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Numerical Solution of Fuzzy Multiple Hybrid Single Retarded Delay Differential Equations

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Abstract: In this Paper, We are combining so many mathematical-cum-engineering topics such as Fuzzy systems, Delay systems and Hybrid Systems under one roof called Numerical Solutions. The fuzzy valued problem was solved numerically and that approximate solution was compared with that of exact solutions. The non fuzzy and fuzzy valued numerical solutions and their graphical illustrations are also provided for the better understanding of the multiple hybrid single retarded delay problems.

Index Terms: Retarded delay, Multiple Hybrid system, Fuzzy system, Numerical Solutions, Runge-Kutta method.

I. INTRODUCTION

Here a multiple hybrid system is modeled with the concept of retarded delay differential equation and we study it using fuzzy numbers. Nowadays hybrid systems play a vital role in communications and retard delay differential equation was considered to be unavoidable in modeling any biological models. Here these two separate mathematical concepts were combined under one roof called fuzzy. We call the system of differential equation as fuzzy multiple hybrid single retarded delay differential equations (FMHSRDDE). The complete system is modeled and explained with a theory following an example. Sambandham and Pederson [9] have gone through the numerical solution of fuzzy hybrid differential equation by using Runge-Kutta method. Al-Rawi and others [14] dealt with numerical method for solving Delay differential equations by RK-4. In [1] Alfredo Bellan and Marino Zennaro explained Numerical methods for delay differential equations. Jayakumar et.al, in [6] have treated fuzzy delay differential equation numerically. D. Prasantha Bharathi, T. Jayakumar and S. Vinoth used this type of Runge-Kutta method to solve various types of delay differential equations [10,11,12,&13]

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Section II reviews fuzzy hybrid retarded delay differential systems. In Section III, the Runge-Kutta method of order four is used for the problem of fuzzy hybrid retarded delay differential equation is discussed. Section IV holds numerical examples to prove the theory.

II. FUZZY MULTIPLE HYBRID SINGLE RETARDED DELAY DIFFERENTIAL EQUATION

With the adequate knowledge of Fuzzy sets, Fuzzy Differential Equation, Hybrid fuzzy differential equations. We are proposing a new system called Fuzzy Multiple Hybrid Single Retarded Delay differential system(FMHSRDDE). Hybrid systems are the systems which uses two are more functions for the effective behavior of the system. Every combined type of differential equations can be considered as hybrid system. But in general, We took only the system that performs continuously with discreteness. The delay differential system is very important in predicting the past with the present data which in turn used to predict the future with the same present data. Suha Najeeb Al Rawi et.al.,[14], classified the delay differential system into Retarded , Neutral and Mixed delay differential equations. Pederson and Sambandham Solved Hybrid systems by Runge-Kutta method [9]. Now we are using Al Rawi's Retarded delay differential system combined with multiples of hybrid system. Fuzzy Multiple Hybrid Systems as the multiple of same hybrid parameter. The importance of this study when the single hybrid systems are exist is, that now a days, physically so many systems are arising with same process is repeated in the same span of time with discontinuities arising then and there between the continues system. With the preliminary knowledge of fuzzy set theory, fuzzy Initial value problem, fuzzy differential equations obtained from [2,3,4,5,7,9 &10],We proceed to solve the FMHSRDDE.

Nowadays Eco Hybrid technology has keep on growing in industrial sectors. Many washing machine, motor vehicles, Electric Gadgets everything uses a hybrid technology. The modeling of hybrid differential equations in fuzzy will definitely find new way of scientific advancements. Let us take a clock as an example in which the seconds are moving between the minutes and it is repeated for every minute.

But change in second cannot occur all of the sudden, there are movement and rest in every micro second to make a movement of second. And suppose if we consider the hour span of time, the seconds are repeated so many times. When the same process is repeated with discrete and continuous behavior in the given interval of time we call it as multiple of same hybrid term. The mathematical form of fuzzy multiple hybrid single retarded delay differential system is given by

$$\begin{cases} \tilde{y}'(t) = \min, \max \ f(t, y(y), y(t-\tau), N\lambda(y_k(t_k))), t \ge t_0 \\ \tilde{y}(t) = \min, \max \ \phi(t), \qquad -\tau \le t \le t_0 \\ \tilde{y}(t_0) = \min, \max \ y_0 \in \phi(t) \end{cases} \quad (1)$$

Where $f: [0,\infty) \times R \times R \times R \to R$, N is a natural number, $\lambda(y_k(t_k)) = \begin{cases} 0, t_k = 0 \\ y_k, t_k > 0 \end{cases}$,

$t_k = k, k = 0,1,2, \ldots \; y(t_k) = y_k$.

