different values of ρ_1 , ρ_2 and n is presented in Table I. It can be seen that as known that the value of n increased, the maximum absolute error decreased. The exact solution compared with the spline solution at $\rho_1 = \frac{1}{6}$, $\rho_2 = \frac{2}{6}$ and $n = 100$ is displayed in Table II. It can be noted that no difference between the two solutions. Solution of problem 1 at $\rho_1 = \frac{1}{6}$, $\rho_2 = \frac{2}{6}$ and $n = 64$ by many numerical methods is illustrated in Table III. it may be determined that there is no difference among the used techniques.

Table-I: The maximal absolute errors in numerical results of example (2) for different values of ρ_1 , ρ_2 and n

	20	40	80	160	320	640
	3.20×10^{-5}	7.99×10^{-6}	1.97×10^{-6}	4.92×10^{-7}	1.23×10^{-7}	3.01×10^{-8}
order	2.02	2.01	2	2	2	……..
	4.20×10^{-5}	1.05×10^{-5}	2.63×10^{-6}	6.57×10^{-7}	1.64×10^{-7}	4.11×10^{-8}
order	1.99	2.00	2.00	2.00	2.00	……..

ρ	2.69×10^{-5}	6.79×10^{-6}	1.71×10^{-6}	4.27×10^{-7}	1.07×10^{-7}	2.67×10^{-8}
order	1.99	1.99	2.00	2.00	2.00	……..

Table-II: Numerical results for example (2) at $\rho_1 = \frac{1}{10}$, $\rho_2 = \frac{4}{10}$, $n = 100$ and $B = 3 - 2\sqrt{2}$

x	Spline solution	Exact solution
0.0	0.3166954606235461	0.31669436764074954
0.1	0.3132669311808239	0.31326585049806327
0.2	0.3030164664719477	0.30301542283229976
0.3	0.2860482475276580	0.28604726530485386
0.4	0.2625320249771198	0.26253112745603310
0.5	0.2326975748498419	0.23269678387383438
0.6	0.1968274700378430	0.19682680569295377
0.7	0.1552486263313899	0.15524810668275627
0.8	0.1083231225542308	0.10832276344446458
0.9	0.0564387875547659	0.05643860246923624
1.0	0.0000000000000000	0.00000000000000000

Table-III: computational results for Example (2) at $\rho_1 = \frac{1}{10}$, $\rho_2 = \frac{4}{10}$, $n = 64$ and $B = 3 - 2\sqrt{2}$

x	Spline solution	Exact solution	[1]	[4]	[5]
0.0	0.31670	0.31670	0.31672	0.31669	0.31643
0.125	0.31134	0.31134	0.31135	0.31134	0.31135
0.250	0.29536	0.29536	0.29535	0.29534	0.29530
0.375	0.26902	0.26901	0.26902	0.26901	0.26897
0.500	0.23270	0.23270	0.23270	0.23269	0.23267
0.625	0.18696	0.18695	0.18696	0.18695	0.18693
0.750	0.13243	0.13243	0.13243	0.13243	0.13242
0.875	0.069853	0.069853	0.069854	0.069852	0.069847
1.0	0.00000	0.00000	0.00000	0.00000	0.00000

V. CONCLUSION

The computational results of Examples 1 and 2 are evaluated corresponding to an iteration index of $k = 5$ by taking $y_0 = 0$ as the initial approximation. with the absolute error criterion, $\left|y_i^{(k+1)} - y_i^{(k)}\right| \leq \omega, \omega \to 0$ for all i is generated, the iterations should be stopped and the evaluated results are the exact solution. it could be mentioned that solutions of the new proposed approach are in good agreement as compared to the answers which acquired with the aid of other numerical techniques. It must be additionally noticed that the results are coinciding with analysis outcomes.it is able to be found based totally on the computational outcomes, that the proposed method has an accuracy of $O(h^2)$. the order of convergence of any method is $O(h^m)$ as if the step size h is decreased by a factor (L), it will resulted in decrease of maximum error by a factor L^m.

It could be concluded that the results acquired from the proposed technique are better in comparison with the results from the usage of the standard finite difference technique on the identical number of knots. moreover, the spline solution has advantages of once the solution has been determined; the data required for spline interpolation among grid points is to be had.

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AUTHORS PROFILE

Dr. Zaki A. Zaki, Assistant Professor of Engineering Mathematics, Electrical Engineering Department, College of Engineering, Jouf University, KSA. Zattwa@ju.edu.sa

Dr. Fathi A. Abd El-Salam, Assistant Professor of Engineering Mathematics, Engineering Mathematics and physics Department, Shoubra Faculty of Engineering, Benha University, Egypt, vittous@yahoo.com

Dr. Abdel Halim M. Azam, Assistant Professor, Civil Engineering Department, College of Engineering, Jouf University, KSA, Amazam@ju.edu.sa