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# Application of Generalized Differential Transform Method to Special Cases of Riccati Differential Equation of Fractional Order

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***Abstract:*** ***In this paper, we acquire the inexact solutions of Special cases of Riccati Differential equation of Fractional order using Generalized Differential Transform Method (GDTM). The fractional derivatives are described in the Caputo sense. Accuracy and competence of the proposed method is verified through numerical solution of some special cases of Riccati Differential equation of fractional order. The obtained results reveal that the performance of the proposed method is specific and predictable.***

***Keywords:*** ***Caputo fractional derivative, Generalized Differential Transform Method, special case of Riccati equation of fractional order.***

## I. INTRODUCTION AND PRELIMINARIES

Fractional Calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to ½ order. The fractional calculus deals with derivatives and integrals in an random order (real of complex). Fractional ordinary differential equations are crucial in many fields, such as many physical phenomena in acoustics, damping laws and electro analytical chemistry, neuron modeling, diffusion processing and material sciences. The Riccati equation was the one of the simplest nonlinear first order differential equation. The Riccati differential equation (RDE) has huge diversity of applications in applied sciences, engineering such as rheology, damping law, diffusion processes and optimal control theory problems.

The general form of some special case of Riccati differential equation of fractional order is

$$D^{\beta}p(x) = f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)$$

where $0 < \beta \leq 1, x > 0$ and $f(x) \neq 0$ with the initial condition p (0) $=p_0$ and the variable coefficients $a(x), b(x)$ and $c(x)$ are continuous functions bounded by L.

(1.1)

The fractional order Riccati differential equation is converted into the classical Riccati differential equation. The application of fractional differential and integral operators in mathematical models has attained enormous significance in current years.

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A number of forms of fractional differential equations have been projected in regular models, and there has been significant interest in developing numerical schemes for their solutions. There are various methods that deal with these types of equations.

DTM is used to find the solution of different kinds of Riccati differential equation such solution of first order , second order and system of Riccati equations [1], Differential Transform Method for solving Linear and nonlinear systems of Ordinary Differential equations [2], Applications to differential Transform method for solving system of Differential equations [3], Solution of Non-Linear Differential Equations by using Differential Transform Method [4], Solution of Riccati Equation with co-efficient by differential Transform method Solution [5], Differential transform method for Quadratic Riccati Differential Equation [6], Linear models of dissipation whose Q is almost frequency independent –II [7], Application of generalized differential transform method to multi-order fractional differential equations [8],Numerical Analysis of Riccati equation using Differential Transform Method, He Laplace Method and Adomain Decomposition Method [9],Generalized Taylor series Method for solving Nonlinear fractional differential equations with Modified Riemann-Liouville Derivatives [10], have been introduced to provide numeric approximations.

***The following definitions, theorems and some basic properties are used in this paper.***

**Definition 1.1:** The Riemann–Liouville fractional derivative operator of order $\beta$ of a function $f(x)$is defined by

$$D^{\beta}f(x) = \frac{1}{\Gamma(m-\beta)}\frac{d^n}{dx^n}\int_{x_0}^{x}\frac{f(\xi)}{(x-\xi)^{\beta-m+1}}d\xi,\ \alpha > 0, x > 0$$

and $m-1<\beta\leq m$, $m\in N$.

**Definition 1.2:** The Caputo fractional derivative operator of order $\beta$ isgiven by

$$D_x^{\beta}f(x) = \frac{1}{\Gamma(m-\beta)}\int_{x_0}^{x}\frac{f^{(m)}(\xi)}{(x-\xi)^{\beta-m+1}}d\xi,\ \alpha > 0, x > 0 \text{ and } m-1<\beta\leq m, m\in N. \quad (1.2)$$

**Definition 1.3:** The generalized Differential transform of the $k^{th}$ derivative of function f(x) in one variable is given by

$$F_{\beta}(k) = \frac{1}{\Gamma(\beta k+1)}[(D_{x_0}^{\beta})^k f(x)]_{x=x_0} \quad (1.3)$$

where, $0 < \beta \leq 1, (D_{x_0}^{\beta})^k = D_{x_0}^{\beta}D_{x_0}^{\beta}D_{x_0}^{\beta}\ldots D_{x_0}^{\beta}(k-\text{times})$ and $F_{\beta}(k)$ is the transformed function.

