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On Graphs with Equal Domination and Chromatic Transversal Domination Numbers

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Abstract— *Let $\chi(G)$ denote the chromatic number of a graph G = (V,E). A dominating set $D \subseteq V(G)$ is a set such that for every vertex $v \in V(G) \setminus D$ there is at least one neighbor in D. The domination number $\gamma(G)$ is the least cardinality of a dominating set of G. A chromatic transversal dominating set(CTDS) of a graph G is a dominating set D which intersects every color class of each χ-partition of G. The chromatic transversal domination number $\gamma_{ct}(G)$ is the least cardinality of a CTDS of G. In this paper, we characterize cubic graphs, block graphs and cactus graphs with equal domination number and chromatic transversal domination number .*

Keywords—*domination number, chromatic transversal domination number, cubic graph, block graph, cactus graph.*

I. INTRODUCTION

Let n and m denote respectively, the order and size of a graph G=(V,E). $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ is called the neighborhood of v in G. The degree $d_G(v)$ of a vertex v is the number of vertices in the neighborhood of v with respect to G. A vertex of degree one is known as a leaf and a vertex adjacent to a leaf is called its support. A strong support vertex is a support adjacent to at least two leaves of G. The complete graph on n vertices we denote by K_n. Let P_n and C_n, respectively, denote a path and a cycle of order n.

Michaelraj et al. [5] introduced a hybrid domination parameter, called the chromatic transversal domination number of a graph. Arumugam and Joseph [2] characterized the families of trees, unicyclic graphs and cubic graphs with domination number equal to the connected domination number. Chen et al.[8] further extended this characterization for the families of block graphs and cactus graphs. Ayyaswamy et al. [3] characterized a family of graphs with equal chromatic transversal domination number and connected domination number. In [11], Sahul Hamid introduced the concept of independent transversal domination in graphs. H.A.Ahangar et al. [1] studied the complexity results of this new domination parameter. L.Benedict Michaelraj et al. [7] obtained some characterization results on dominating chromatic partition-covering number of graphs. S. Balamurugan et al.[4] characterized the family of graphs for which neighbourhood chromatic domination number is two.
Motivated by the results in [2,8] we study the equality of domination number and chromatic transversal domination number for the classes of cubic graphs, block graphs and cactus graphs.

A proper coloring of a graph G is an assignment of colors to the vertices such that adjacent vertices do not share the same color. The chromatic number $\chi(G)$ is the minimum colors used to color the vertices of G. A partition of V(G) into χ independent sets {V_1,V_2,…,V_χ} is called a χ-partition of G, where each V_i is the color class that represents the color i for i = 1,2,…,$\chi(G)$. A vertex v of G is a critical vertex if $\chi(G-v) < \chi(G)$, where $\chi(G)$ is the chromatic number of G. If every vertex of G is critical, then G is called a χ-critical graph.

For a subset $D \subseteq V(G)$, if every vertex of $V(G) \setminus D$ has a neighbor in D, then D is called a dominating set of G. The domination number $\gamma(G)$ is the least cardinality of a dominating set of G. One may refer [8] for a detailed survey about domination in graphs. A chromatic transversal dominating set(CTDS) of a graph G is a dominating set D which intersects every color class of each χ-partition of G. The chromatic transversal domination number $\gamma_{ct}(G)$ is the least cardinality of a CTDS of G. For a subset S of V(G), $N_G(S)$ is the collection of vertices adjacent to some vertex in S and $N_G[S] = N_G(S) \cup S$. A graph G is cubic if it is 3-regular. A vertex v of a connected graph G is said to be a cut-vertex if G-v is disconnected. A connected subgraph B of G is a block, if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cut-vertex. A block B of G is called an end-block, if B contains at most one cut-vertex of G; such cut-vertex is called an end-block cut-vertex. A block graph G is a connected graph in which every block is complete. A graph G is called a cactus graph if every edge of G belongs to at most one cycle. A cycle in a cactus graph G is called an end cycle if it contains exactly one cut-vertex of G. The clique number $\omega(G)$is the order of a maximum clique in G.

Theorem 1. 1 ([6]): *Let G be a connected graph. We have $\gamma_{ct}(G) = n$ if and only if G is χ-critical.*

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Theorem 1.2([6])**:** *Let G be a connected bipartite graph of order n ≥3 with bipartition (X,Y), where |X| ≤|Y|. Then $\gamma_{ct}(G)$ = $\gamma(G)+1$ if and only if every vertex of X is a strong support vertex.*

Let $\mathfrak{I}$ be a family of trees T such that T is either K_2, or T is such that all its support vertices are strong and every vertex at an even distance from a support vertex is also a support vertex.

