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Multi Node Tandem Queuing Model with Bulk Arrivals Having Geometric Arrival Distribution

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Abstract: **In this paper a K-node forked queuing model with load *dependent service rates is analysed. Here it is assumed that the customers arrive to the first queue in batches and wait for service. After getting service at first service station with some probability they may join any one of the (K-1) parallel queues which are connected to first queue in series and exit from the system after getting service. It is assumed that the arrival and service completions follow Poisson processes and service rates depend on number of customers in the queue connected to it. The influence of Geometrically distributed bulk arrivals on this queuing model is studied. Sensitivity analysis of the system behaviour with regards to the arrival rates and load dependent service distribution parameters is carried out. The influence of these parameters on system performance measures such as average number of customers, waiting time of customer, variation of number of customers in each queue, throughput of each service station, utilization of each server are derived explicitly when arrivals follow a Geometric distribution. Simulations are carried out to illustrate the result.***

Keywords : : Bulk arrivals, Geometric Distribution , Forked queuing model, Poisson Process Load dependent service rates, Performance of system.

I. INTRODUCTION

In this paper we are going to study a multi node tandem queuing model with bulk arrivals The model describes a system in which the bulk arrivals are Geometrically distributed. Ever since the basic work of Erlang [9] the formation of mathematical models for queuing systems has gained momentum and several models have been designed and analysed to understand the real time problems. The earlier queuing models are based on the applications to communication networks[11],[13-17],[20],[22],[29],[30],[33-34].Due to development in technology and communication the need of appropriate models for situations like communication networks[17],[20], ATM scheduling[39], Production[21] and Transportation systems[36],[38] have made researchers to explore the behaviour of such systems .The main aim of these models is monitoring and controlling of the systems under consideration..

Bulk arrivals is a common phenomenon observed in nature beginning with migration of populations, grazing of animals etc., Handling situations of bulk arrivals is a different activity. This requires organization and handling of masses and skill providing services uniformly. Thus a proper understanding of situation is desired. For some interesting models on bulk arrivals we refer the readers to [19],[20],[23],[27],[40-42].

In order to streamline the bulk arrivals, forked queuing models are proposed.[11-17],[44],[45].The customers arrived at a service point (first node) are diverted to (K-1) nodes for various services .The models are developed and studied with various distribution processes (Uniform, Poisson, Geometric, Binomial etc.,)to closely represent the real time situation.

To further simplify the service activity and to establish direct relation between the first node and the secondary nodes tandem queuing models are designed. In a Tandem queuing model , the output of first queue formulates the input for the other. These models could be load dependent (situations where service time is adjusted w.r.t .the number of customers) [11-17] , [22],[23],[44-45]or load independent [17],[8],[10],[4],[36].

In recent work [18] , the present authors studied a multi node tandem queuing model with load dependent service rates. It is established that the state dependent service rates have significant influence on performance measures. Modifying the model further the authors allowed bulk arrivals with the model in [44] the sensitivity of the model revealed that the arrival rates and load dependent service time distribution parameters play a vital role on the performance measures of the system. In both these models the arrivals are assumed to follow Poisson distribution. In the present paper we wish to consider a multi node tandem queuing model with bulk arrivals that follow a geometric distribution. The motivation for considering a geometric distribution stems from the fact that Geometric distribution is more applicable in population models, econometrics and return of investment etc.,[36]. Further geometric distribution has the important property of being memory less in the sense that the failure at a given point does not depend on the failures at previous instants directly observed. For some interesting models with Geometric distribution one may refer to M.L.Chaudry etal.,[40], P.Vijayalakshmi etal [41],S.H.Chang etal[42], Bhagavathi Devi etal[43].Thus the number of failures at a particular service point need not be the criterion for arriving at a node.

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Further since Geometric distribution is useful for determining the likelihood of success given a limited number of trials, it attracts more applicability in real world situation in which unlimited trials are rear and useless.

With this background we are going to consider here Geometrically distributed bulk arrivals for the model studied in [44]. It is assumed that the K servers are connected in tandem where the customers arrive to the first Queue in batches and after getting service at first server they may join at any one of the (K-1) Queues which are in parallel with certain probability.

The paper is organised as follows. In section 2 we describe the model under consideration. In section 3 characteristics of the model such as performance analysis for general arrival distribution as well as Geometric distribution in particular are studied. Restricting to four servers we have studied the performance analysis of the model in section 4 for a clear understanding of the system. A numerical illustration is provided in section 5. In section 6 sensitivity of performance measures is carried out. Section 7 deals with steady state analysis of the model under general arrivals as well as Geometric arrivals. A comparative study between transient and steady state is the content of section 8.The performance measures are pictorially exhibited through graphs in section 9.

II. II.QUEUEING MODEL WITH GENERAL BULK SIZE DISTRIBUTION

In this section a queuing model with K buffers $B_1, B_2, \dots B_k$ of infinite capacity and K servers $S_1, S_2, \dots S_k$ connected as forked network is considered. It is assumed that the customers arrive in bathes to the first queue and after getting service at first server they may join any of the (K-1) queues connected to the servers $S_2, S_3, \dots S_k$ which are parallel and connected to first server in tandem , with some probability i.e., the customers after getting service at S_1 in batches may join second buffer with probability $\theta_1$or third buffer with probability $\theta_2$or K th buffer with probability θ_{k-1}.Let us assume that the actual number of customers in any arriving module is a random variable X with probability C(X).Let $\lambda_x$be the arrival rate of batches of size x and λ is the composite arrival rate. Then $\lambda = \sum \lambda_x$.Therefore the arrival process follows a compound Poisson process with arrival rate $\lambda . E(x)$ .Further it is assumed that the service completion in each service station is random and follows a Poisson process with service rates $\mu_1 \mu_2 \mu_3 \dots \mu_k$ respectively. Here we assume that service rate in each server is linearly dependent on the content of buffer connected to it and queue discipline is first come first serve(FCFS).

Let $P(n_1, n_2, \dots, n_k; t)$ be the probability that there are $n_1$costumers in first queue, n_2 customers in second queue and n_k customers in k^{th} th queue at time t. The customers arrive in batches of size X .The probability generating function of X is $C(Z) = \sum_{m=1}^{\infty} C_m z^m$.