**Definition 1.4:** The Inverse generalized transform of $F_{\beta}(k)$ is defined by

$$f(x) = \sum_{k=0}^{\infty} F_{\beta}(k)(x-x_0)^{\beta k} \quad (1.4)$$

**Definition 1.5**: The Riemann –Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x)$is given by

$$I^{\alpha} f(x) = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \alpha > 0,$$

$$I^0 f(x) = f(x) \qquad (1.5)$$

The operator $I^{\alpha}$ satisfies the following properties, for $\alpha, \beta > 0$ and $m \geq -1$

i) $I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x)$,
ii) $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x)$,
iii) $I^{\alpha} x^m = \frac{\Gamma(m+1)x^{m+n}}{\Gamma(m+\alpha+1)}$,
iv) $D^{\alpha} x^m = \frac{\Gamma(m+1)x^{m-\alpha}}{\Gamma(m-\alpha+1)}$, where m>0 , m$\in R$

Fundamental properties of the Generalized Differential Transform

Let $F_{\beta}(k), G_{\beta}(k)$and $H_{\beta}(k)$ be the generalized differential transforms of the functions f(x), g(x) and h(x) respectively, followed by

1. If $f(x) = g(x) \pm h(x)$, then$F_{\beta}(k) = G_{\beta}(k) \pm H_{\beta}(k)$
2. If $f(x) = ag(x)$, then $F_{\beta}(k) = aG_{\beta}(k)$, where a is a constant
3. If $f(x) = g(x)h(x)$, then $F_{\beta}(k) = \sum_{r=0}^{k} G_{\beta}(r) H_{\beta}(k-r)$
4. If $f(x) = D_{x_0}^{\beta} g(x), then$ $F_{\beta}(k) = \frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)} G_{\beta}(k+1)$
5. If $f(x) = (x-x_0)^{\eta}, \eta = n\beta, n \in Z,$ then $F_{\beta}(k) = \delta\left(k - \frac{\eta}{\beta}\right)$,
   where $\delta(k) = \begin{cases} 1, \text{if } k = 0 \\ 0, \text{if } k \neq 0 \end{cases}$

***Theorem 1.1:(Generalized Taylor's Series)***

Assume$(D_a^{\beta})^k f(x) \in C[a,b]$,
$k = 0,1,2,\ldots,n+1$, where $0 < \alpha \leq 1$, then

$$f(x) = \sum_{r=0}^{n} \frac{(x-a)^{r\beta}}{\Gamma(r\beta+1)} \left((D_a^{\beta})^r f\right)(a) + \frac{((D_a^{\beta})^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\beta}$$

with $a \leq \xi \leq x, \forall\, x \in [a,b]$.

## II. EXISTENCE AND UNIQUENESS

In this section, we prove the existence and uniqueness of the solution of FRDE (1.1) where the fractional derivative is defined in Caputo's sense. For $\alpha = 1$, the fractional –order Riccati differential equation converts into the classical Riccati differential equation.

***Definition 2.1:*** Let $K = [0,\eta]$, $\eta < \infty$ and C(K) be the class of all continuous functions defined on K with sup-norm $\|p\| = \sup_{x \in K} |p(x)|$

***Remark 2.1:*** Assume that the solution p(x) of fractional –order Riccati differential equation (1.1) belongs to the space S ={p ∈ R: $|p| \leq c$, c is any constant}, in order to study the existence and uniqueness of the initial value problem

**Definition 2.2:** The Space of integral functions $L_1[0,\eta]$ in the interval $[0,\eta]$ is defined as

$$L_1[0,\eta] = \left\{u(t): \int_0^{\eta} |u(t)|dt < \infty\right\}.$$

**Theorem 2.1: Existence Theorem**

Suppose h(x, p(x)) is continuous function on some region K=[0,$\eta$]×R→ $R$ and let $Q_m = \{\, p \in C\,(K) : |p(x)| \leq p_m \,\}$. Since h is continuous on a closed and bounded in K. i.e., $\exists\, L > 0 \ni |h(x,p(x)| \leq L, \ \forall (x,p(x)) \in K$ then the IVP (1.1) has at least one solution p = p(x) in the interval [0,$\delta$] for suitable $\delta \leq 1$.