Theorem 1.3([6])

Let T be a tree. Then $\gamma_{ct}(T) = \gamma(T)$ if and only if $T \in \mathfrak{I}$.

Let $\mathcal{A}$ be the following family of graphs.

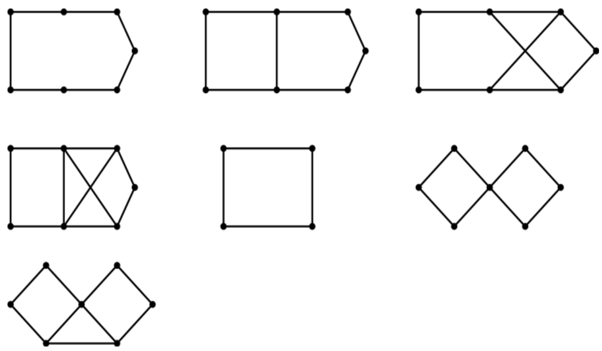

Fig.1

We have the following upper bound on the domination number of a graph.

Theorem 1.4:

If G is a connected graph with $\delta(G) \geq 2$ and $G \notin \mathcal{A}$, then $\gamma(G) < \dfrac{2n}{5}$.

II. MAIN RESULTS

A. Cubic graphs

Let G be a connected cubic graph with at least one odd cycle C. For each cycle C of smallest length in G, let $G'(C) = G - N_G[C]$. By $\gamma(G'(C))$ we mean a γ-set of $G'(C)$ with respect to G. That is a γ-set of $G'(C)$ with respect to G may contain a vertex of $G'(C)$ or a vertex of $N_G[C] \setminus V(C)$ or both.

Proposition 2.1: Let $G \neq K_4$ be a connected cubic graph with at least one odd cycle. Let C be the set of all odd cycles of smallest length in G. Then $\gamma_{ct}(G) =$ min{|C|+ $\gamma(G'(C))$: $C \in$ C }.

Proof: Let T be any subset of G. If the subgraph $<T>$ induced by T does not contain an odd cycle, then we can find a χ-partition of G for which all the vertices of T can be colored with at most two colors. This implies T is not a transversal for this χ-partition. Thus any transversal of all χ-partitions of G must contain an odd cycle. Therefore for any $C \in$ C, T=V(C) \ S, where S is a γ-set of $G'(C) = G - N_G[C]$ is a CTDS of G. Hence $\gamma_{ct}(G) =$ min{|C|+ $\gamma(G'(C))$: $C \in$ C }.

Example 2.2: For this cubic graph G given in fig.2, C= $\{C_1, C_2, C_3, C_4, C_5\}$ where $C_1 = \langle v_1, v_2, v_3, v_4, v_5 \rangle$, $C_2 = \langle v_1, v_2, v_2', u_2, v_1' \rangle$, $C_3 = \langle v_4, v_4', u_5, v_5', v_5 \rangle$, $C_4 = \langle v_1, v_1', u_3, v_5', v_5 \rangle$ and $C_5 = \langle v_3, v_4, v_3', u_7, v_4' \rangle$.

