Abstract: This article, we study sufficient conditions for the controllability of second-order impulsive neutral integro-differential evolution systems with an infinite delay in Banach spaces by using the theory of cosine families of bounded linear operators and fixed point theorem.

Keywords: Nonlinear ordinary differential operators, Impulsive optimal control problems; evolution equations.

2000 Subject Classification: 34L30, 49N25, 37L05.

I. INTRODUCTION

The study of impulsive functional differential systems is related to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are executed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place instantaneously in the form of impulses. The theory of impulsive systems provides a common framework for mathematical modeling of many real world phenomena. Moreover, these impulsive phenomena can also be found in fields such as information science, electronics, fed-batch culture in fermentative production, robotics and telecommunications (see [1, 5, 14, 16, 18, 19, 21] and references therein).

In recent years, the study of impulsive control systems has received increasing interest. Due to its importance several authors have investigated the controllability of impulsive systems (see [2, 6, 7, 10, 20]).

Motivated by the effort of the after mentioned papers [2, 11, 18], the primary inspiration driving this manuscript is mainly concerned with the study of controllability of second order impulsive partial neutral integro-differential system of the form

The control function \( u(\cdot) \) is given in \( L^2(J,U) \), a Banach space of admissible control functions with \( U \) as a Banach space and \( B : U \to X \) as a bounded linear operator. For \( \forall u \in J, \vartheta_k \) represents the function \( \vartheta_k : (-\infty,0] \to X \) defined by \( \vartheta_k(\theta) = \vartheta(\theta + t), -\infty < \theta \leq 0 \) which belongs to some abstract phase space B defined axiomatically, \( \theta, \vartheta_k : J \times J \times B \to X; a_r, a_l : J \times J \times B \to X; B \to X; I_0 : B \to X \) are appropriate functions and will be specified later. 0 < \( u_1 < \ldots < u_m < a \) are fixed numbers and the symbol \( \Delta \vartheta_k(\vartheta) \) represents the jump of a function \( \vartheta \) at \( \vartheta_k \), which is defined by \( \Delta \vartheta_k(\vartheta) = \vartheta(\vartheta_k) - \vartheta^{(a)}(\vartheta_{k-1}) \).

Throughout the text we will assume that \( A(\cdot) \) generates an evolution operator \( S(u,s') \).

II. PRELIMINARIES

This section we review some basic concepts, notation, and properties required to find our main results. Nowadays there has been an increasing interest in studying the theoretical non-autonomous second order initial value problem where \( A(u') : D(A(u')) \subseteq \vartheta \to X; u \in J = [0,a] \) is a closed densely defined operator and \( f : J \to X \) is a suitable function. Equations of this form have been considered in several papers. We refer the reader to [15, 17] and the references therein. In the majority of works, the existence of results to the problem (5)-(6) is related to the existence of an evolution operator \( S(u,s') \) for the homogeneous equation,

\[
\hat{\vartheta}'(u') = A(u')\vartheta(u'),
\]

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Let as assume that the domain of $A(u')$ is a subspace of $D$ dense in $X$ and not dependent of $u'$, and for each $\forall \theta \in D$ the function $u' \mapsto A(u')\theta$ is continuous. The fundamental solution for the second-order evolution equation (7), has been developed by Kozak [13], and we will use the following concept of evolution operator.

**Definition 2.1** A family $S$ of a bounded linear operator $S(u,s') : J \times X \to L(\theta)$ is called an evolution operator for (7), if the following conditions are satisfied:

1. For each $\forall \theta \in X$, the mappings $(u,s') \mapsto S(u,s') \in C$ and
   
   \[
   (\theta) \quad \text{For each } u' \in [0, s), S(u, u') = 0.
   \]

2. For all $u, s' \in [0, a]$, and for each $\forall \theta \in X$,
   \[
   \frac{\partial}{\partial u} S(u,s')|_{u=s} = 0
   \]

3. For all $u, s' \in [0, a]$, if $\forall \theta \in D(A)$, then $S(u,s')$ is of class $\mathcal{C}$ and
   \[
   (i) \quad \frac{\partial^2}{\partial u^2} S(u,s') = A(u)S(u,s')\theta,
   \]
   \[
   (ii) \quad \frac{\partial^2}{\partial s^2} S(u,s') = S(u,s')A(s)\theta.
   \]