Proof: From the properties of fractional calculus, IVP (1.1) can be written in the subsequent form

$$I^{1-\beta} \frac{dp(x)}{dx} = f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x) \qquad (2.1)$$

Operating with $I^{\beta}$ , we get the integral equation

$$p(x) = p_0 + I^{\beta}(f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)) \qquad (2.2)$$

when x = 0

$$I^{\beta}\left(f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)\right) = 0$$

then $p(x) = p_0 + (\frac{x^{\beta}}{\Gamma(\beta+1)}(f_0 p_0^2 + a_0 p_0 + a_0 b_0 x_0^{\beta} + b_0^2 f_0) + I^{\beta+1}[2f(x)p(x)p'(x) + p^2(x)f^1(x) + a^1(x)p(x) + a(x)p^1(x) + a^1(x)b(x)x^{\beta} + a(x)b^1(x)x^{\beta} + a(x)b(x)\beta x^{\beta-1} + 2b(x)b^1(x)f(x) + b^2(x)f^1(x)]$

which implies that $p(x) \in C(K)$

From (2.2), we can write

$$\frac{dp}{dx} = \frac{d}{dx} I^{\beta}[f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)]$$

$$I^{1-\beta}\frac{dp}{dx} = I^{1-\beta}\frac{d}{dx} I^{\beta}[f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)]$$

$$D_C^{\beta} p(x) = \frac{d}{dx} I[f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)]$$

$$D_C^{\beta} p(x) = f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)$$

and $p(0) = p_0$

Then the integral equation (2.2) is equivalent to the initial value problem (1.1)

Let $h(x, p(x)) = f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)$

Since $h(x,p): [0,\eta] \times R \to R$ and $Q_m = \{p \in C(K) :$ :p(x)≤pm

for $p \in Q_m$, we have $\left|\frac{\partial h}{\partial p}\right| = |2f(x)p(x) + a(x)|$

$\left|\frac{\partial h}{\partial p}\right| \leq |2Lp_m + L| = L_1$

There fore $h(x,p(x))$ satisfies Lipchitz condition with respect to the second argument with Lipchitz constant $L_1 = 2Lp_m + L$

Hence there exists at least one solution to the IVP (1.1)

***Theorem 2.2: Uniqueness Theorem***

For every initial value problem

$$D^{\beta} p(x) = f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^{\beta} + b^2(x)f(x)$$

where $0 < \beta \leq 1, x > 0$ and $f(x) \neq 0$ with the initial condition p (0) = $p_0$ and the variable coefficients $a(x), b(x)$ and $c(x)$ are continuous functions bounded by L, there is an unique solution

$p \in C(K), p^1 \in \{p \in L_1[0,\eta], \|p\| = |p(x)|_{L_1}\}$

Proof: Let the operator F

$Fp(x) = p_0 + I^\beta\left(f(x)p^2(x) + a(x)p(x) + a(x)b(x)x^\beta + b2xfx\right.$ (2.3)

Or equivalently

$Fp(x) = p_0 + \int_0^x \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} f(u,p(u)du$

Let $p(x) \in Q_m, x_1, x_2 \in K$ such that $0<x_1 < x_2$ then

$$|Fp(x_2) - Fp(x_1)| = \left|\frac{1}{\Gamma(\beta)}\int_0^{x_2}(x_2-u)^{\beta-1} f(u,p(u))du - \frac{1}{\Gamma(\beta)}\int_0^{x_1}(x_1-u)^{\beta-1} f(u,p(u))du\right|$$

$=\left|\frac{1}{\Gamma(\beta)}\int_0^{x_1}(x_2-u)^{\beta-1} f(u,p(u))du - \frac{1}{\Gamma(\beta)}\int_0^{x_1}(x_1-\right.$

$u\beta-1f(u,pu)du+1\Gamma(\beta)x2x1x2-u\beta-1f(u,pu)du$

$$\le \frac{L_1}{\Gamma(\beta)}\int_0^{x_1}(x_1-u)^{\beta-1} - (x_2-u)^{\beta-1} + \frac{L_1}{\Gamma(\beta)}(x_2-u)^{\beta-1}du$$

$= \frac{L_1}{\Gamma(\beta+1)}[2(x_2-x_1)^\beta + x_1^\beta - x_2^\beta]$

$\le \frac{2L_1}{\Gamma(\beta+1)}[2(x_2-x_1)^\beta$ where $L_1 = sup_{x\epsilon K}|f(x,p(x)|$

which implies that F:C(K)→ C(K)

