‘River Water DO Mechanics & New DO-Sag Equation’

Ravindra Sharma, M.K. Verma, Meena Murmu

Nomenclature-
Following terms have been used to define DO-Sag Equation by alternative algorithm
a) DO_{SH} – Reference DO Model made by Streeter Phelps equation
b) DO_{new} – DO Model made using new equation (quadratic polynomial)
c) De-oxidation Coefficient – k_1
d) Re-oxygenation Coefficient – k_2
e) A- Re-aeration Coefficient in DO_{new} Model

Abstract: DO modeling by Streeter Phelps equation [1] is most popular method to determine the water quality of a River. To compute DO by Streeter Phelps equation River coefficients k_1 and k_2 (de-oxidation and re-oxygenation) are required. Determination of these coefficients is tedious because it requires field observation of river velocity and depth over a long period of time at river site. To avoid maximum field work in calculating DO of River water DO Modeling approach is developed by combining Lab analysis of water samples DO with field data, e.g. river velocity and depth. Streeter Phelps (1925) developed the 1st important water quality model describing the BOD-DO relationship in a stream. In their pioneering work the simplest system was considered, in which biodegradable waste is discharges to the stream and consumes oxygen, atmospheric re-aeration being the only source of oxygen. The model is based on complicated solution of differential equation for above process. The equation is derived assuming River coefficients k_1 and k_2 as exponential function of time variation. The authors have simplified the derivation of DO-Sag equation [4] by replacing the exponential function with a quadratic polynomial. To explain the use of new equation, authors have defined the geometry of DO curve known as ‘River Water DO Mechanics’. Also in this paper, new equation is applied to make ‘Shivnath River water DO Model’ with data taken by the author as part of his Ph. D. research work. The results justify the acceptance of new modified equation for River Water Quality Assessment.

Key words- DO Modeling, River coefficients of new DO-Sag equation: De-oxidation (k_1), Re-oxygenation (k_2) & Super Aeration (A), River Water DO Mechanics

I. INTRODUCTION

A. DO mechanics defined from Streeter Phelps equation (1925):

Basics of river water DO mechanics [4] can be defined from DO-Sag curve introduced by Streeter and Phelps in 1925. Appendix- Fig. (1) shows the geometry of DO-Sag curve with Cartesian coordinates marked as (Day, DO : for x, y). The curve shows initial DO-point ’B’ (0, 6), DO deficit point ’C’ (2, 2.4) and DO after re-aeration point ’D’ (5, 5.5), over particular distance of river reach about 100 km. Let DO-Sag curve coordinates be expressed as:

X- Axis represents No. of days or distances
Y- Axis represents DO values in mg/L

Coordinates of points on ideal DO-Sag curve are expressed as follows:

i) Point O- Origin (0, 0) → (d=0.DO=0)
ii) Point A- Coordinates (0, 10) → fully DO saturated river water in contact with atmosphere
iii) Point B- Coordinates (0, 6) → DO of river surface water before waste water stream joins
iv) Point C- Coordinates (2, 2.4) → DO minimum level due to de-oxygenation process and become dangerous for aquatic life. The process takes 2 days for completion. Thereafter re-oxygenation begins.
v) Point D- Coordinates (5, 5.5) → Re-aeration continues and DO becomes equal to point C.
vi) Point E- Coordinates (10, 10) → Re-aeration continues to wards super saturation zone and DO finally after 10 days becomes 8.8 mg/L.

The DO-Sag curve obtained by above coordinates is known as Reference DO Model with ideal River data.

II. 2. Formulation of DO-Sag Equation by alternative algorithm-

Close examination of DO-Sag curve of Reference DO model with ideal River data (Fig.(1) shows that the path described by unit polluted mass rolling can be divided into three branches viz:

i) Stage -I: The decaying vector BC where Oxygen of rolling ball is reduced as it mix-up with the main river stream. This stage continues for 1st 3 days.

ii) Stage- II: Re-aeration vector CD is the stage when the rolling ball acquires Oxygen from Main River. This stage is known as re-oxygenation and it ends in 5 days. The path traced out by the rolling ball is resembles to a quadratic polynomial given by:

DO = k_1 d^2 + k_2 d + C  

Where k_1 - de-oxidation coefficient
k_2 - re-oxygenation coefficient

d - is time period (in days) between test of two water samples

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C- is constant for sample temp (20°C)

By experimentation it is found that \( k_1 = 2 \ v \) (river mean velocity-\( v \)) and \( k_2 \ = \ h \), depth of river bed. Therefore DO of River Water at minimum 3 locations are required to determine \( k_1 \), \( k_2 \) and \( C \) of Equation (1).

iii) Stage- III: The rolling ball move ahead from Re-aeration stage-II to DO saturation stage.