4. For all $u, s' \in [0, a]$, if $\forall \theta \in D(A)$, then $\frac{\partial}{\partial s} S(u,s') \in D[A]$
   \[
   \frac{\partial^2}{\partial s^2} S(u,s') = \frac{\partial}{\partial s} S(u,s')A(s)\theta.
   \]

5. For each $u' \in [0, a]$, $S(u,s')$ is bounded on $[0, a]$, and $S(u,s')$ is continuous.

Throughout this problem we assume that there exists an evolution operator $S(u,s')$ associated to the operator $A(u')$. To abbreviate the text, we introduce the operator

\[
C(u,s) = -\frac{\partial}{\partial s} S(u,s).
\]

In addition, we set $N$ and $\tilde{N}$ for positive constants such that

\[
\sup_{0 < s, u' < a} \|C(u,s)\| \leq \tilde{N}
\]

In the literature a number of methods have been discussed to establish the existence of the evolution operator $S(\cdot, \cdot)$. In particular, a very studied situation is that $A(u')$ is that perturbation of an operator $A$ that generates a cosine operator function. In this reason, below we briefly analysis some essential properties of the theory of cosine functions. Let $A : D(A) \subseteq \theta \to X$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(u'))_{u' \in R}$ in Banach space $\theta$. We denote by $(S(u'))_{u' \in R}$ the sine function associated with $(C(u'))_{u' \in R}$ which is defined by

\[
S(u') = \int_0^{u'} C(s)ds, \quad \forall u' \in X
\]

and $u' \in R$. We refer them to [3, 23] for the necessary concepts about cosine functions. After that we only mention a few results and notations about this matter needed to establish our results. It is immediate that

\[
C(u)\vartheta - A \int_0^{u} S(s)ds,
\]

for all $\forall \theta \in X$. The notation $D(A)$ stands for the domain of the operator $A$ endowed with the norm graph $k_{D(A)} = k_{D} + k_{A D}$, $D(A)$.

Moreover, in this work, $E$ is the space formed by the vectors $\forall \theta$ for which $C(u')\theta$ is of class $\mathcal{C}$ on $R$. It was proved by Kisyński [12] that $E$ endowed with the norm $k_{D(A)} = k_{D} + k_{A D}$, $k_{A D}$, $\forall \theta \in E$, is a Banach space. The operator-valued function

\[
\mathcal{H}(u) = \begin{bmatrix}
C(u) & S(u) \\
A S(u) & C(u)
\end{bmatrix}
\]

is a strongly continuous group of bounded linear operators on the space $E \times \theta$ generated by

\[
\mathcal{H}(u) = \begin{bmatrix}
0 & 1 \\
A & 0
\end{bmatrix}
\]

defined on $D(A) \times \theta$. From this, it follows that $S(u') : X \to E$ is a bounded linear map such that the operator valued maps $S(\cdot)$ is strongly continuous and $A S(u') : E \to X$ is a bounded linear operator such that $A S(\theta) \to 0$ as $u' \to 0$, for each $\theta \in E$. Furthermore, if $\vartheta : [0, \infty) \to X$ is a locally integrable function, then the function

\[
\vartheta(\cdot) = \int_0^{\cdot} S(\theta - s)\vartheta(s)ds
\]

defines an $E$-valued continuous function. The existence of solutions for the second order abstract Cauchy problem,

\[
\begin{cases}
S(u') = A S(u') + h(u'), & 0 \leq u' \leq a \\
P(u, S(a)) = k \in N
\end{cases}
\]

is called a mild solution of (7)-(8) and that when $u \in E, \vartheta(\cdot)$ is continuously differentiable with $\vartheta(\cdot) + h(\cdot) \in [0, a]$. Published By:

Blue Eyes Intelligence Engineering & Sciences Publication

Retrieval Number: paper_id/2019©BEIESP

DOI: 10.35940/ijrte.xxxxx.xxxxxx

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In addition, if $\lambda \in D(A),\omega \in E$ and $f$ is a continuously differentiable function, then the function $\dot{\lambda}(\cdot)$ is a solution of the initial value problem (11)-(12).