Suppose p, q∈ $Q_m$

$$|Fp(x) - Fq(x)| = \left|\int_0^x \frac{(x-u)^{\beta-1}}{\Gamma(\beta)}[f(u,p(u) - f(u,g(u)]du\right|$$

$$\le L\int_0^x \frac{(x-u)^{\beta-1}}{\Gamma(\beta)}|p(u) - g(u)|$$

$$\|Fp(x) - Fq(x)\| \le Lsup_{x\epsilon K}|p(u) - g(u|\int_0^x \frac{(x-u)^{\beta-1}}{\Gamma(\beta)}du$$

$$\le L\|p(x) - q(x)\|\frac{B^\beta}{\Gamma(\beta+1)}$$

If $\frac{B^\beta}{\Gamma(\beta+1)} < 1$ then the operator F defined in equation (2.3) is contradiction and the theorem is proved.

## III.SOLUTION OF SOME SPECIAL CASES OF RICCATI EQUATIONS OF FRACTIONAL ORDER

**Example 1**: $y_x^\alpha = ay^2 + bx^n$ where $0 < \alpha \le 1$ , n = 2 and y (0) =1 (3.1)

Apply GDTM to equation (3.1), we get

$$Y_\beta\left(k+\frac{\alpha}{\beta}\right) = \frac{\Gamma\left(\frac{k}{4}+1\right)}{\Gamma[\frac{1}{4}(k+1)+1]}[a\sum_{r=0}^{k} Y_\beta(r)Y_\beta(k-r) + b\delta(k-\frac{n}{\beta})] \quad (3.2)$$

$$\alpha = \frac{1}{4}, \beta = \frac{1}{4}$$

From (3.2),

For k = 0, $Y_{\frac{1}{4}}(1) = \frac{\Gamma(1)}{\Gamma(\frac{1}{4}+1)}a = \frac{4}{\Gamma(\frac{1}{4})}a = 1.1032a$

For k = 1, $Y_{\frac{1}{4}}(2) = \frac{\Gamma(\frac{1}{4}+1)}{\Gamma(\frac{1}{2}+1)}\left[2aY_{\frac{1}{4}}(0)Y_{\frac{1}{4}}(1)\right] = 2.2566a^2$

For k = 2, $Y_{\frac{1}{4}}(3) = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{3}{4}+1)}\left[2aY_{\frac{1}{4}}(0)Y_{\frac{1}{4}}(2) + aY_{\frac{1}{4}}(1)Y_{\frac{1}{4}}(1)\right] = 6.9252a^3$

For k = 3, $Y_{\frac{1}{4}}(4) = \frac{\Gamma(\frac{3}{4}+1)}{\Gamma(1+1)}\left[2aY_{\frac{1}{4}}(0)Y_{\frac{1}{4}}(3) + 2aY_{\frac{1}{4}}(1)Y_{\frac{1}{4}}(2)\right] = 13.8076a^4$ and so on

The solution of (1) is $y(x) = \sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k)x^{\frac{k}{4}}$

$= Y_{\frac{1}{4}}(0)x^{\frac{0}{4}} + Y_{\frac{1}{4}}(1)x^{\frac{1}{4}} + Y_{\frac{1}{4}}(2)x^{\frac{1}{2}} + Y_{\frac{1}{4}}(3)x^{\frac{3}{4}} + Y_{\frac{1}{4}}(4)x + \ldots$

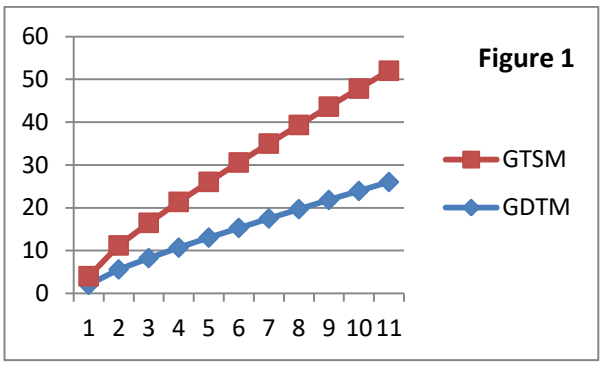
$y(x) = 1 + 1.1032a + 2.2566a^2x^{\frac{1}{2}} + 6.9252a^3x^{\frac{3}{4}} + 13.8076a^4x + \ldots$

which converges to the exact solution by GTSM is

$y(x) = 1 + 1.1032a + 2.2566a^2x^{\frac{1}{2}} + 6.9252a^3x^{\frac{3}{4}} + 13.8076a^4x + \ldots$