After 5 days, DO of rolling ball moves from re-aeration zone to DO saturation zone. It is given by the exponential curve Eqn.(2)

\[
C_{s} = DO_{3} + vA \quad ^{d} \quad \text{----------------(2)}
\]

Where \( v \) : is mean velocity of river water
\( d \) : is time period between two samples

\( DO_3 \) : DO value of water sample on 5th day

A : Aeration Coefficient of the River

DO Model resulted by Eqn. (1) and (2) when combined gives New DO-Sag equation

3. DORef Model - from DORef Model Fig. (1.1), DO coordinates are shown in Table (1),

<table>
<thead>
<tr>
<th>Days</th>
<th>DO (mg/L)</th>
<th>DOsaturated</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2.4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>5.4</td>
<td>10</td>
</tr>
</tbody>
</table>

Therefore equations of DO_{New} Model De-oxidation and Re-aeration coefficients \( k_1 \) and \( k_2 \), can be expressed from eqn. (1) as:

\[
DO_n = k_1 \ d^2 + k_2 \ d + DOs \quad ^{-----------(3)}
\]

Where, \( n \) – number of days 1 to 5

Substituting \( d \) as 0, 2, 5 we have following equations (3)

\[
k_1(0)^2 + k_2(0) + DOs = 6 \quad \text{Therefore initial Coordinate (0, 6) i.e. (d, DO)}
\]

\[
k_1(2)^2 + k_2(2) + 6 = 2.4 \quad \text{-----------(4)}
\]

\[
k_1(5)^2 + k_2(5) + 6 = 6 \quad \text{-----------(4)}
\]

Solving set of equations (5) we have \( k_1 = 0.6 \), \( k_2 = -3 \)

Since \( k_1 = 2 \ v \), and \( k_2 = -h \), the river parameter we have \( v = 0.3 \ m/s \) and \( h = 3 \ m \)

Note- (-)ve sign of \( h \) show that the river depth is measured from the River Surface to its bed.

Hence DO Model equation of Ideal River is:

\[
DO_n = 0.6 \ d^2 - 3 \ d + 6 \quad ^{-----------------(-5)}
\]

Therefore DO\(_3\), DO\(_2\) etc. can be computed from eqn. (5). It takes 5 days to complete de-oxidation and re-aeration process and DO mass balance is achieved. Thereafter DO is further computed from DO\(_3\) to super saturation level given by equation-

\[
DO = DO_3 + k_1A^d \quad ^{-----------(6)}
\]

On substitution \( A = 1.36 \)

Where \( A \) = Super aeration coefficient

Therefore DO\(_n\), DO\(_2\) etc can be computed from eqn. (6)

The combined effect Eqn. (5) & (6) gives DO_{New Model}

III. DONew Model by new Equation

The main advantage of DO_{New} Model equation is that there is no need to make field observation at river site to determine mean velocity of river flow and its depth. Determination of River coefficients \( k_1 \), \( k_2 \) and \( A \) of DO_{New} Model equation takes care of velocity and depth components.

DORef model is made using Streeter Phelps eqn. Fig.(1) at initial point 'B', Deficit Point 'C' and Re-aeration point 'D' have been used for determine Coefficients \( k_1 \) and \( k_2 \) as explained below to determine of hydraulic parameters (v-velocity & depth-h) at site, is not required. The computation is explained below:

5 days DO coordinates are tabulated in Table (2) from Fig.(1) DORef Model

Table (2)- DO coordinate of DORef Model Fig. (1.1)

<table>
<thead>
<tr>
<th>Days</th>
<th>DO (mg/L)</th>
<th>DOsaturated</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.00</td>
<td>8.8</td>
</tr>
<tr>
<td>1</td>
<td>3.00</td>
<td>8.8</td>
</tr>
<tr>
<td>2</td>
<td>2.20</td>
<td>8.8</td>
</tr>
<tr>
<td>3</td>
<td>3.40</td>
<td>8.8</td>
</tr>
<tr>
<td>4</td>
<td>3.90</td>
<td>8.8</td>
</tr>
<tr>
<td>5</td>
<td>5.40</td>
<td>8.8</td>
</tr>
</tbody>
</table>