The function $\dot{u} \mapsto B(\dot{u})\dot{\omega}$ is continuously differentiable in $E$ for each $\lambda \in \mathcal{E}$. It has been assumed now that $A(\dot{u}) = A + B(\dot{u})$ where $B(\cdot): R \rightarrow L(E,\mathcal{E})$ is a map such that $\dot{\lambda}(\cdot)$ is continuously differentiable in $E$. This is clear that the same argument allows us to conclude that Eq. (14), with the initial condition (12) has a unique solution $\dot{\lambda}(-s)$ such that the function $\dot{u} \mapsto \dot{\lambda}(\cdot)$ is continuously differentiable in $E$. It follows from (13) that

$$\dot{\lambda}(u,s) = C'(\dot{u} + \dot{\lambda}(s)) + \int_{0}^{s} S'((u - \tau),\dot{\lambda}(s))d\tau$$

and, applying the Gronwall - Bellman lemma we infer that

$$k\dot{\lambda}(u,s) \leq M\sup_{x \in J}(\dot{\lambda}(u,s))$$

We define the operator $S(\dot{u},s) = \dot{\lambda}(u,s)$, It follows from the previous estimate that $S(\dot{u},s)$ is a bounded linear map on $E$. Since $E$ is dense in $X$, we can extend $S(\dot{u},s)$ to $X$. We keep the notation $S(\dot{u},s)$ for this extension. It is well known that, exception the case $dim(X) < \infty$, the cosine function $C(\cdot)$ cannot be compact for all $\lambda \in R$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(\lambda)$ is very often a compact operator for all $\lambda \in R$. This motivates the result [18], Theorem 1.2.

We now consider some notions and definitions concerning impulsive differential equations. A function $\dot{\lambda}: [\tau,\tau] \rightarrow X$ is said to be a normalized piecewise continuous function on $[\tau,\tau]$. If $\dot{\lambda}$ is piecewise continuous and left continuous on $(\tau,\tau]$. We denote by $PC([\tau,\tau],X)$ the space of normalized piecewise continuous functions from $[\tau,\tau]$ to $X$. In particular, we introduce the space $PC$ formed by all normalized piecewise continuous functions $\dot{\lambda}:[0,\tau] \rightarrow X$ such that $\dot{\lambda}(\cdot)$ is continuous at $t \in (\tau,\tau]$. For $\dot{\lambda}(u,s)$ and $\dot{\lambda}(u,s)$, exists, for $k = 1,2,...,m$. In this paper, we always assume that $PC$ is endowed with the norm $\|\dot{\lambda}\|_{PC} = \sup_{s \in J}|\dot{\lambda}(s)|$. It is clear that $(PC,k,\mathcal{K})$ is a Banach space.

In what follows, we put $u_0 = 0, u_{k+1} = \alpha$ and, for $\lambda \in \mathcal{E}$, we denote by $\mathcal{F}_k$, for $k = 0,1,...,m$, the function $\mathcal{F}_k \in C([x_{k+1},x_{k+1}];\mathcal{E})$ given by $\mathcal{F}_k(u) = \mathcal{F}(u)$ for $u \in (u_{k+1},u_{k+1})$ and $\mathcal{F}_k(u_{k+1}) = lim \mathcal{F}(u_{k+1})$. Moreover, for a set $E \subseteq PC$, we denote by $E^k$, for $k = 0,1,...,m$, their $u \mapsto u^{-}$. Set $E^k = \mathcal{F}_k \subseteq \mathcal{E}$.

**Lemma 2.1** A set $E \subseteq PC$ is relatively compact in PC if and only if each $E_k, k = 0,1,...,m$,

is relatively compact in $C([x_{k+1},x_{k+1}];\mathcal{E})$.

In this work we will employ an axiomatic definition of the phase space $B$, similar to the one used in [9] and suitably modify to treat retarded impulsive differential equations. Moreover, $B$ will denote the vector space of functions defined from $(-\infty,0]$ into $\mathcal{E}$ endowed with a seminorm denoted $k_{\lambda}$ and such that the following axioms are hold:

(A) If $\mathcal{F}(\cdot,\cdot) \rightarrow \mathcal{F}, \lambda > 0$, is such that $\mathcal{F}_k \in B$ and $\mathcal{F}(\mu,\mu+b) \in PC([\mu,\mu+b],X)$ then, for every $u \in [\mu,\mu+b)$, the following conditions are hold:

(i) $\mathcal{F}_k \in B$

(ii) $k\mathcal{F}(u)k \leq Hk\mathcal{F}_k k_B$

(iii) $k\mathcal{F}_k k_B \leq K(u)\sup_{k \in \mu-\mu} \sup_{k \in s \leq s \leq u} \mathcal{F}_k m(u) \mathcal{M} k\mathcal{F}_k k_B$

where $H > 0$ is a constant; $K,M : [0,\infty) \rightarrow [1,\infty)$ is continuous, $M$ is locally bounded and $H, K, M$ are independent of $\mathcal{F}(\cdot)$.