**Table 1**

| x | GDTM | GTSM | Error |
|---|---|---|---|
| 0 | 2 | 2 | 0.00000 |
| 0.1 | 5.593479 | 5.593479 | 0.00000 |
| 0.2 | 8.235129 | 8.235129 | 0.00000 |
| 0.3 | 10.68069 | 10.68069 | 0.00000 |
| 0.4 | 13.01818 | 13.01818 | 0.00000 |
| 0.5 | 15.28355 | 15.28355 | 0.00000 |
| 0.6 | 17.49591 | 17.49591 | 0.00000 |
| 0.7 | 19.66698 | 19.66698 | 0.00000 |
| 0.8 | 21.80456 | 21.80456 | 0.00000 |
| 0.9 | 23.9142 | 23.9142 | 0.00000 |
| 1 | 26 | 26 | 0.00000 |




***Example 2:*** $y_x^\alpha = f(x)\,y^2 + ay - abx^\alpha - b^2f(x)$, where $0 < \alpha \le 1$, $f(x) \ne 0$ and y (0) =1 (3.3)

Apply GDTM to equation (3.3), we have

$$Y_\beta\left(k+\frac{\alpha}{\beta}\right) = \frac{\Gamma(\alpha k+1)}{\Gamma[\alpha k+\beta+1]}[f(x)\sum_{r=0}^{k} Y_\beta(r)Y_\beta(k-r) + aY_\beta(k) - ab\delta\left(k-\frac{\alpha}{\beta}\right) - b^2f(x)] \quad (3.4)$$

**Case (i) when f(x) is a constant function i.e.,** $f(x) = 1, \alpha = \frac{3}{4}, \beta = \frac{3}{4}$

From (3.4),

$Y_{\frac{3}{4}}(k+1) = \frac{\Gamma\left(\frac{3}{4}k+1\right)}{\Gamma[\frac{3}{4}k+\frac{3}{4}+1]}[\sum_{r=0}^{k} Y_{\frac{3}{4}}(r)Y_{\frac{3}{4}}(k-r) + aY_{\frac{3}{4}}(k) - ab\delta(k-1) - b^2]$

At k = 0, $Y_{\frac{3}{4}}(1) = \frac{\Gamma(1)}{\Gamma(\frac{3}{4}+1)}[Y_{\frac{3}{4}}(0)Y_{\frac{3}{4}}(0) + aY_{\frac{3}{4}}(0) - b^2]$

$= 1.3636 + 1.3636a - 1.3636b^2$

At k =1, $Y_{\frac{3}{4}}(2) = \frac{\Gamma\left(\frac{3}{4}+1\right)}{\Gamma\left(\frac{3}{2}+1\right)}[2Y_{\frac{3}{4}}(0)Y_{\frac{3}{4}}(1) + aY_{\frac{3}{4}}(0) - ab\delta(k-1) - b^2]$

$= 1.5043+2.2564a + 0.7521a^2 - 1.5043b^2 - 0.7521ab^2 - 0.5516ab]$

At k =2 $Y_{\frac{3}{4}}(3) = \frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{9}{4}+1\right)}[2Y_{\frac{3}{4}}(0)Y_{\frac{3}{4}}(2) + Y_{\frac{3}{4}}(1)Y_{\frac{3}{4}}(1) + aY_{\frac{3}{4}}(2) - ab\delta(k-1) - b^2$

$Y_{\frac{3}{4}}(3) = 3.2992+5.0297a + 2.1630a^2 - 3.8744b^2 - 1.7304ab^2 - 0.6345ab - 0.3172a^2b - 0.4326a^2b^2 + 0.4326a^3$

and so on.