Excel Worksheet- computing template for coefficients \( k_1 \), \( k_2 \) and \( A \) have been prepared for eqn. (1) stated above. It resulted in set of 5- linear equations in the form stated below:

\[
DO_n = k_1 \ d^2 + k_2 \ d + DOs \quad ^{-----------(7)}
\]

5. Comparison of Results- DORef Model & DO_{New Model}

a) DORef Model Coefficients- \( k_1 = 0.12, k_2 = 0.41 \)

b) DO_{New Model} Model Coefficients- \( k_1 = 0.0.62, k_2 = 3.12 \ A = 1.138 \)

DO_{New Model} Equation- \( DO_n = 0.62 \ d^2 - 3.12 \ d + 6 \)

DO_{New Model} in Super aeration zone – equation: \( DO_A = DO_3 + 0.62 \ x \ 1.38^d \)

Additional Features of DO-Sag new equation-

i) River coefficients \( k_1 \), \( k_2 \) determination in case of DORef model by Streeter Phelps equation and new DO_{New} equation are represented by bar graph in Fig.(3).

ii) In case of DO_{New} Model coefficients \( k_1 \), \( k_2 \) and \( A \ (de-oxidation/ Re-oxygenation, Super- aeration -A) \) are determined from lab tests. Solution of quadratic equation using the results gives coefficient \( k_1 \) and \( k_2 \). It is found that \( k_2 = 2 \ v \) (mean velocity of River) and \( k_2 = \) mean depth (depth-h). Hence there is no need to determine them at river site.
iii) Error comparison of $DO_{\text{New}}$ Model with $DO_{\text{Ref}}$

DO values of $DO_{\text{Ref}}$ Model and $DO_{\text{New}}$ Model are compared by determination of mean square deviation from true values tabulated in Table (3).

**Table (3)- Mean square error of New DO Eqn. Vs Reference DO Model**

<table>
<thead>
<tr>
<th>Days</th>
<th>$DO_{\text{New}}$</th>
<th>$DO_{\text{Ref}}$</th>
<th>$Error^2$</th>
<th>$DO$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.00</td>
<td>6.00</td>
<td>0.00</td>
<td>8.80</td>
</tr>
<tr>
<td>1</td>
<td>3.50</td>
<td>3.20</td>
<td>0.09</td>
<td>8.80</td>
</tr>
<tr>
<td>2</td>
<td>2.24</td>
<td>2.40</td>
<td>0.03</td>
<td>8.80</td>
</tr>
<tr>
<td>3</td>
<td>2.22</td>
<td>3.00</td>
<td>0.61</td>
<td>8.80</td>
</tr>
<tr>
<td>4</td>
<td>3.44</td>
<td>4.20</td>
<td>0.58</td>
<td>8.80</td>
</tr>
<tr>
<td>5</td>
<td>5.90</td>
<td>5.50</td>
<td>0.16</td>
<td>8.80</td>
</tr>
<tr>
<td>6</td>
<td>6.74</td>
<td>6.10</td>
<td>0.41</td>
<td>8.80</td>
</tr>
<tr>
<td>7</td>
<td>7.05</td>
<td>7.00</td>
<td>0.00</td>
<td>8.80</td>
</tr>
<tr>
<td>8</td>
<td>7.46</td>
<td>7.25</td>
<td>0.05</td>
<td>8.80</td>
</tr>
<tr>
<td>9</td>
<td>8.03</td>
<td>8.00</td>
<td>0.00</td>
<td>8.80</td>
</tr>
<tr>
<td>10</td>
<td>8.80</td>
<td>8.20</td>
<td>0.36</td>
<td>8.80</td>
</tr>
</tbody>
</table>

$m.s.e. = 0.32$

It is apparent from above analysis that mean square error of $DO_{\text{New}}$ with respect to $DO_{\text{Ref}}$ Model is 0.32. It implies that:

Error in $DO_{\text{New}}$ Model (made by quadratic equation) may be $\pm 0.32$ mg/L. Thus deviation from true values is very small.

Therefore $DO_{\text{Ref}}$ Model and $DO_{\text{New}}$ Model are in close resemblance obvious from Fig.(2). Hence new equation is acceptable for study of River DO Model for water quality assessment.