III. FOURTH ORDER-RUNGE KUTTA METHOD

Numerical solutions play a vital role in finding the better approximate values of non-linear problems whenever it was not easy to find the analytic solutions. Also some complicated linear systems can also be solved by numerical solution techniques. The numerical solution of FMHSRDDE was found by using fourth order Runge kutta method. We choose RK-4 since it is the minimum order method with better accuracy than Euler methods. For our system the Runge-Kutta method is given by

$$\tilde{K}_1 = \min, \max \ (hf(t, y(t), y(t-\tau), N\lambda(y_k(t_k))))$$
$$\tilde{K}_2 = \min, \max \ (hf(t+h/2, y(t)+\tilde{K}_1/2, y(t-\tau)+\tilde{K}_1/2, N\lambda(y_k(t_k)+\tilde{K}_1/2)))$$
$$\tilde{K}_3 = \min, \max \ (hf(t+h/2, y(t)+\tilde{K}_2/2, y(t-\tau)+\tilde{K}_2/2, N\lambda(y_k(t_k)+\tilde{K}_2/2)))$$
$$\tilde{K}_4 = \min, \max \ (hf(t+h, y(t)+\tilde{K}_3, y(t-\tau)+\tilde{K}_3, N\lambda(y_k(t_k)+\tilde{K}_3)))$$
$$\tilde{y}_{n+1} = \tilde{y}_n + h/6.(\tilde{K}_1 + 2\tilde{K}_2 + 2K_3 + K_4)$$

This algorithm is derived from RK-4 algorithms provided for fuzzy differential equations [1], Hybrid differential equation [9] and delay differential equation [2,6&11].

IV. NUMERICAL EXAMPLES:

In below, two types of problems are solved both analytically and numerically. The numerical solutions are obtained by using the above method described in Section 3. The exact solution was solved in a usual way of solving ordinary DDE. In first and second problem the delay term was taken as 1 and hybrid term was taken as $|Sin(\pi t)|$ and $|Sin(3\pi t)|$ and the multiple term was taken as 2 and 3 respectively.

Example 4.1

Let us consider a below problem which is in the form of (1).

$$\begin{cases} \tilde{y}'(t) = (0.75+0.25r, 1.125-0.125r) \\ \qquad y(t-1) + 2m(t)\lambda_k y(t_k), \qquad 0 \le t \le 2 \\ \tilde{y}(t) = (0.75+0.25r, 1.125-0.125r)e^t, \quad -1 \le t \le 0 \\ m(t) = |Sin(\pi t)|, \qquad 0 \le t \le 2 \end{cases} \quad (2)$$

The exact Solution of (2) is given by

$$\tilde{Y}(t) = (0.75 + 0.25r, 1.125 - 0.125r) \begin{cases} e^t & t < 0 \\ -\dfrac{(-2e^2 + \pi - e\pi - e^t\pi + 2e^2\mathrm{Cos}[\pi t])}{e\pi} & 0 \le t < 1 \\ \dfrac{-e\pi^2 + e^2\pi^2 + e^t\pi^2 + 2e^3\pi t - e\pi^2 t + e^2\pi^2 t - 2e^3\pi\mathrm{Cos}[\pi t] + 2e^3\mathrm{Sin}[\pi t]}{e^2\pi^2} & 1 \le t \le 2 \end{cases} \quad (3)$$

In Table:1, We are comparing the numerical solutions and exact solutions.