Then difference differential equations governing the system are

$$\frac{\partial P}{\partial t}(n_1, n_2, \dots n_k; t) = -[\lambda + \sum_{i=1}^{k} n_i \mu_i] P(n_1, n_2, \dots, n_k; t) + (n_1+1)\mu_1 [\theta_1 P(n_1+1, n_2 - 1, n_3, \dots, n_k; t) + \theta_2 P(n_1+1, n_2, n_3 - 1, \dots, n_k; t) + \dots + \theta_{k-1} P(n_1+1, n_2, \dots, n_k - 1; t)] + (n_2+1)\mu_2 P(n_1, n_2+1, n_3, \dots, n_k; t) + (n_3+1)\mu_3 P(n_1, n_2, n_3+1, \dots, n_k; t) + \dots + (n_k+1)\mu_k P(n_1, n_2, \dots, n_k+1; t) + \lambda \sum_{m=1}^{n_1} C_m P(n_1 - m, n_2, \dots, n_k, t) \quad (1)$$

$$\frac{\partial P}{\partial t}(0, n_2, \dots n_k; t) = -[\lambda + \sum_{i=2}^{k} n_i \mu_i] P(0, n_2, \dots, n_k, t) + \mu_1 [\theta_1 P(1, n_2 - 1, n_3, \dots, n_k; t) + \theta_2 P(1, n_2, n_3 - 1, \dots, n_k; t) + \dots + \theta_{k-1} P(1, n_2, n_3, \dots, n_k - 1; t)] + (n_2+1)\mu_2 P(0, n_2+1, n_3, \dots, n_k; t) + (n_3+1)\mu_3 P(0, n_2, n_3+1, \dots, n_k; t) + \dots + (n_k+1)\mu_k P(0, n_2, n_3, \dots, n_k+1; t) \quad (2)$$

$$\frac{\partial P}{\partial t}(n_1, 0, \dots n_k; t) = -[\lambda + \sum_{i=1, i\neq 2}^{k} n_i \mu_i] P(n_1, 0, \dots, n_k, t) + (n_1+1)\mu_1 [\theta_2 P(n_1+1, 0, n_3 - 1, \dots, n_k; t) + \dots + \theta_{k-1} P(n_1+1, n_2, \dots, n_k - 1; t)] + \mu_3 P(n_1, 0, n_3+1, \dots, n_k; t) + \dots + (n_k+1)\mu_k P(n_1, 0, \dots, n_k+1; t) + \lambda \sum_{m=1}^{n_1} C_m P(n_1 - m, 0, \dots, n_k, t) \quad (3)$$

$$\frac{\partial P}{\partial t}(n_1, n_2, \dots, 0; t) = -[\lambda + \sum_{i=1}^{k-1} n_i \mu_i] P(n_1, n_2, \dots, 0, t) + (n_1+1)\mu_1 [\theta_1 P(n_1+1, n_2 - 1, n_3, \dots, 0; t) + \theta_2 P(n_1+1, n_2, n_3 - 1, \dots, 0; t) + \dots + \theta_{k-2} P(n_1+1, n_2, \dots, n_{k-1} - 1, 0; t)] + (n_2+1)\mu_2 P(n_1, n_2+1, n_3, \dots, 0; t) + (n_3+1)\mu_3 P(n_1, n_2, n_3+1, \dots, 0; t) + \dots + (n_k+1)\mu_k P(n_1, n_2, \dots, 1; t) + \lambda \sum_{m=1}^{n_1} C_m P(n_1 - m, n_2, \dots, n_{k-1}, 0; t) \quad (4)$$

$$\frac{\partial P}{\partial t}(0, 0, \dots n_k; t) = -[\lambda + \sum_{i=3}^{k} n_i \mu_i] P(0, 0, \dots, n_k, t) + \mu_1 [\theta_2 P(1, 0, n_3 - 1, \dots, n_k; t) + \dots + \theta_{k-1} P(1, 0, \dots, n_k - 1; t)] + \mu_2 P(0, 1, n_3, \dots, n_k; t) + (n_3+1)\mu_3 P(0, 0, n_3+1, \dots, n_k; t) + \dots + (n_k+1)\mu_k P(0, 0, n_3, \dots, n_k+1; t) \quad (5)$$

$$\frac{\partial P}{\partial t}(0, n_2, \dots n_{k-1}, 0; t) = -[\lambda + \sum_{i=2}^{k-1} n_i \mu_i] P(0, n_2, \dots, n_{k-1}, 0; t) + \mu_1 [\theta_1 P(1, n_2 - 1, n_3, \dots, n_{k-1}, 0; t) + \theta_2 P(1, n_2 + 1, n_3, \dots, n_{k-1}, 0; t) + \dots + \theta_{k-2} P(1, n_2, \dots, n_{k-1} - 1, 0; t)] + + (n_3+1)\mu_3 P(0, n_2, n_3+1, \dots, 0; t) + \dots + (n_k+1)\mu_k P(0, n_2, \dots, 1; t) \quad (6)$$

$$\frac{\partial P}{\partial t}(0, 0, 0, n_4, \dots n_k; t) = -[\lambda + \sum_{i=4}^{k} n_i \mu_i] P(0, 0, 0, n_4, \dots n_k; t) + \mu_1 \theta_3 P(1, 0, 0, n_4 - 1, \dots, n_k; t) + \dots + \mu_k \theta_{k-1} P(1, 0, 0, n_4, \dots, n_k - 1; t)] + \mu_2 P(0, 1, 0, n_4, \dots, n_k; t) + \mu_3 P(0, 0, 1, n_4, \dots, n_k; t) + \dots + (n_k+1)\mu_k P(0, 0, 0, n_4, \dots, n_k+1; t) \quad (7)$$

$$\frac{\partial P}{\partial t}(0, 0, n_3, \dots n_{k-1}, 0; t) = -[\lambda + \sum_{i=3}^{k-1} n_i \mu_i] P(0, 0, n_3, \dots, n_{k-1}, 0; t) + \mu_1 [\theta_2 P(1, 0, n_3 - 1, \dots, 0; t) + \dots + \theta_{k-2} P(1, 0, n_3, \dots, n_{k-1} - 1, 0; t)] + \mu_2 P(0, 1, n_3, \dots, n_{k-1}, 0; t) + (n_3+1)\mu_3 P(0, 0, n_3+1, \dots, n_{k-1}, 0; t) + \dots + \mu_k P(0, 1, 0, n_3, \dots, 1; t) \quad (8)$$

$$\frac{\partial P}{\partial t}(n_1, 0, 0, n_4, \dots, 0; t) = -[\lambda + \sum_{i=1, i\neq 2,3}^{k-1} n_i \mu_i] P(n_1, 0, 0, n_4, \dots, 0; t) + (n_1+)\mu_1 [\theta_3 P(n_1+1, 0, 0, n_4 - 1, \dots, 0; t) + \dots + \theta_{k-2} P(n_1+1, 0, 0, \dots, n_{k-1} - 1, 0; t)] + \mu_2 P(n_1, 1, 0, n_4, \dots, 0; t) + \mu_3 P(n_1, 0, 1, n_4, \dots, 0; t) + \dots + \mu_k P(n_1, 0, 0, n_4, \dots, 1; t) + \lambda \sum_{m=0}^{n_1} C_m P(n_1 - m, 0, 0, n_4, \dots, t) \quad (9)$$

$$\frac{\partial P}{\partial t}(0, 0, \dots, 0; t) = -\lambda P(0, 0, \dots, 0; t) + \mu_2 P(0, 1, 0, \dots, 0; t) + \mu_3 P(0, 0, 1, \dots, 0; t) + \dots + \mu_k P(0, 0, \dots, 0, 1; t) \quad (10)$$

Let $P(Z_1, Z_2, \dots, Z_k; t) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_k=0}^{\infty} p(n_1, n_2, \dots n_k; t)\, z_1^{n_1} z_2^{n_2} \dots z_k^{n_k}$ be probability generating function of $p(n_1, n_2, \dots n_k; t)$.