IV. 6 Season-wise comparison of Results-

$DO_{\text{Ref}}$ Model Vs Shivnath River $DO_{\text{New}}$ Model

Following DO model equation season have been resulted while processing Shivnath River Data observed during Jan. to Dec. 2017. The processing was done in Excel Worksheets.

i) DO equation for winter season:

$$DO_a = 0.36 \, d^2 - 2.25 \, d + 6.76$$

DO variation in super-aeration zone:

$$DO = DO_a + k_1 \cdot (1.64)^d$$

ii) DO equation for Summer season:

$$DO_a = 0.22 \, d^2 - 1.86 \, d + 7.35$$

DO variation in super-aeration zone:

$$DO = DO_a + k_1 \cdot (1.89)^d$$

iii) DO equation for Rainy season:

$$DO_a = 0.31 \, d^2 - 2.04 \, d + 7.67$$

DO variation in super-aeration zone:

$$DO = DO_a + k_1 \cdot (1.63)^d$$

iv) DO equation for Autumn season:

$$DO_a = 0.42 \, d^2 - 2.69 \, d + 7.82$$

DO variation in super-aeration zone:

$$DO = DO_a + k_1 \cdot (1.52)^d$$

DO values computed by new equation in Excel Worksheet. The extract of results of worksheet are shown in Table (4). Season-wise curves are compared with $DO_{\text{Ref}}$ model in Fig. (4). Results show that excepting summer, the DO variation in all seasons resemble to $DO_{\text{Ref}}$ model curve.

Table-(4): Summary of $DO_{\text{New}}$ Model of all season Vs $DO_{\text{Ref}}$ Model

<table>
<thead>
<tr>
<th>Days</th>
<th>$DO_{\text{Ref}}$</th>
<th>$DO_{\text{Winter}}$</th>
<th>$DO_{\text{Summer}}$</th>
<th>$DO_{\text{Rainy}}$</th>
<th>$DO_{\text{Autumn}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.00</td>
<td>6.76</td>
<td>7.35</td>
<td>7.67</td>
<td>7.83</td>
</tr>
<tr>
<td>1</td>
<td>3.20</td>
<td>4.87</td>
<td>5.71</td>
<td>5.94</td>
<td>6.10</td>
</tr>
<tr>
<td>2</td>
<td>2.40</td>
<td>3.70</td>
<td>4.51</td>
<td>4.83</td>
<td>4.99</td>
</tr>
<tr>
<td>3</td>
<td>3.00</td>
<td>3.25</td>
<td>3.75</td>
<td>4.34</td>
<td>4.50</td>
</tr>
<tr>
<td>4</td>
<td>4.20</td>
<td>3.52</td>
<td>3.43</td>
<td>4.47</td>
<td>4.63</td>
</tr>
<tr>
<td>5</td>
<td>5.50</td>
<td>4.51</td>
<td>3.55</td>
<td>5.22</td>
<td>5.38</td>
</tr>
<tr>
<td>6</td>
<td>6.10</td>
<td>5.10</td>
<td>3.96</td>
<td>5.73</td>
<td>5.86</td>
</tr>
<tr>
<td>7</td>
<td>7.00</td>
<td>5.48</td>
<td>4.33</td>
<td>6.04</td>
<td>6.19</td>
</tr>
<tr>
<td>8</td>
<td>7.25</td>
<td>6.10</td>
<td>5.03</td>
<td>6.57</td>
<td>6.70</td>
</tr>
<tr>
<td>9</td>
<td>8.00</td>
<td>7.12</td>
<td>6.33</td>
<td>7.41</td>
<td>7.47</td>
</tr>
<tr>
<td>10</td>
<td>8.80</td>
<td>8.80</td>
<td>8.80</td>
<td>8.80</td>
<td>8.84</td>
</tr>
</tbody>
</table>

V. 7. Results & Discussion-

i) Results of Shivnath River coefficients $k_1$, $k_2$ and A are analyzed below to produce the bar chart. Season-wise variation of River Coefficients $k_1$, $k_2$ and A of New DO equation of Shivnath River have been tabulated in Table (5).