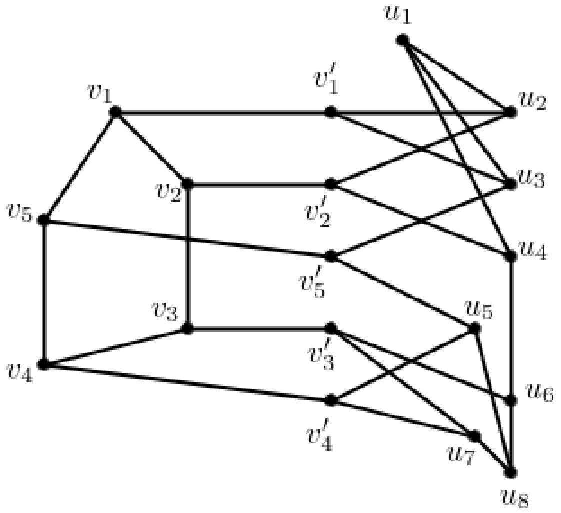

Fig.2

Let T be a transversal of all χ-partitions of G. If $T = C_1$, then a γ-set of $G'(C_1)$ is $\{u_1, u_8\}$. If $T = C_2$, then a γ-set of $G'(C_2)$ is $\{v_4', v_5', u_6\}$. If $T = C_3$, then a γ-set of $G'(C_3)$ is $\{v_3', u_3, v_2'\}$. If $T = C_4$, then a γ-set of $G'(C_4)$ is $\{v_3', u_6, v_2', u_4'\}$. Similarly, if $T = C_5$, then a γ-set of $G'(C_5)$ is $\{v_1', u_2, u_4, v_5'\}$. Therefore, $\gamma_{ct}(G) =$ min {5+2,5+3,5+4}=7.

Theorem 2.3: Let G be a connected cubic graph with at least one odd cycle. Then $\gamma_{ct}(G) = \gamma(G)$ if and only if there exists an odd cycle C of smallest length, say k in G such that the following conditions hold:

(i) $G' \neq \phi$

(ii) $N_G[C] \setminus V(C)$ is independent with cardinality k

(iii) If S is a γ-set of $G'(C)$ with respect to G, then $N[S] \subseteq V(G'(C))$.

Proof: Assume that $\gamma_{ct}(G) = \gamma(G)$. Then $G \neq K_4$, since $\gamma(K_4) = 1$ whereas $\gamma_{ct}(K_4) = 4$.

Let C be an odd cycle of smallest length k such that

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$\gamma_{ct}(G) = |V(C)| + \gamma(G'(C))$.

Let $V(C) = \{v_1, v_2, ..., v_k\}$ and $N_G(C) \setminus V(C) = \{v_1', v_2', ..., v_r'\}$ where $r \le k$. Suppose condition (i) fails. That is $G' = \phi$. Now |V(G)|=2k, $\delta(G) = 3 > 2$ and $G \notin A$, in view of Theorem 1.4, $\gamma(G) \le \frac{4k}{5} < k.$ But $\gamma_{ct}(G) = k$.

This is a contradiction.

If $|N_G[C] \setminus V(C)| < k$, then there exist vertices v_i' and v_j' such that $v_i' = v_j'$ for some i and j, $i \ne j$. Therefore $(V(C) \setminus \{v_i, v_j\}) \cup \{v_i'\} \cup S$ is a dominating set of G where S is a γ-set of $G'(C)$ with respect to G. Hence $\gamma(G) \le |C| - 1 + |S| < \gamma_{ct}(G)$, a contradiction. Further, if $N_G[C] \setminus V(C)$ is not independent, let v_i' be adjacent to v_j' in G. Then $V(C) \setminus \{v_i'\} \cup S$ where S is a γ-set of $G'(C)$, is a dominating set with cardinality < $\gamma_{ct}(G)$, a contradiction.

Suppose condition (iii) fails in G. Then there exists a vertex $v \in S$ such that $vv_i' \in E(G)$ for some i=1,2,3,…,k. This implies $(V(C) \setminus \{v_i\}) \cup S$ is a dominating set of G and so $\gamma(G) \le k - 1 + |S| <$ min{ $|C| + |S| : C \in$ C}, a contradiction.

The converse is obvious.

B. Block graphs

Let G be a block graph with $\omega(G) \ge 3$ and K be a maximal clique of G. We define the set S_K to contain vertices $v \in V(K)$ such that:

(a) v is either an end-block cut-vertex or

(b) for at least one path P in $G - N_G(K)$ from v, the vertex w in P with $d(v, w) = 3$ is in every γ-set of G.

Question: How to identify such a vertex w which is in every γ-set of G?

If P is a shortest path from w to a cut-vertex w_l of an end-block such that every vertex w_k on P with $d(w, w_k) \equiv 0$ (mod 3) is in every γ-set of G, in particular, if $d(w, w_l) \equiv 0$ (mod 3), then w is in every γ-set of G. We give below a few examples satisfying this criteria.

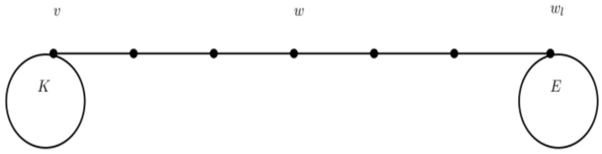

Fig.3: A block graph G with maximal clique K, end-block E and end-block cut-vertex w_l.