Multiplying equations (1) to (10)with probability generating function and summing over $n_1, n_2, \dots n_k$ from 0 to ∞ we get the Joint Probability generating function of number of

customers in first, second,….kth queues respectively at any time t as

$$P(Z_1,Z_2,\ldots Z_k;t) = exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\ldots\sum_{r_k=0}^{r_{k-1}}(-1)^{r_2}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\ldots\binom{r_{k-1}}{r_k}C_m \left\{(z_1-1)+\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}+\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}+\ldots+\frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_1-r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_2-r_3}\ldots\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_3-r_4}\left\{\frac{\theta_{k-2}\mu_1(z_{k-1}-1)}{\mu_{k-1}-\mu_1}\right\}^{r_{k-1}-r_k}\left\{\frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_k}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\ldots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\ldots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right]$$

III. CHARACTERISTICS OF THE MODEL

Putting $z_1=0, z_2=0,\ldots. z_k=0$ in (11) and expanding we get the probability that the k-server system is empty at any time t.as

$$P(0,0,..0;t) = \exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\ldots\sum_{r_k=0}^{r_{k-1}}(-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\ldots\binom{r_{k-1}}{r_k}C_m \left(1+\frac{\theta_1\mu_1}{\mu_2-\mu_1}+\frac{\theta_2\mu_1}{\mu_3-\mu_1}+\ldots+\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_1-r_2}\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^{r_3-r_4}\ldots\left(\frac{\theta_{k-2}\mu_1}{\mu_{k-1}-\mu_1}\right)^{r_{k-1}-r_k}\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^{r_2-r_3}\left(\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_k}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\ldots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\ldots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right] \quad (12)$$

III.A. PERFORMANCE ANALYSIS OF FIRST QUEUE

Putting $z_2=1, z_3=1,\ldots. z_k=1$ in (11) we get probability generating function of first queue size as

$$P(Z_1;t) = exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}C_m\binom{m}{r_1}(z_1-1)^{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (13)$$

Mean number of customers in first buffer is $E(N_1)=L_1(t)=\frac{\lambda}{\mu_1}(1-e^{-\mu_1 t})E(X)$ (14)

Where E(X) is the mean of batch size arrivals to first queue given by $E(X)=\sum_{m=1}^{\infty}m.C_m$

Putting $Z_1=0$ in (13) we get the probability that the first queue is empty as

$$P(0,,,,,;t) = exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (15)$$

Utilization of first server is $U_1(t)=1-P(0,,,,,;t)$

$$=1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (16)$$

Throughput of first server is $Thp1(t)=\mu_1.U_1(t)$

$$=\mu_1\left\{1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\} \quad (17)$$

Average waiting time of a customer in first queue is

$$W_1(t)=\frac{L_1(t)}{Thp_1(t)}=\frac{\left(\frac{\lambda}{\mu_1^2}\right)(1-e^{-\mu_1 t})E(X)}{\left\{1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\}} \quad (18)$$

Variation of the number of customers in first queue is

$$V(Z_1)=V_1(t)=\lambda\sum_{m=2}^{\infty}\left[\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right] \quad (19)$$

Coefficient of variation of the number of customers in first queue is

$$CV_1(t)=\frac{\sqrt{V_1(t)}}{L_1(t)}x100=\frac{\sqrt{\lambda\sum_{m=2}^{\infty}\left[\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right]}}{\frac{\lambda}{\mu_1}(1-e^{-\mu_1 t})E(X)}x100 \quad (20)$$

III.B. PERFORMANCE ANALYSIS OF ith QUEUE FOR i =2,3,..k

Putting $z_1=1, z_2=1, z_3=1,.., z_{k-1}=1$ we get probability generating function of ith buffer size distribution as $P(z_i;t)$

$$=exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_i=0}^{r_1}(-1)^{r_i}\binom{m}{r_1}\binom{r_1}{r_i}C_m\left\{\frac{\theta_{i-1}\mu_1(z_i-1)}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_1-r_i)+\mu_i r_i)t}}{\mu_1(r_1-r_i)+\mu_i r_i}\right\}\right] \quad (21)$$

Mean number of customers in ith queue is

$$E(N_i)=L_i(t)=\left(\frac{\lambda\theta_{i-1}}{\mu_i}\right)\left[1-\left\{\frac{\mu_i e^{-\mu_1 t}-\mu_1 e^{-\mu_i t}}{\mu_i-\mu_1}\right\}\right]E(X) \quad (22)$$

Where E(X) is the mean of batch size arrivals to first queue and is given by $E(X)=\sum_{m=1}^{\infty}m.C_m$

Putting $z_i=0$ in (21)we get the probability that the ith queue is empty as

$$P(,,,0,,;t) = exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_i=0}^{r_1}(-1)^{r_i+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_i}\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_1-r_i)+\mu_i r_i)t}}{\mu_1(r_1-r_i)+\mu_i r_i}\right\}\right] \quad (23)$$

Utilization of ith server is $U_i(t)=1-P(,,,0,,;t)=$

$$1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_i=0}^{r_1}(-1)^{r_i+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_i}\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_1-r_i)+\mu_i r_i)t}}{\mu_1(r_1-r_i)+\mu_i r_i}\right\}\right] \quad (24)$$

Throughput of ith server is $Thp_i(t)=\mu_i.U_i(t)=$

$$\mu_i.\left[1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_i=0}^{r_1}(-1)^{r_i+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_i}\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_1-r_i)+\mu_i r_i)t}}{\mu_1(r_1-r_i)+\mu_i r_i}\right\}\right]\right] \quad (25)$$

Average waiting time of a customer in ith queue is

$$W_i(t)=\frac{L_i(t)}{Thp_i(t)}=\frac{\left(\frac{\lambda\theta_{i-1}}{\mu_i^2}\right)\left[1-\left\{\frac{\mu_i e^{-\mu_1 t}-\mu_1 e^{-\mu_i t}}{\mu_i-\mu_1}\right\}\right]E(X)}{\left[1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_i=0}^{r_1}(-1)^{r_i+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_i}\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_1-r_i)+\mu_i r_i)t}}{\mu_1(r_1-r_i)+\mu_i r_i}\right\}\right]\right]} \quad (26)$$

Variance of the number of customers in ith queue is

$$V(z_i)=V_i(t)=\lambda\left[\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_i+\mu_1)t}}{\mu_i+\mu_1}\right)+\left(\frac{1-e^{-2\mu_i t}}{\mu_i}\right)\right\}+\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_i t}}{\mu_i}\right)\right\}E(X)\right] \quad (27)$$