The solution of (3.3) is

$y(x) = \sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k)x^{\frac{3k}{4}}$

$= Y_{\frac{3}{4}}(0)x^{\frac{0}{4}} + Y_{\frac{3}{4}}(1)x^{\frac{3}{4}} + Y_{\frac{3}{4}}(2)x^{\frac{3}{2}} + Y_{\frac{3}{4}}(3)x^{\frac{9}{4}} + \ldots$

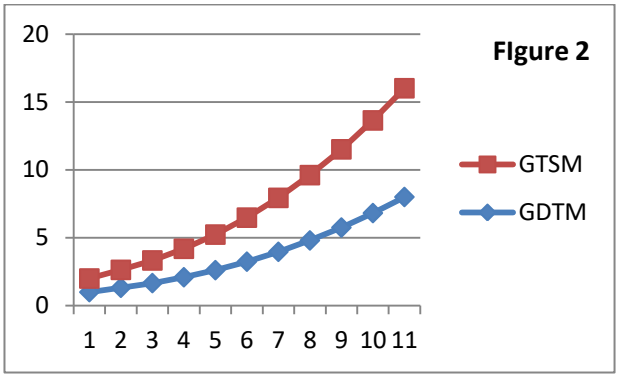
$= 1+ [1.3636 + 1.3636a - 1.3636b^2]x^{\frac{3}{4}} + [1.5043+2.2564a + 0.7521a^2 - 1.5043b^2 - 0.7521ab^2 - 0.5516ab]\, x^{\frac{3}{2}} + [3.2992+5.0297a + 2.1630a^2 - 3.8744b^2 - 1.7304ab^2 - 0.6345ab - 0.3172a^2b - 0.4326a^2b^2 + 0.4326a^3]\, x^{\frac{9}{4}} + \ldots$

which converges to the exact solution by GTSM is

$y(x) = 1+ [1.3636 + 1.3636a - 1.3636b^2]x^{\frac{3}{4}} + [1.5043+2.2564a + 0.7521a^2 - 1.5043b^2 - 0.7521ab^2 - 0.5516ab]\, x^{\frac{3}{2}} + [3.2992+5.0297a + 2.1630a^2 - 3.8744b^2 - 1.7304ab^2 - 0.6345ab - 0.3172a^2b - 0.4326a^2b^2 + 0.4326a^3]\, x^{\frac{9}{4}} + \ldots$

**Table 2**

| x | GDTM | GTSM | Error |
|---|---|---|---|
| 0 | 1 | 1 | 0.00000 |
| 0.1 | 1.318527 | 1.318527 | 0.00000 |
| 0.2 | 1.665564 | 1.665564 | 0.00000 |
| 0.3 | 2.095003 | 2.095003 | 0.00000 |
| 0.4 | 2.617892 | 2.617892 | 0.00000 |
| 0.5 | 3.240855 | 3.240855 | 0.00000 |
| 0.6 | 3.968822 | 3.968822 | 0.00000 |
| 0.7 | 4.805821 | 4.805821 | 0.00000 |
| 0.8 | 5.755319 | 5.755319 | 0.00000 |
| 0.9 | 6.820385 | 6.820385 | 0.00000 |
| 1 | 8.0038 | 8.0038 | 0.00000 |




***Case (ii) when $f(x)$ is exponential function***

**i.e, $f(x) = e^x$, $\alpha = \frac{1}{4}, \beta = \frac{1}{4}$**

From (3.4),

$Y_{\frac{1}{4}}(k+1) = \frac{\Gamma\left(\frac{1}{4}k+1\right)}{\Gamma[\frac{1}{4}k+\frac{1}{4}+1]}[\sum_{r=0}^{k}\sum_{s=0}^{r}\frac{1}{r!}Y_{\frac{1}{4}}(s)Y_{\frac{1}{4}}(k-r) + aY_{\frac{1}{4}}(k) - ab\delta(k-1) - b^2\frac{1}{k!}]$ (3.5)

From (3.5),

At k = 0, $Y_{\frac{1}{4}}(1) = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{4}+1\right)}[1 + a - b^2] = 1.1032 + 1.1032a - 1.1032b^2$

At k =1, $Y_{\frac{1}{4}}(2) = \frac{\Gamma\left(\frac{1}{4}+1\right)}{\Gamma\left(\frac{1}{2}+1\right)}[\sum_{r=0}^{1}\sum_{s=0}^{r}\frac{1}{r!}Y_{\frac{1}{4}}(s)Y_{\frac{1}{4}}(1-r) + aY_{\frac{1}{4}}(1) - ab\delta(1-1) - b^2\frac{1}{0!}]$

$Y_{\frac{1}{4}}(2) = 3.2791 + 3.3847a - 3.2791b^2 + 1.1282a^2 - 1.1282ab^2 - 1.0227ab$

and so on.