Let $T_K = \{v \in K :$ there is a path P from v such that the vertex v_1 in $N_G(v) \cap P$ dominates the vertex v and $v_2 \in N_G(v_1) \cap P\}$ and $U_K = \{v \in N_G(K) \setminus V(K) :$ there are at least two paths $P_1 = < v = v_0, v_1, ..., v_r >$ and $P_2 = < v = u_0, u_1, ..., u_s >$ such that both v_1 and u_1 are dominated by v }.

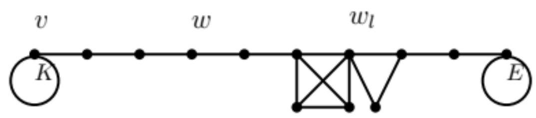

Fig.4: Block graph G with maximal clique K, end-block

E and a cut-vertex w_l of two blocks $\ne K_2$.

Lemma 3.1: Let G be a block graph with $\omega(G) \ge 3$.

(i) If K is a maximal clique such that $|S_K|$ is maximum, then there exists a $\gamma_{ct}(G)$-set containing K.

(ii) If no such K exists, that is, $|S_K| = 0$ or $|S_K|$ is the same for all maximal cliques K, then for any maximal clique K for which $|T_K|$ is minimum, then there exists a $\gamma_{ct}(G)$-set containing K.

(iii) If conditions (i) and (ii) fails, then for any maximal clique K in G, there exists a $\gamma_{ct}(G)$-set containing K.

Proof: (i) If $v \in S_K$ is an end cut-vertex, then v is needed to dominate all its neighbors and so v will be in every γ-set of G. On the other hand, if v satisfies the condition (b) of S_K, then w with d(v,w)=3 on any path $P = < v = v_0, v_1, v_2, w, ..., v_r >$ from v will be in every $\gamma - set$ of G and so v_2 will be dominated by w. Therefore, to dominate v_1, either v or v_1 must be included in every γ-set of G. Let $|S_{K_1}| > |S_{K_2}|$. Let C denote the set of all end-cutvertices in $G - (K_1 \cup K_2)$. Now $(G - N_G(K_2)) \cup U_{K_2}$ contains C and S_{K_1}. Similarly, $(G - N_G(K_1)) \cup U_{K_1}$ contains C and S_{K_2}. Therefore $|S_{K_1}| > |S_{K_2}|$ implies $|D_1| \le |D_2|$ where D_1 and

D_2 are the γ -sets of $(G-N_G(K_1))\cup U_{K_1}$ and $(G-N_G(K_2))\cup U_{K_2}$ respectively.

(ii) Let K_1 and K_2 be two maximal cliques such that $|T_{K_1}|<|T_{K_2}|$. Thus among the CTD sets $V(K_1)\cup E_1$ and $V(K_2)\cup E_2$ of G, $|V(K_1)\cup E_1|$ $\leq|V(K_2)\cup E_2|$. This implies the condition (ii) is true.

If conditions (i) and (ii) fail, then for any two maximal cliques K_1 and K_2, $|F_1|=|F_2|$ where F_1 and F_2 are γ -sets of $(G-N_G(K_1))\cup U_{K_1}$ and $(G-N_G(K_2))\cup U_{K_2}$ respectively. Therefore, $|V(K_1)\cup F_1|=|V(K_2)\cup F_2|=\gamma_{ct}(G)$.

Theorem 3.2: Let G be a block graph with $\omega(G)\geq 3$. Then $\gamma_{ct}(G)=\gamma(G)$ if and only if there exists a maximal clique K such that $|S_K|=|V(K)|=\omega(G)$.

Proof: Let us assume that $\gamma_{ct}(G)=\gamma(G)$. Let K be a maximal clique satisfying the condition either (i), (ii) or (iii) of Lemma 3.1. Then $\gamma_{ct}(G)=|V(K)|+\gamma((G-N_G(K))\cup U_K)$. But $\gamma(G)=\gamma_G(K)+\gamma((G-N_G(K))\cup U_K)$ where $\gamma_G(K)$ is the domination number of K with respect to G.

Therefore $\gamma_{ct}(G)=\gamma(G)$ if and only if $|V(K)|=\gamma_G(K)$. This implies every vertex of K dominates at least one of its neighbors. This is possible if and only if $|S_K|=|V(K)|=\omega(G)$.