Coefficient of variation of the number of customers in ith queue is

$$\frac{\sqrt{\left[\lambda\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_i+\mu_1)t}}{\mu_i+\mu_1}\right)+\left(\frac{1-e^{-2\mu_i t}}{\mu_i}\right)\right\}+\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_i t}}{\mu_i}\right)\right\}E(X)\right]}}{\left(\frac{\lambda\theta_{i-1}}{\mu_i}\right)\left[1-\left\{\frac{\mu_i e^{-\mu_1 t}-\mu_1 e^{-\mu_i t}}{\mu_i-\mu_1}\right\}\right]E(X)}x100 \quad (28)$$

III.C.PERFORMANCE MEASURES OF THE MODEL WHEN BATCH SIZE DISTRIBUTION IS GEOMETRIC

Let $P(n_1,n_2,..,n_k;t)$ be the probability that there are $n_1$customers in first queue, n_2 customers in second queue and n_k customers in k^{th} th queue at time t. It is assumed that the customers arrive to the queue in batches of size 'm'. In other words the arrival of customers follows compound Poisson doubly truncated Geometric Distribution with parameters 'p and q'.(q=1-p).

The probability mass function of Geometric batch size distribution is $C_m = \frac{q^{m-1}p}{1-q^A}$ $for\, m = 1,2,..A\ and\ 0 < p < 1.$ The mean number of customers in batch with Geometric distribution is $E(X) = \frac{1-q^A-pAq^A}{(1-q)(1-q^A)}$ and the probability generating function is $C(Z) = \sum_{m=1}^{\infty} C_m z^m$.

The performance of the model is influenced by the batch size arrival distribution.

Then the Joint Probability generating function of number of customers in first, second,….kth queues respectively at any time t as $P(Z_1, Z_2, ...Z_k; t) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \dots \sum_{r_k=0}^{r_{k-1}} (-1)^{r_2} \binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}..\binom{r_{k-1}}{r_k}\left(\frac{q^{m-1}p}{1-q^A}\right)\right.$$
$$\left\{(z_1-1) + \frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1} + \frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1} + .. + \frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_1-r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_2-r_3}$$
$$\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_3-r_4} \dots \left\{\frac{\theta_{k-2}\mu_1(z_{k-1}-1)}{\mu_{k-1}-\mu_1}\right\}^{r_{k-1}-r_k}\left\{\frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_k}$$
$$\left.\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+..+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+..+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right] \quad (29)$$

III.D.CHARACTERISTICS OF THE MODEL

Putting $z_1 = 0, z_2 = 0, \dots. z_k = 0$ in (29) and expanding we get the probability that the k-server system is empty at any time t as $P(0,0,..0;t)=$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \dots \sum_{r_k=0}^{r_{k-1}} (-1)^{r_2+r_1} \binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}..\binom{r_{k-1}}{r_k}\left(\frac{q^{m-1}p}{1-q^A}\right)\right.$$
$$\left(1 + \frac{\theta_1\mu_1}{\mu_2-\mu_1} + \frac{\theta_2\mu_1}{\mu_3-\mu_1} + .. + \frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_1-r_2}\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^{r_2-r_3}\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^{r_3-r_4} \dots$$
$$\left(\frac{\theta_{k-2}\mu_1}{\mu_{k-1}-\mu_1}\right)^{r_{k-1}-r_k}$$
$$\left.\left(\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_k}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+..+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+..+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right] \quad (30)$$

III.E.PERFORMANCE ANALYSIS OF FIRST QUEUE

Putting $z_2 = 1, z_3 = 1, \dots. z_k = 1$ in (29) we get probability generating function of first queue size as

$$P(Z_1;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)(z_1-1)^{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (31)$$

Mean number of customers in first buffer is

$$L_1(t) = \left[\frac{\lambda}{\mu_1}\right]\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)(1-e^{-\mu_1 t}) \quad (32)$$

Putting $Z_1 = 0$ in (4) we get the probability that the first queue is empty as

$$P(0,.,.,.;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (33)$$

Utilization of first server is

$$U_1(t) = 1 - P(0,.,.,.;t) = 1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (34)$$

Throughput of first server is

$$Thp1(t) = \mu_1 . U_1(t) = \mu_1\left\{1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\} \quad (35)$$

Average waiting time of a customer in first queue is

$$W_1(t) = \frac{L_1(t)}{Thp_1(t)} = \frac{\left[\frac{\lambda}{\mu_1^2}\right]\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)(1-e^{-\mu_1 t})}{\left\{1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\}} \quad (36)$$

Variance of number of customers in first queue is

$$V_1(t) = \lambda \sum_{m=2}^{A} \left[\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) + \left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right] \quad (37)$$

Coefficient of variation of number of customers in first queue is

$$CV_1(t) = \frac{\sqrt{V_1(t)}}{L_1(t)} x100 = \frac{\sqrt{\lambda \sum_{m=2}^{A} \left[\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) + \left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right]}}{\left[\frac{\lambda}{\mu_1}\right]\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)(1-e^{-\mu_1 t})} x100 \quad (38)$$

III.F.PERFORMANCE ANALYSIS OF ith QUEUE FOR i =2,3,..k

Putting $z_1 = 1, z_2 = 1, z_3 = 1, \dots z_{i-1} = 1$ in (29) we get probability generating function of **ith** queue size distribution as $P(Z_i;t) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_{i-1}=1}^{m} \sum_{r_i=0}^{r_{i-1}} (-1)^{r_i} \binom{m}{r_{i-1}}\binom{r_{i-1}}{r_i}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_{i-1}\mu_1(z_i-1)}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_{i-1}-r_i)+\mu_i r_i)t}}{\mu_1(r_{i-1}-r_i)+\mu_i r_i}\right\}\right] \quad (39)$$

Mean number of customers in **ith** queue is

$$L_i(t) = \left[\left(\frac{\lambda\theta_{i-1}}{\mu_i}\right)\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)\left\{1 - \frac{(\mu_i e^{-\mu_1 t} - \mu_1 e^{-\mu_i t})}{(\mu_i-\mu_1)}\right\}\right] \quad (40)$$

Putting $z_i = 0$ in (29) we get probability that the **ith** queue is empty as

$$P(.,..0,.;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_{i-1}=1}^{m} \sum_{r_i=0}^{r_{i-1}} (-1)^{r_i+r_1} \binom{m}{r_{i-1}}\binom{r_{i-1}}{r_i}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_{i-1}-r_i)+\mu_i r_i)t}}{\mu_1(r_{i-1}-r_i)+\mu_i r_i}\right\}\right] \quad (41)$$

Utilization of **ith** server is $U_i(t) = 1 - P(.,.,.,.0;t) =$

$$1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_{i-1}=1}^{m} \sum_{r_i=0}^{r_{i-1}} (-1)^{r_i+r_1} \binom{m}{r_{i-1}}\binom{r_{i-1}}{r_i}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_{i-1}-r_i)+\mu_i r_i)t}}{\mu_1(r_{i-1}-r_i)+\mu_i r_i}\right\}\right] \quad (42)$$