(B) The space $B$ is complete.

**Remark 2.1** In impulsive functional differential systems, the map $[\mu,\mu+b] \rightarrow \mathcal{F}, \lambda \mapsto \mathcal{F}$, is in general discontinuous. For this reason, this property has been omitted from our description of the phase space $B$.

Now we include that some of our proofs are based on the following well-known result [4], Theorem 6.5.4.

**Lemma 2.2** (Leray-Schauder Alternative) Let $D$ be a closed convex subset of a normed linear space $X$ such that $0 \in D$. Let $F : D \rightarrow D$ be a completely continuous map. Then the set $\{\mathcal{F} \in D : \mathcal{F} = F(\mathcal{F}), for some 0 < \lambda < 1\}$ is unbounded or the map $F$ has a fixed point in $D$.

The terminology and notations are generally used in functional analysis. In particular, for Banach spaces $(Z,k,k)$, $(W,k,k)$, the notation $L(Z,W)$ stands for the Banach space of bounded linear operators from $Z$ into $W$ and we abbreviate to $L(Z)$ whenever $Z = W$. By $\mathcal{S}(A)$ (respectively $\varphi(A)$) we denote the spectrum (respectively the resolvent set) of a linear operator $A$. Moreover, $B_+(\mathcal{S},Z)$ denotes the closed ball with center at $\mathcal{S}$ and radius $r > 0$ in the space $Z$.

**Remark 2.2** In what follows the notation $g(a)$ stands for the space...
Theorem 3.1 If the hypothesis (H1)-(H6) are satisfied, then the impulse second order system (14)-(17) is controllable on $J$. Proof. Using the assumption (H3), we define the control function

$$u(\tilde{u}) = W^{-1}(z, a, s, B, \psi, \theta, \phi, 0, 0, 0, 0, 0)$$

and let $u(\tilde{u}) = \sup \{u : \tilde{u} \in \mathbb{R}^n \}$. The bounded invertible operator $W^{-1}$ is defined by $W^{-1}(z, a, s, B, \psi, \theta, \phi, 0, 0, 0, 0, 0)$. The function $W^{-1}$ is bounded and hence $W^{-1}$ is a bounded operator. Therefore, the impulse second order system (14)-(17) is controllable on $J$. The proof is complete.
The function \( \phi(x, \varphi) \rightarrow \theta(\varphi, x) \) is continuous for almost everywhere \( x \in J \).

For each \( x \in J \), the function \( \theta(x, \varphi) := B \) and the function \( F \rightarrow \theta(\varphi, x) \) is strongly measurable.

There is a positive continuous function \( \mu : [0, \infty) \rightarrow (0, \infty) \) and a continuous non decreasing function \( \psi : R \rightarrow (0, \infty) \) such that

\[
\int_0^{\infty} \left( \max(0, \mu(s) - \psi(s)) \right) \psi(s) ds < \infty.
\]

Theorem 3.2 Assume that \( \theta, \) verify condition (A1) and condition (A2), (H3) are satisfied.

Also, the following condition hold:

(a) For every \( \varphi \in J \) and every \( r > 0 \), the set \( U_r(\varphi, \psi) = \{ u(s) \mid \psi(u(s)) < r \} \) is an open subset of \( B \).