TABLE:1 ERROR ANALYSIS OF NON-FUZZY SOLUTIONS

t.	Exact Solutions	Numerical Solution	Error Analysis
0	1	1	0
0.1	1.12338750242	1.123387791083	0.0000002886
0.2	1.41194789811	1.411949022070	0.0000011239
0.3	1.84204841314	1.842050837431	0.0000024242
0.4	2.37668654952	2.376690612072	0.0000040625
0.5	2.96916317740	2.969169055968	0.0000058785
0.6	3.56771016798	3.567717862779	0.0000076947
0.7	4.12062014671	4.120629480398	0.0000093336
0.8	4.58137685446	4.581387489545	0.0000106350
0.9	4.91328461073	4.913296082618	0.0000114718
1.0	5.09314447655	5.093156239057	0.0000117624
1.1	5.11318203088	5.113192953158	0.0000109222
1.2	4.99284704927	4.992856714784	0.0000096655
1.3	4.77165937947	4.771667552393	0.0000081729
1.4	4.49949624207	4.499502890349	0.0000066482
1.5	4.23176093041	4.231766228994	0.0000052985
1.6	4.02401594703	4.024020260747	0.0000043137
1.7	3.92660648677	3.926610334677	0.0000038479
1.8	3.97980118146	3.979805186096	0.0000040046
1.9	4.20992645332	4.209931279804	0.0000048264
2.0	4.62687363535	4.626879649797	0.0000060144

By considering the fuzzy values corresponding to the above values the following plot, Figure:1 is obtained. As a sample the fuzzy value of $t = 2$ is given in table2.

Table:2 Error Analysis of Fuzzy Solutions

t = 2	Exact Solution		Approximate Solution		Error Analysis	
r.	Min	Max	Min	Max	Min	Max

0	3.470155	5.205232	3.470159	5.205239	0.000004	0.000007
0.1	3.585827	5.147396	3.585831	5.147403	0.000004	0.000007
0.2	3.701498	5.089560	3.701503	5.089567	0.000005	0.000007
0.3	3.817170	5.031725	3.817175	5.031731	0.000005	0.000006
0.4	3.932842	4.973889	3.932847	4.973895	0.000005	0.000006
0.5	4.048514	4.916053	4.048519	4.916059	0.000005	0.000006
0.6	4.164186	4.858217	4.164191	4.858223	0.000005	0.000006
0.7	4.279858	4.800381	4.279863	4.800387	0.000005	0.000006
0.8	4.395529	4.742545	4.395535	4.742551	0.000006	0.000006
0.9	4.511201	4.684709	4.511207	4.684715	0.000006	0.000006
1.0	4.626873	4.626873	4.626879	4.626879	0.000006	0.000006

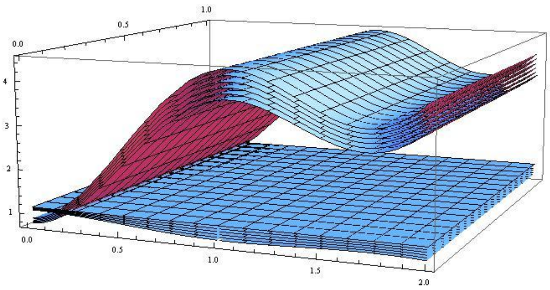

Figure:1: Fuzzy Solutions at $t \in [0, 2]$, $r \in [0, 1]$

Example:4.2

$$\begin{cases} \tilde{y}'(t) = (0.75 + 0.25r, 1.125 - 0.125r) \\ \quad y(t-1) + 3m(t)\lambda_k y(t_k), & 0 \le t \le 2 \\ \tilde{y}(t) = (0.75 + 0.25r, 1.125 - 0.125r)e^t & -1 \le t \le 0 \\ m(t) = |Sin(3\pi t)|, & 0 \le t \le 2 \end{cases} \quad (4)$$