Throughput of **ith** server is $Thp_i(t) = \mu_i . U_i(t) =$ (43)

Average waiting time of a customer in **ith** queue is $W_i(t) = \frac{L_i(t)}{Thp_i(t)} =$

$$\mu_i\left[1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_{i-1}=1}^{m} \sum_{r_i=0}^{r_{i-1}} (-1)^{r_i+r_1} \binom{m}{r_{i-1}}\binom{r_{i-1}}{r_i}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right\}^{r_i}\left\{\frac{1-e^{-(\mu_1(r_{i-1}-r_i)+\mu_i r_i)t}}{\mu_1(r_{i-1}-r_i)+\mu_i r_i}\right\}\right]\right] \quad (43)$$

Average waiting time of a customer in **ith** queue is $W_i(t) = \frac{L_i(t)}{Thp_i(t)} =$

$$\frac{\left[\left(\frac{\lambda\theta_{i-1}}{\mu_i^2}\right)\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)\left[1 - \frac{(\mu_i e^{-\mu_1 t} - \mu_1 e^{-\mu_i t})}{(\mu_i-\mu_1)}\right]\right]}{\left[1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_{i-1}=1}^{m} \sum_{r_i=0}^{r_{i-1}} (-1)^{r_i+r_1} \binom{m}{r_{i-1}}\binom{r_{i-1}}{r_i}\left(\frac{q^{m-1}p}{1-q^A}\right)\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)^{r_i}\left[\frac{1-e^{-(\mu_1(r_{i-1}-r_i)+\mu_i r_i)t}}{\mu_1(r_{i-1}-r_i)+\mu_i r_i}\right]\right]\right]} \quad (44)$$

Variance of number of customers in **ith** queue is $V_i(t) =$

$$\lambda\left[\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right).\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_i+\mu_1)t}}{\mu_i+\mu_1}\right) + \left(\frac{1-e^{-2\mu_i t}}{\mu_i}\right)\right\}\right.$$
$$\left.\left\{\frac{\theta_{i-1}\mu_1}{(\mu_i-\mu_1)}\right\}\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right) - \left(\frac{1-e^{-\mu_i t}}{\mu_i}\right)\right\}\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)\right]$$

Coefficient of variation of number of customers in **ith** queue is

$$CV_i(t) = \frac{\sqrt{V_i(t)}}{L_i(t)} x100 =$$
$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_{i-1}\mu_1}{\mu_i-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_i+\mu_1)t}}{\mu_i+\mu_1}\right) + \left(\frac{1-e^{-2\mu_i t}}{\mu_i}\right)\right\}\left\{\left(\frac{\theta_{i-1}\mu_1}{(\mu_i-\mu_1)}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right) - \left(\frac{1-e^{-\mu_i t}}{\mu_i}\right)\right\}\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)\right]}}{\left[\left(\frac{\lambda\theta_{i-1}}{\mu_i}\right)\left(\frac{1-q^A-pAq^A}{(1-q)(1-q^A)}\right)\left[1 - \frac{(\mu_i e^{-\mu_1 t} - \mu_1 e^{-\mu_i t})}{(\mu_i-\mu_1)}\right]\right]}$$

X100 (46)

IV. NUMERICAL ILLUSTRATION

For numerical illustration we take k=4 and calculate equations and performance measures

The Joint Probability generating function of number of customers in first, second, third and fourth queues respectively at any time t is $P(Z_1, Z_2, Z_3, Z_4; t) =$

$$exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_2}\sum_{r_3=0}^{r_3}\sum_{r_4=0}^{r_3}(-1)^{r_2}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\binom{r_3}{r_4}C_m \left\{(z_1-1)+\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}+\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}+\frac{\theta_3\mu_1(z_4-1)}{\mu_4-\mu_1}\right\}^{r_1-r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_2-r_3}\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_3-r_4}\left\{\frac{\theta_3\mu_4(z_4-1)}{\mu_4-\mu_1}\right\}^{r_4}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4 r_4}\right\}\right]$$

(47)

IV.A.CHARACTERISTICS OF THE MODEL

Putting $z_1 = 0, z_2 = 0, z_3 = 0, z_4 = 0$ in (47) and expanding we get the probability that the 4-server system is empty at any time t as $P(0,0,0,0;t)=$

$$exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_2}\sum_{r_3=0}^{r_3}\sum_{r_4=0}^{r_3}(-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\binom{r_3}{r_4}C_m \left(1+\frac{\theta_1\mu_1}{\mu_2-\mu_1}+\frac{\theta_2\mu_1}{\mu_3-\mu_1}+..+\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_1-r_2}\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^{r_2-r_3}\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^{r_3-r_4}\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^{r_4}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4 r_4}\right\}\right]$$

(48)

IV.B.PERFORMANCE ANALYSIS OF FIRST QUEUE

Putting $z_2 = 1, z_3 = 1, z_4 = 1$ in(48) we get probability generating function of first queue size as

$$P(Z_1;t) = exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m} C_m \binom{m}{r_1}(z_1-1)^{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (49)$$

Mean number of customers in first queue is

$$E(N_1) = L_1(t) = \left(\frac{\lambda}{\mu_1}\right)(1-e^{-\mu_1 t})E(X) \quad (50)$$

Where E(X) is the mean of batch size arrivals to first queue and is given by $E(X) = \sum_{m=1}^{\infty} m.C_m$

Putting $Z_1 = 0$ we get the probability that the first queue is empty as

$$P(0,.,.,.;t) = exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]$$

(51)

Utilization of first server is $U_1(t) = 1 - P(0,.,.,.;t) =$

$$1 - exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (52)$$

Throughput of first server is $Thp1(t) = \mu_1.U_1(t) =$

$$\mu_1\left\{1 - exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\} \quad (53)$$

Average waiting time of a customer in first queue is

$$W_1(t) = \frac{L_1(t)}{Thp_1(t)} = \frac{\left(\frac{\lambda}{\mu_1}\right)(1-e^{-\mu_1 t})E(X)}{\left\{1-exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m}(-1)^{r_1}C_m\binom{m}{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\}} \quad (54)$$

Variance of the number of customers in first queue is

$$V(Z_1) = V_1(t) = \lambda \sum_{m=2}^{\infty}\left[\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right] \quad (55)$$

Coefficient of variation in number of customers in first queue is

$$CV_1(t) = \frac{\sqrt{V_1(t)}}{L_1(t)}x100 = \frac{\sqrt{\lambda \sum_{m=2}^{\infty}\left[\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right]}}{\left(\frac{\lambda}{\mu_1}\right)(1-e^{-\mu_1 t})E(X)}x100 \quad (56)$$

IV.C. PERFORMANCE ANALYSIS OF SECOND QUEUE

Putting $z_1 = 1, z_3 = 1, z_4 = 1$ in (48)we get probability generating function of SECOND queue size distribution as P

$$(Z_2;t) = exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_1}(-1)^{r_2}C_m\binom{m}{r_1}\binom{r_1}{r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]$$