(b) The maps \( U_r, J \rightarrow X \) is completely continuous and there exist positive constants \( \alpha_r, \beta_r \) such that for all \( \varphi \in B, \psi \in [0, \infty) \),

\[ \int_0^1 u(s) \varphi(s) ds < \infty \]
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Denoting by \( \omega_\lambda(t) \) the right-hand side of the previous inequality, we see that

\[
\omega_\lambda(t) \leq \frac{K_a N}{1 - \mu} \left[ \sum \left( \int \left( \phi_1(t) \| u(t) \| \right) dt + \phi_2(t) \right) + N \left( \| u(t) \| \right) \right]
\]

and subsequently, upon integrating over \([0, u^*] \), we obtain

\[
\int_0^u \frac{ds}{\Omega(s)} \leq \frac{K_a N}{1 - \mu} \int_0^u p(s) ds \leq \frac{K_a N}{1 - \mu} \int_0^u p(s) ds < \int_0^\infty \frac{ds}{\Omega(s)}
\]

This estimate permits us to conclude that the set of functions \( \{ \omega_\lambda : \lambda \in (0, 1) \} \) is bounded and, in turn, that \( \{ y_\lambda : \lambda \in (0, 1) \} \) is bounded in \( \theta_2(a) \). Next we show that \( \Gamma \) is completely continuous. To clarify this proof, we decompose \( \Gamma \) in the form

\[
\Gamma = \Gamma_1 + \Gamma_2,
\]

where

\[
\begin{align*}
\Gamma_1(u) &= \int_0^u S(u, s) \left[ \phi_2(s, y_s + \phi_s) \right] + \int_0^s a_2(s, \tau, y_s + \phi_s) \right] + B \left( u(s) \right) \right] ds \\
\Gamma_2(u) &= \int_0^u C(u, s) \left[ \phi_1(s, y_s + \phi_s) \right] + \int_0^u a_1(s, \tau, y_s + \phi_s) \right] + B \left( u(s) \right) \right] ds
\end{align*}
\]

Using the hypotheses, condition (b) and Lemma 2.1, we obtain that \( \Gamma_1 \) is continuous and that \( \Gamma_2 \) is completely continuous. In order to use the Ascoli-Arzela theorem we prove that \( \Gamma_1 \) takes bounded sets into relatively compact ones. As above, \( B_1 = B \left( u(s) \right) \) and \( \left[ y_0 + \phi_1 \right] \). Also, \( k(B_0 u(s)) \) is \( \frac{1}{B_0} \).

From the mean value theorem, we get the set \( \{ \gamma (u^* : y \in B \left( u(s) \right) \} \) is relatively compact for each \( \gamma \in J \).

Moreover, from

\[
\begin{align*}
\Gamma_1(u + h) - \Gamma_1(u) &= \int_0^u S(u, s) \left[ \phi_2(s, y_s + \phi_s) \right] + \int_0^s a_2(s, \tau, y_s + \phi_s) \right] + B \left( u(s) \right) \right] ds \\
\Gamma_2(u + h) - \Gamma_2(u) &= \int_0^u C(u, s) \left[ \phi_1(s, y_s + \phi_s) \right] + \int_0^u a_1(s, \tau, y_s + \phi_s) \right] + B \left( u(s) \right) \right] ds
\end{align*}
\]

and using that \( S(s, s) \) and \( C(s, s) \) verifies a Lipschitz condition, we obtain that

\[
\Gamma_1(u + h) - \Gamma_1(u) \leq \left[ \| h \| N \int_0^u \left( | \phi_2(s, y_s + \phi_s) \right) + \| a_2(s, \tau, y_s + \phi_s) \right] + B \left( u(s) \right) \right] ds
\]

which implies that \( \Psi \Gamma (u^* + h) - \Psi \Gamma (u^*) \rightarrow 0 \) as \( h \rightarrow 0 \) uniformly for \( y \in B \left( u(s) \right) \). From this we infer that \( \Gamma (u^* + h) \) is relatively compact in \( g(a) \) and consequently that \( \Gamma \) is completely continuous. This completes the proof of the assertion that the map \( \Gamma \) is completely continuous.

By an application of Lemma 2.1, we conclude that there exists a fixed point \( y \) of \( \Psi \). It is clear that the function \( \Psi y + \phi \) is a mild solution of the system (1)-(4). This completes the proof.

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IV. CONCLUSION

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

APPENDIX

It is optional. Appendices, if needed, appear before the acknowledgment.

ACKNOWLEDGMENT

It is optional. The preferred spelling of the word “acknowledgment” in an American English is without an “e” after the “g.” Use the singular heading even if you have many acknowledgments. Avoid expressions such as “One of us (S.B.A.) would like to thank...” Instead, write “F. A. Author thanks...” Sponsor and financial support acknowledgments are placed in the unnumbered footnote on the first page.

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