The solution of (3.3) is $y(x) = \sum_{k=0}^{\infty} Y_{\frac{1}{4}}(k)x^{\frac{3k}{4}}$

$= Y_{\frac{1}{4}}(0)x^{\frac{0}{4}} + Y_{\frac{1}{4}}(1)x^{\frac{1}{4}} + Y_{\frac{1}{4}}(2)x^{\frac{1}{2}} + \ldots$

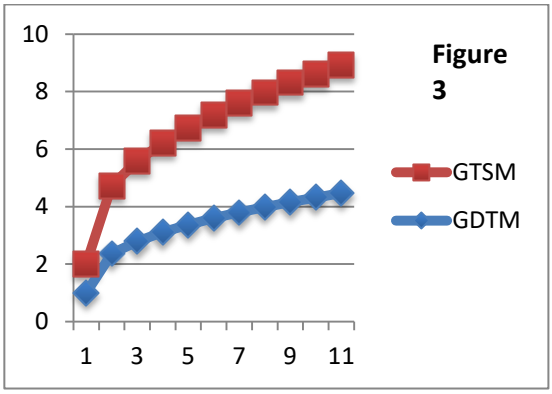
$= 1+ (1.1032 + 1.1032a - 1.1032b^2)x^{\frac{1}{4}} + (3.2791 + 3.3847a - 3.2791b^2 + 1.1282a^2 - 1.1282ab^2 - 1.0227ab)x^{\frac{1}{2}} + \cdots$

which converges to the exact solution by GTSM is

$y(x) = 1+(1.1032 + 1.1032a - 1.1032b^2)x^{\frac{1}{4}} + (3.2791 + 3.3847a - 3.2791b^2 + 1.1282a^2 - 1.1282ab^2 - 1.0227ab)x^{\frac{1}{2}} + \cdots$

**Table 3**

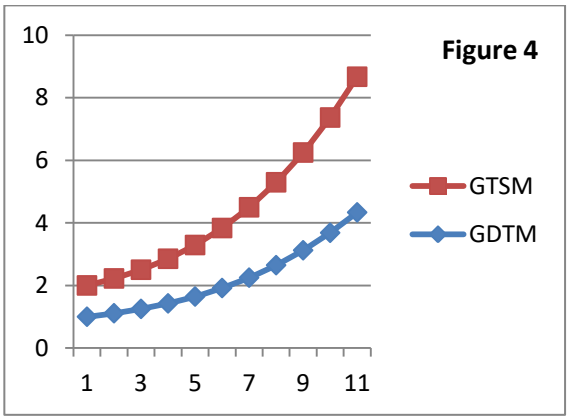
| x | GDTM | GTSM | Error |
|---|---|---|---|
| 0 | 1 | 1 | 0.00000 |
| 0.1 | 2.367305 | 2.367305 | 0.00000 |
| 0.2 | 2.794073 | 2.794073 | 0.00000 |
| 0.3 | 3.11018 | 3.11018 | 0.00000 |
| 0.4 | 3.371203 | 3.371203 | 0.00000 |
| 0.5 | 3.597863 | 3.597863 | 0.00000 |
| 0.6 | 3.800537 | 3.800537 | 0.00000 |
| 0.7 | 3.985278 | 3.985278 | 0.00000 |
| 0.8 | 4.155979 | 4.155979 | 0.00000 |
| 0.9 | 4.315311 | 4.315311 | 0.00000 |
| 1 | 4.4652 | 4.4652 | 0.00000 |




***Example 3:*** $y_x^{\alpha}$ = y $^2$ + x $f(x)$ y + $f(x)$,
where f(x) = $x^2$, $\alpha = 1, \beta = 1$ and $y(0) = 1$ (3.6)
Apply GDTM to equation (3.6),
$Y_1(k+1) = \frac{\Gamma(\alpha k+1)}{\Gamma[\alpha k+\beta+1]}[\sum_{r=0}^{k} Y_1(r)Y_1(k-r) +$
*r=0ks=0rδr−3Y1k−r+δk−2* (3.7)
From (3.7),
For k = 0, $Y_1(1) = 1$
For k = 1, $Y_1(2) = 1$
For k = 2, $Y_1(3) = \frac{4}{3}$
and so on.
The solution of (3.6) is y(x) = $\sum_{k=0}^{\infty} Y_1(k)x^k$
= $Y_1(0)x^0$ +$Y_1(1)x^1$+$Y_1(2)x^2 + Y_1(3)x^3$ +…
= $1 + x + x^2 + \frac{4}{3}x^3 + \cdots$
which converges to the accurate solution by GTSM is y(x)
= $1+x + x^2 + \frac{4}{3}x^3 + \cdots$