(57)

Mean number of customers in second queue is

$$E(N_2) = L_2(t) = \left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left\{1-\frac{(\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t})}{(\mu_2-\mu_1)}\right\}\right]E(X) \quad (58)$$

Where $E(X)$ is the mean of batch size arrivals at second queue and $E(X) = \sum_m m.C_m$

Putting $z_2 = 0$ we get the probability that the second queue is empty as

$$P(.,0,.,.;t) = exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_1}(-1)^{r_2+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_2}\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (59)$$

Utilization of second server is $U_i(t) = 1 - P(.,0,.,.;t)$

$$= 1 - exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_1}(-1)^{r_2+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_2}\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (60)$$

Throughput of second server is $Thp_2(t) = \mu_2.U_2(t) =$

$$\mu_2\left[1 - exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_1}(-1)^{r_2+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_2}\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]\right] \quad (61)$$

Average waiting time of a customers in the second queue is $W_2(t)$

$$= \frac{L_2(t)}{Thp_2(t)} = \frac{\left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left(1-\left(\frac{\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t}}{(\mu_2-\mu_1)}\right)\right)\right]E(X)}{\left[1-exp\left[\lambda \sum_{m=1}^{\infty}\sum^{m}\sum_{r_1=1}^{r_1}\sum_{r_2=0}^{r_1}(-1)^{r_2+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_2}\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]\right]} \quad (62)$$

Variation of the number of customers in second queue is

$$V(z_2) = V_2(t) = \lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_2+\mu_1)t}}{\mu_2+\mu_1}\right)+\left(\frac{1-e^{-2\mu_2 t}}{\mu_2}\right)\right\}+\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_2 t}}{\mu_2}\right)\right\}\right]E(X) \quad (63)$$

Coefficient of variation of the number of customers in second queue $CV_2(t) = \frac{\sqrt{V_2(t)}}{L_2(t)}x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_2+\mu_1)t}}{\mu_2+\mu_1}\right)+\left(\frac{1-e^{-2\mu_2 t}}{\mu_2}\right)\right\}+\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_2 t}}{\mu_2}\right)\right\}\right]E(X)}}{\left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left(1-\frac{(\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t})}{(\mu_2-\mu_1)}\right)\right]E(X)}x100 \quad (64)$$

IV.D.PERFORMANCE ANALYSIS OF THIRD QUEUE

Putting $z_1 = 1, z_3 = 1, z_4 = 1$ in (48) we get probability generating function of third queue size distribution as $P(Z_3;t) =$

$$exp\left[\lambda \sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_3=0}^{r_1}(-1)^{r_3}C_m\binom{m}{r_1}\binom{r_1}{r_3}\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (65)$$

Mean number of customers in third queue is

$$E(N_3) = L_3(t) = \left[\left(\frac{\lambda\theta_2}{\mu_3}\right)\left\{1-\left(\frac{\mu_3 e^{-\mu_1 t}-\mu_1 e^{-\mu_3 t}}{\mu_3-\mu_1}\right)\right\}\right]E(X) \quad (66)$$

Where $E(X)$ is the mean of batch size arrivals at third queue and $E(X) = \sum_m m.C_m$

Putting $z_3 = 0$ we get the probability that the third queue is empty as

$P(.,.,0,.;t) =$

$$exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_3=0}^{r_1}(-1)^{r_3+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_3}\left\{\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_3)+\mu_3r_3)t}}{\mu_1(r_1-r_3)+\mu_3r_3}\right\}\right] \quad (67)$$

Utilization of third server is $U(t) = 1 - P(.,.,0,.;t) =$

$$1 - exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_3=0}^{r_1}(-1)^{r_3+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_3}\left\{\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_3)+\mu_3r_3)t}}{\mu_1(r_1-r_3)+\mu_3r_3}\right\}\right] \quad (68)$$

Throughput of third server is $Thp_3(t) = \mu_3.U_3(t) =$

$$\mu_3\left[1 - exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_3=0}^{r_1}(-1)^{r_3+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_3}\left\{\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_3)+\mu_3r_3)t}}{\mu_1(r_1-r_3)+\mu_3r_3}\right\}\right]\right] \quad (69)$$

Average waiting time of a customer in third queue is $W_3(t) = \frac{L_3(t)}{Thp_3(t)} =$

$$\frac{\left[\left(\frac{\lambda\theta_3}{\mu_3^2}\right)\left\{1-\frac{(\mu_3e^{-\mu_1t}-\mu_1e^{-\mu_3t})}{(\mu_3-\mu_1)}\right\}\right]E(X)}{\left[1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_3=0}^{r_1}(-1)^{r_3+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_3}\left\{\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_3)+\mu_3r_3)t}}{\mu_1(r_1-r_3)+\mu_3r_3}\right\}\right]\right]} \quad (70)$$

Variation of the number of customers in third queue is $V(z_3) = V_3(t) =$

$$\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_3+\mu_1)t}}{\mu_3+\mu_1}\right)+\left(\frac{1-e^{-2\mu_3t}}{\mu_3}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_3t}}{\mu_3}\right)\right\}E(X)\right] \quad (71)$$

Coefficient of variation of the number of customers in third queue is $CV_3(t) = \frac{\sqrt{V_3(t)}}{L_3(t)}x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_3+\mu_1)t}}{\mu_3+\mu_1}\right)+\left(\frac{1-e^{-2\mu_3t}}{\mu_3}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_3t}}{\mu_3}\right)\right\}E(X)\right]}}{\left[\left(\frac{\lambda\theta_3}{\mu_3}\right)\left\{1-\left(\frac{\mu_3e^{-\mu_1t}-\mu_1e^{-\mu_3t}}{\mu_3-\mu_1}\right)\right\}\right]E(X)}x100 \quad (72)$$

IV.E.PERFORMANCE ANALYSIS OF FOURTH QUEUE

Putting $z_1 = 1, z_2 = 1, z_3 = 1$ in (49)we get probability generating function of fourth queue size distribution as $P(Z_4;t)$

$=$

$$exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4}C_m\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_1(z_4-1)}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_4)+\mu_4r_4}\right\}\right] \quad (73)$$

Mean number of customers in fourth queue is

$$E(N_4) = L_4(t) = \left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left\{1-\left(\frac{\mu_4e^{-\mu_1t}-\mu_1e^{-\mu_4t}}{\mu_4-\mu_1}\right)\right\}\right]E(X) \quad (74)$$

Where $E(X)$ is the mean of batch size arrivals at fourth queue and $E(X) = \sum_m m.C_m$

Putting $z_4 = 0$ we get the probability that the fourth queue is empty as $P(.,.,.,0;t)$

$=$

$$exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_4)+\mu_4r_4}\right\}\right] \quad (75)$$