C. Cactus graphs

Let G be a cactus graph with an odd cycle C of smallest length. Similar to block graphs we define the sets S_C, T_C and U_C as follows:

$S_C=\{v\in C: v$ is either a cut-vertex of an end cycle of length $l\equiv 0,2$ (mod 3), or for at least one path P in $G-N_G[C]$ from v, the vertex w in P with $d(v,w)=3$ is in every γ -set of $G\}$.

$T_C=\{v\in C$: there is a path from v for which the vertex $v_1\in N_G(v)\cap P$ dominates the vertices v and v_2 $\in N_G(v_1)\cap P$ or v is a cut-vertex of an end cycle of length $l\equiv 1$ (mod 3)$\}$ = $V(C)\setminus S_C$.

$U_C=\{v\in N_G[C]\setminus V(C)$: there are at least two paths $P_1=<v=v_0,v_1,\ldots,v_r>$ and $P_2=<v=u_0,u_1,\ldots,u_s>$ such that both v_1 and u_1 are dominated by $v\}$.

Let T be the set of all odd cycles in a cactus graph G for which S_C is maximum.

Lemma 4.1: Let G be a cactus graph with at least one odd cycle.

(i) If C is an odd cycle of least length such that $|S_C|$ is maximum, then there exists a $\gamma_{ct}(G)$ -set containing C.

(ii) If no such C exists, that is, $|S_C|=0$ or there are more than one odd cycle of least length with the same $|S_C|$, then for any such C for which $|T_C|$ is minimum there exists a $\gamma_{ct}(G)$ -set containing C.

(iii) If both conditions (i) and (ii) fail, then for any odd cycle of least length C, there exists a $\gamma_{ct}(G)$ -set containing C.

Proof: Proof of this lemma is just similar to that of Lemma 3.1. For the sake of completeness we prove (ii).

Let C_1 and C_2 be two odd cycles in T of smallest length, say k, such that $|S_{C_1}|=|S_{C_2}|$ and C_1 has more number of cut vertices of end cycles of lengths $l\equiv 1$ (mod 3) than those of C_2. Then $V(C_1)\cup D_1$ and $V(C_2)\cup D_2$ are ctd-sets of G. where D_1 and D_2 are γ-sets of $(G-N_G(C_1))\cup U_{C_1}$ and $(G-N_G(C_2))\cup U_{C_2}$ respectively. As C_2 has less number of cut vertices of end cycles of lengths $l\equiv 1$ (mod 3), we have $|D_2|\leq |D_1|$. This implies $\gamma_{ct}(G)=|V(C)\cup D|$ where C is an odd cycle in T of smallest length k such that C has the least number of cut vertices of end cycles of lengths $l\equiv 1\ (mod\ 3)$ and D is a γ -set of $(G-N_G[C])\cup U_C$.

Theorem 4.2: Let G be a cactus graph with at least one odd cycle. Then $\gamma_{ct}(G)=\gamma(G)$ if and only if there exists an odd cycle C of least length in T such that $|S_C|=|V(C)|$.

Proof: Let us assume that $\gamma_{ct}(G)=\gamma(G)$. Then there exists an odd cycle CT. By Lemma 4.1,

$\gamma_{ct}(G)=|V(C)|+\gamma((G-N_G[C])\cup U_C)$. But $\gamma(G)=\gamma_G(C)+\gamma((G-N_G[C])\cup U_C)$. Let D be a $\gamma_{ct}(G)$ -set. Suppose $|S_C|\neq|V(C)|$. Then $T_C\neq\emptyset$. Let $v\in T_C$. Then we have the following two cases.

Case 1: v is a cut-vertex of an end cycle of length $l\equiv 1\ (mod\ 3)$.

Let $C'=<v=u_1,u_2,\ldots,u_{3r+1}>$ be an end cycle of length 3r+1 containing v. Then the neighbors of v in C' namely u_2 and u_{3r+1} are dominated by u_3 and u_{3r} respectively. Therefore for any $\gamma_{ct}(G)$ -set D containing $V(C)$, $D\setminus\{v\}$ is a γ- set of G. This implies $\gamma_{ct}(G)>\gamma(G)$, a contradiction.

Case 2: Let $P=<v=v_0,v_1,\ldots,v_s>$ be a path in $G-N_G[C]$ from v such that v and v_2 are dominated by v_1. Then $D\setminus\{v\}$ is a γ -set of G implying $\gamma_{ct}(G)>$

$\gamma(G)$, a contradiction. Thus $|T_C| = 0$ which implies $|S_C| = |V(C)|$.

III. ACKNOWLEDGMENT

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