**Table 4**

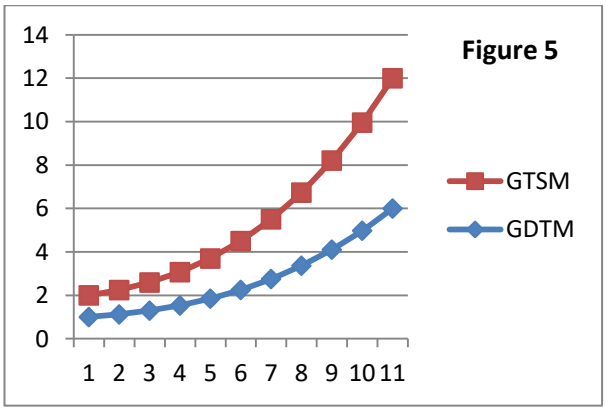
| x | GDTM | GTSM | Error |
|---|---|---|---|
| 0 | 1 | 1 | 0.00000 |
| 0.1 | 1.111333 | 1.111333 | 0.00000 |
| 0.2 | 1.250667 | 1.250667 | 0.00000 |
| 0.3 | 1.426 | 1.426 | 0.00000 |
| 0.4 | 1.645333 | 1.645333 | 0.00000 |
| 0.5 | 1.916667 | 1.916667 | 0.00000 |
| 0.6 | 2.248 | 2.248 | 0.00000 |
| 0.7 | 2.647333 | 2.647333 | 0.00000 |
| 0.8 | 3.122667 | 3.122667 | 0.00000 |
| 0.9 | 3.682 | 3.682 | 0.00000 |
| 1 | 4.333333 | 4.333333 | 0.00000 |




***Example 4:*** $y_x^{\alpha} = f(x)y^2$+an$x^{n-1} - a^2x^{2n}f(x)$,
$0 < \alpha \leq 1$, n = 2, f(x) = 1 and $\alpha = 1$ (3.8)
with initial condition y (0) =1
i.e., $y_x^{\alpha} = y^2$+2a$x^1 - a^2x^4$ (3.9)
Apply GDTM to equation example (3.9), we get
$Y_1(k+1) = \frac{\Gamma(\alpha k+1)}{\Gamma[\alpha k+\beta+1]}[\sum_{r=0}^{k} Y_1(r)Y_1(k-r) +$
*2aδk−1*−a2*δk−4* (3.10)
From (3.10),
For k = 0, $Y_1(1) = 1$
For k = 1, $Y_1(2)$ =(1+a)
For k = 2, $Y_1(3) = \frac{4+2a}{3}$ and so on.
The solution of (3.8) is y(x) = $\sum_{k=0}^{\infty} Y_1(k)x^k$
= $Y_1(0)x^0$ +$Y_1(1)x^1$+$Y_1(2)x^2 + Y_1(3)x^3$ +…
= $1 + x + (1+a)x^2 + (\frac{4+2a}{3})x^3 + \cdots$
which converges to the exact solution by GTSM is y(x) =
$1 + x + (1+a)x^2 + (\frac{4+2a}{3})x^3 + \cdots$

**Table 5**

| x | GDTM | GTSM | Error |
|---|---|---|---|
| 0 | 1 | 1 | 0.0000 |
| 0.1 | 1.122 | 1.122 | 0.0000 |
| 0.2 | 1.296 | 1.296 | 0.0000 |
| 0.3 | 1.534 | 1.534 | 0.0000 |
| 0.4 | 1.848 | 1.848 | 0.0000 |
| 0.5 | 2.25 | 2.25 | 0.0000 |
| 0.6 | 2.752 | 2.752 | 0.0000 |
| 0.7 | 3.366 | 3.366 | 0.0000 |
| 0.8 | 4.104 | 4.104 | 0.0000 |
| 0.9 | 4.978 | 4.978 | 0.0000 |
| 1 | 6 | 6 | 0.0000 |




## IV.CONCLUSION

In the present study, we included the existence and uniqueness theorem on Fractional order Riccati Differential Equation. Also the approximate solution of Special cases of Riccati Differential equation of Fractional order was obtained by using the Generalized Differential Transform Method (GDTM). The mathematical results gained by the projected process is in superior agreement with the accurate solution.

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