Utilization of fourth server is $U_4(t) = 1 - P(.,.,.,0;t)$

$= 1 -$

$$exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_4)+\mu_4r_4}\right\}\right] \quad (76)$$

Throughput of fourth server is $Thp_4(t) = \mu_4.U_4(t) =$

$$\mu_4\left[1 - exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_4)+\mu_4r_4}\right\}\right]\right] \quad (77)$$

Average waiting time of a customer in fourth queue is $W_4(t) = \frac{L_4(t)}{Thp_4(t)}$

$$= \frac{\left[\left(\frac{\lambda\theta_3}{\mu_4^2}\right)\left\{1-\frac{(\mu_4e^{-\mu_1t}-\mu_1e^{-\mu_4t})}{(\mu_4-\mu_1)}\right\}\right]E(X)}{\left[1-exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}C_m\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{(\mu_1(r_1-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_4)+\mu_4r_4}\right\}\right]\right]} \quad (78)$$

Variance of the number of customers in fourth queue is $V(z_4) = V_4(t)=$

$$\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_4+\mu_1)t}}{\mu_4+\mu_1}\right)+\left(\frac{1-e^{-2\mu_4t}}{\mu_4}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_4t}}{\mu_4}\right)\right\}E(X)\right] \quad (79)$$

Coefficient of variation of the number of customers in fourth queue $CV_4(t) = \frac{\sqrt{V_4(t)}}{L_4(t)}x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2\sum_{m=1}^{\infty}\binom{m}{2}.C_m\left\{\left(\frac{1-e^{-2\mu_1t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_4+\mu_1)t}}{\mu_4+\mu_1}\right)+\left(\frac{1-e^{-2\mu_4t}}{\mu_4}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)\left\{\left(\frac{1-e^{-\mu_1t}}{\mu_1}\right)+\left(\frac{1-e^{-\mu_4t}}{\mu_4}\right)\right\}E(X)\right]}}{\left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left\{1-\left(\frac{\mu_4e^{-\mu_1t}-\mu_1e^{-\mu_4t}}{\mu_4-\mu_1}\right)\right\}\right]E(X)}x100 \quad (80)$$

IV.F .NUMERICAL ILLUSTARTION WITH GEOMETRIC DISTRIBUTION

It is assumed that the customers arrive to the queue in batches of size m and the probability generating function of m is $C(Z) = \sum_{m=1}^{\infty}C_mz^m$. The performance of the model is influenced by the batch size arrival distribution. It is assumed that the customers in any arriving model is random and follows Geometric distribution with parameters A and p. Which means the number of customers in a batch follows Doubly Truncated Geometric distribution with parameter p, i.e., the probability that the batch consists of x customers is $\frac{q^{m-1}p}{1-q^A}$,where q =1-p .The probability mass function of Doubly Truncated Geometric batch size distribution is $C_m = \frac{q^{m-1}p}{1-q^A}, m = 1,2,..A$ and $0 < p < 1$.The mean of Geometric distribution is $E(X) = \frac{1-q^A-pAq^A}{p(1-q^A)}$.

Then the Joint Probability generating function of number of customers in first, second, third and fourth queues respectively at any time t $P(Z_1,Z_2,Z_3,Z_4;t) =$

$$exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}(-1)^{r_2}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\binom{r_3}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{(z_1-1)+\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}+\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}+\frac{\theta_3\mu_1(z_4-1)}{\mu_4-\mu_1}\right\}^{r_1-r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_2-r_3}\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_3-r_4}\left\{\frac{\theta_3\mu_4(z_4-1)}{\mu_4-\mu_1}\right\}^{r_4}\left\{\frac{1-e^{(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4r_4}\right\}\right] \quad (84)$$

Putting $z_1 = 0, z_2 = 0,....z_k = 0$ in (84) and expanding we get the probability that the

4-server system is empty at any time t as $P(0,0,0.0;t) =$

$$exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\sum_{r_4=0}^{r_3}(-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\binom{r_3}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left(1+\frac{\theta_1\mu_1}{\mu_2-\mu_1}+\frac{\theta_2\mu_1}{\mu_3-\mu_1}+..+\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_1-r_2}\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^{r_2-r_3}\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^{r_3-r_4}\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^{4}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4r_4)t}}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\mu_3(r_3-r_4)+\mu_4r_4}\right\}\right] \quad (85)$$

IV.G.PERFORMANCE MEASURES OF FIRST QUEUE

Putting $z_2 = 1, z_3 = 1, z_4 = 1$ in (84) we get probability generating function of first queue size as

$$P(Z_1;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \left(\frac{q^{m-1}p}{1-q^A}\right)\binom{m}{r_1}(z_1-1)^{r_1}\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (86)$$

Mean number of customers in first buffer is

$$L_1(t) = \left[\frac{\lambda}{\mu_1}\right](1-e^{-\mu_1 t})E[X] = \left[\frac{\lambda}{\mu_1}\right]\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}(1-e^{-\mu_1 t}) \quad (87)$$

Putting $Z_1 = 0$ in (86) we get the probability that the first queue is empty as

$$P(0,.,.,.;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1}\binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (88)$$

Utilization of first server is

$$U_1(t) = 1 - P(0,.,.,.;t) = 1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1}\binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right] \quad (89)$$

Throughput of first server is

$$Thp1(t) = \mu_1 . U_1(t) = \mu_1 . \left\{1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1}\binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\} \quad (90)$$

Average waiting time of a customer in first queue is

$$W_1(t) = \frac{L_1(t)}{Thp_1(t)} = \frac{\left[\frac{\lambda}{\mu_1^2}\right]\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}(1-e^{-\mu_1 t})}{\left\{1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1}\binom{m}{r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{1-e^{-\mu_1 r_1 t}}{\mu_1 r_1}\right\}\right]\right\}} \quad (91)$$

Variance of number of customers in first queue is

$$V(Z_1) = V_1(t) = \lambda \sum_{m=1}^{A} \left[\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) + \left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right] \quad (92)$$

Coefficient of variation of number of customers in first queue is

$$CV_1(t) = \frac{\sqrt{V_1(t)}}{L_1(t)} x100 = \frac{\sqrt{\lambda \sum_{m=1}^{A} \left[\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) + \left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)\right\}\right]}}{\left[\frac{\lambda}{\mu_1}\right]\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}(1-e^{-\mu_1 t})} x100 \quad (93)$$

IV.H.PERFORMANCE MEASURES OF SECOND QUEUE

Putting $z_1 = 1, z_3 = 1, z_4 = 1$ in (84)we get probability generating function of **SECOND** queue size distribution as

$$P(Z_2;t) = \exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2}\binom{m}{r_1}\binom{r_1}{r_2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (94)$$

Mean number of customers in **SECOND** queue is

$$L_2(t) = \left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{1-\left(\frac{\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t}}{\mu_2-\mu_1}\right)\right\}\right] \quad (95)$$

Probability that the **SECOND** queue is empty is $P(.,0,.,.;t) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (96)$$