The exact solution of (4) is given by

$$\tilde{Y}(t) = 0.75 + 0.25r, 1.125 - 0.125r$$

$$\begin{cases} e^t & t < 0 \\ -\dfrac{-e^2 + \pi - e\pi - e^t\pi + e^2 Cos[3\pi t]}{e\pi} & 0 \le t < 1 \\ \dfrac{-3e\pi^2 + 3e^2\pi^2 + 3e^t\pi^2 + 3e^3\pi t - 3e\pi^2 t + 3e^2\pi^2 t - 3e^3\pi Cos[3\pi t] + e^3 Sin[3\pi t]}{3e^2\pi^2} & 1 \le t \le 2 \end{cases}$$

t=2	Exact Solution		Approximate Solution		Error Analysis	
r.	Min	Max	Min	Max	Min	Max
0	2.821213	4.231819	2.821462	4.232194	0.000249	0.000375
0.1	2.915253	4.184799	2.915511	4.185169	0.000258	0.000370
0.2	3.009294	4.137779	3.009560	4.138145	0.000266	0.000366
0.3	3.103334	4.090759	3.103609	4.091121	0.000275	0.000362
0.4	3.197375	4.043738	3.197657	4.044096	0.000282	0.000358
0.5	3.291415	3.996718	3.291706	3.997072	0.000291	0.000354
0.6	3.385455	3.949698	3.385755	3.950048	0.000300	0.000350
0.7	3.479496	3.902678	3.479804	3.903023	0.000308	0.000345
0.8	3.573536	3.855658	3.573852	3.855999	0.000316	0.000341
0.9	3.667577	3.808637	3.667901	3.808974	0.000324	0.000337
1.0	3.761617	3.761617	3.761950	3.761950	0.000333	0.000333

(5)

Similar to previous example, In the following table (Table:3), The non-fuzzy values are compared from which the fuzzy values are found using the ideas given in Section 2.

Table:3 Error Analysis of Non-Fuzzy Solutions

t.	Exact Solutions	Numerical Solution	Error Analysis
0	1	1	0.
0.1	1.39536149383	1.395461854795	0.0001003609
0.2	2.21408430450	2.214403006380	0.0003187018
0.3	2.81686917955	2.817344196840	0.0004750172
0.4	2.74619496620	2.746635403730	0.0004404375
0.5	2.10390719797	2.104150670664	0.0002434726
0.6	1.46768979245	1.467736300513	0.0000465080
0.7	1.41528742144	1.415299350381	0.0000119289
0.8	2.04872848920	2.048896734627	0.0001682454
0.9	2.91079866046	2.911185248298	0.0003865878
1.0	3.36263251769	3.363119468449	0.0004869507
1.1	3.12011608741	3.120489842987	0.0003737555
1.2	2.52360938641	2.523788903217	0.0001795168

1.3	2.22401825283	2.224102696825	0.0000844439
1.4	2.63121506974	2.631402061374	0.0001869916
1.5	3.57652253671	3.576945227075	0.0004226903
1.6	4.45221237102	4.452829668498	0.0006172974
1.7	4.71301646003	4.713646907771	0.0006304477
1.8	4.32619470259	4.326666077312	0.0004713747
1.9	3.79911566797	3.799406966338	0.0002912983
2.0	3.76161765591	3.761950502909	0.0003328469

Similar to Table:2, Table:4 is given by taking fuzzy values at $t = 2$. For all the values at $t \in [0,2]$, $r \in [0,1]$, Figure:2 is given.

Table:4 Error Analysis of Fuzzy Solutions

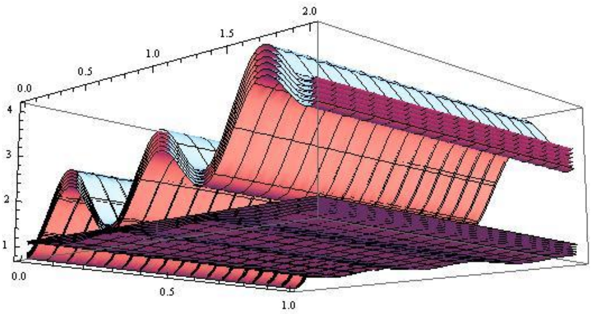

Figure:2: Fuzzy Solutions at $t \in [0, 2]$, $r \in [0, 1]$

V. CONCLUSION

The combination of single retarded delay and multiples of hybrid system was studied with the numerical examples. The value of N is taken as natural number because it should not be less than or equal to zero. Suppose if it is negative the change may be imaginary, which is meaningless while modeling a physical system. When it is considered as zero, the system will become pure delay differential equation such that no hybrid terms exist. From the table values of both the problems and from the error analysis study we could see that the numerical solution coincides well with the exact solutions.

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