Table No.(5)- Seasonal variation of Coefficients ($k_1$, $k_2$, A) of $DO_{\text{New}}$ Model

<table>
<thead>
<tr>
<th>No.</th>
<th>Season</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Winter</td>
<td>0.36</td>
<td>2.25</td>
<td>1.640</td>
</tr>
<tr>
<td>2</td>
<td>Summer</td>
<td>0.22</td>
<td>1.86</td>
<td>1.890</td>
</tr>
<tr>
<td>3</td>
<td>Rainy</td>
<td>0.31</td>
<td>2.04</td>
<td>1.630</td>
</tr>
<tr>
<td>4</td>
<td>Autumn</td>
<td>0.42</td>
<td>2.69</td>
<td>1.520</td>
</tr>
</tbody>
</table>

Coefficients of Shivnath River for four seasons (Jan. to Dec. 2017). Bar Graph is given in Fig. (5). It predicts DO values for Winter, Summer, Rainy and Autumn season of year 2017.

ii) Coefficients $k_2$ needs to be modified to determine sector area covered in each season. Thus data is arranged season-wise as shown in Table (6) to depict the seasonal variation of DO model. The Table (6) also gives processed data for making the Pie chart Fig.(5).
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Table No.(6)- Seasonal variation of DO_{New} Model Coefficients (k_1, k_2, A)

<table>
<thead>
<tr>
<th>No.</th>
<th>Season</th>
<th>% Area</th>
<th>K1</th>
<th>K2</th>
<th>A</th>
<th>K_m</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Winter</td>
<td>27.21</td>
<td>0.36</td>
<td>2.25</td>
<td>1.640</td>
<td>1.660</td>
<td>2.66</td>
</tr>
<tr>
<td>2</td>
<td>Summer</td>
<td>15.63</td>
<td>0.22</td>
<td>1.66</td>
<td>1.590</td>
<td>2.33</td>
<td>0.257</td>
</tr>
<tr>
<td>3</td>
<td>Rainy</td>
<td>21.56</td>
<td>0.31</td>
<td>2.04</td>
<td>1.630</td>
<td>2.45</td>
<td>0.379</td>
</tr>
<tr>
<td>4</td>
<td>Autumn</td>
<td>36.24</td>
<td>0.42</td>
<td>2.69</td>
<td>1.520</td>
<td>3.07</td>
<td>0.845</td>
</tr>
</tbody>
</table>

3. System Performance/ Efficiency Indicator - It is an accurate method to compute central angles of each season wise sector area from Coefficients k_1 & k_m determined for New DO-Sag equation.

Test criteria- As there are 4 seasons in which central angle of each sector lies therefore:

I) Central angle of each area vector Δ is nearly 90°.

II) Let each pair of coefficients k_1 and k_m be considered season wise.

III) Estimated angle value (Acute & Obtuse) and in each set there are 4 options.

IV) Compare the central angle (90°) with above options and select the value depending upon k_m > k_1, in each sector as >45° & <90°, or >90° & <135°.

V) The computation of efficiency index is illustrated in Excel worksheet with grouping of coefficients in Table (6) to process data and determine DO model efficiency index shown in Fig.(7).

Table No.(7)- Seasonal Efficiency Index of Shivnath River DO model

<table>
<thead>
<tr>
<th>System Performance/ Efficiency Test of DO model</th>
<th>Shivnath River coefficients k_1 &amp; k_m seasonwise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Season</td>
<td>k_1</td>
</tr>
<tr>
<td>Winter</td>
<td>0.36</td>
</tr>
<tr>
<td>Summer</td>
<td>0.22</td>
</tr>
<tr>
<td>Rainy</td>
<td>0.31</td>
</tr>
<tr>
<td>Autumn</td>
<td>0.42</td>
</tr>
<tr>
<td>Deficit</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>360</td>
</tr>
</tbody>
</table>

It ranges from 10.13% in Summer to 38.54% in Autumn. The designed model efficiency is 95%.

VI. CONCLUSION-

Streeter-Phelps (1925) developed DO Model to determine River coefficients k_1 & k_2 by using lab test results of water samples for DO presence and DO deficit and hydraulic parameters like river velocity, depth etc. Computation of k_1 & k_2 are performed in Excel worksheet. The computation results are presented in Table No. (4), (5), (6) and (7) for months Jan. To Dec. 2017. Finally variation of k_1 & k_2 for Shivnath is presented by means of following Statistical Analysis tools (see Appendix- Figures & Graphs):

A) Bar graph- to represent variation of k_1 and k_2 for months Jan. To Dec. 2017.

B) Pie chart and

C) Shivnath DO model season-wise study by Statistical efficiency index system

REFERENCES-


Appendix- Figs. & Graphs

Fig.(1)- DO Mechanics to derive new DO-Sag equation as quadratic polynomial

Fig.(2)- Reference DO Model Vs New Equation DO

Fig.(3)- Variation of Coefficients
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