Utilization of **SECOND** server is $U_2(t) = 1 - P(.,0,.,.;t) =$

$$1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (97)$$

Throughput of **SECOND** server is $Thp_2(t) = \mu_2 . U_2(t) =$

$$\mu_2\left[1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]\right] \quad (98)$$

Average waiting time of a customer in **SECOND** queue is

$$\frac{\left[\left(\frac{\lambda\theta_1}{\mu_2^2}\right)\left\{1-\left(\frac{\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t}}{\mu_2-\mu_1}\right)\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\right]}{\left[1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_2)+\mu_2 r_2)t}}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]\right]} \quad (99)$$

Variance of number of customers in **SECOND** queue is $V_2(t) =$

$$\lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2 \sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right).\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_2+\mu_1)t}}{\mu_2+\mu_1}\right) + \left(\frac{1-e^{-2\mu_2 t}}{\mu_2}\right)\right\} + \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right) - \left(\frac{1-e^{-\mu_2 t}}{\mu_2}\right)\right\}\right] \quad (100)$$

Coefficient of variation of number of customers in **SECOND** queue $CV_2(t) = \frac{\sqrt{V_2(t)}}{L_2(t)} x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2 \sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right).\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_2+\mu_1)t}}{\mu_2+\mu_1}\right) + \left(\frac{1-e^{-2\mu_2 t}}{\mu_2}\right)\right\} + \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right) - \left(\frac{1-e^{-\mu_2 t}}{\mu_2}\right)\right\}\right]}}{\left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{1-\left(\frac{\mu_2 e^{-\mu_1 t}-\mu_1 e^{-\mu_2 t}}{\mu_2-\mu_1}\right)\right\}\right]} x100 \quad (101)$$

IV.I.PERFORMANCE ANALYSIS OF THIRD QUEUE

Putting $z_1 = 1, z_2 = 1, z_4 = 1$ in (84)we get probability generating function of **THIRD** queue size distribution as

$$P(Z_3;t) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3}\binom{m}{r_1}\binom{r_1}{r_3}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (102)$$

Mean number of customers in **THIRD** queue is $L_3(t) =$

$$\left[\left(\frac{\lambda\theta_2}{\mu_3}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{1-\left(\frac{\mu_3 e^{-\mu_1 t}-\mu_1 e^{-\mu_3 t}}{\mu_3-\mu_1}\right)\right\}\right] \quad (103)$$

Probability that the **THIRD** queue is empty is $P(.,.,0,.;t) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1}\binom{m}{r_1}\binom{r_1}{r_3}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (104)$$

Utilization of **THIRD** server is $U_3(t) = 1 - P(.,.,0,.;t) =$

$$1 - \exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1}\binom{m}{r_1}\binom{r_1}{r_3}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (105)$$

Throughput of **THIRD** server is $Thp_3(t) = \mu_3 . U_3(t) =$

$$\mu_3 . \left[1 - \exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1}\binom{m}{r_1}\binom{r_1}{r_3}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right]\right] \quad (106)$$

Average waiting time of a customer in **THIRD** queue is

$$\frac{\left[\left(\frac{\lambda\theta_1}{\mu_3^2}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{1-\left(\frac{\mu_3 e^{-\mu_1 t}-\mu_1 e^{-\mu_3 t}}{\mu_3-\mu_1}\right)\right\}\right]}{\left[1-\exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1}\binom{m}{r_1}\binom{r_1}{r_3}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_3)+\mu_3 r_3)t}}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right]\right]} \quad (107)$$

Variance of number of customers in **THIRD** queue is

$$V_3(t) = \lambda\left[\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^2 \sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right).\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right) - 4\left(\frac{1-e^{-(\mu_3+\mu_1)t}}{\mu_3+\mu_1}\right) + \left(\frac{1-e^{-2\mu_3 t}}{\mu_3}\right)\right\} + \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right) - \left(\frac{1-e^{-\mu_3 t}}{\mu_3}\right)\right\}\right] \quad (108)$$

Coefficient of variation of number of customers in

THIRD queue is $CV_3(t) = \frac{\sqrt{V_3(t)}}{L_3(t)} x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)^2 \sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_3+\mu_1)t}}{\mu_3+\mu_1}\right)+\left(\frac{1-e^{-2\mu_3 t}}{\mu_3}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_3-\mu_1}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)-\left(\frac{1-e^{-\mu_3 t}}{\mu_3}\right)\right\}\right]}}{\left[\left(\frac{\lambda\theta_3}{\mu_3}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left[1-\left(\frac{\mu_3 e^{-\mu_1 t}-\mu_1 e^{-\mu_3 t}}{\mu_3-\mu_1}\right)\right]\right]} x100$$

(109)

IV.J.PERFORMANCE ANALYSIS OF FOURTH QUEUE

Putting $z_1 = 1, z_2 = 1, z_3 = 1$ in (84)we get probability generating function of **FOURTH** queue size distribution as $P(Z_4; t) =$

$$exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4}\binom{m}{r_1}\binom{r_1}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_3\mu_1(z_4-1)}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]$$

(110)

Mean number of customers in **FOURTH** queue is

$$L_4(t) = \left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{1-\left(\frac{\mu_4 e^{-\mu_1 t}-\mu_1 e^{-\mu_4 t}}{\mu_4-\mu_1}\right)\right\}\right] \quad (111)$$

Probability that the **FOURTH** queue is empty is $P(.,.,.,.,0;t)=$

$$exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}\binom{m}{r_1}\binom{r_1}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]$$

(112)

Utilization of **FOURTH** server is $U_4(t) = 1 - P(.,.,.,.,0;t) = 1 -$

$$-exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}\binom{m}{r_1}\binom{r_1}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]$$

(113)

Throughput of **FOURTH** server is $Thp_4(t) = \mu_4 . U_4(t) =$

$$\mu_4\left[1-exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}\left(\frac{q^{m-1}p}{1-q^A}\right)\binom{m}{r_1}\binom{r_1}{r_4}\left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]\right]$$

(114)

Average waiting time of a customer in **FOURTH** queue is $W_4(t) = \frac{L_4(t)}{Thp_4(t)} =$

$$\frac{\left[\left(\frac{\lambda\theta_3}{\mu_4^2}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left[1-\left(\frac{\mu_4 e^{-\mu_1 t}-\mu_1 e^{-\mu_4 t}}{\mu_4-\mu_1}\right)\right]\right]}{\left[1-exp\left[\lambda\sum_{m=1}^{A}\sum_{r_1=1}^{m}\sum_{r_4=0}^{r_1}(-1)^{r_4+r_1}\binom{m}{r_1}\binom{r_1}{r_4}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1}\left\{\frac{1-e^{-(\mu_1(r_1-r_4)+\mu_4 r_4)t}}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]\right]}$$

(115)

Variance of number of customers in **FOURTH** queue is

$$V_4(t) = \lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2\sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right).\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_4+\mu_1)t}}{\mu_4+\mu_1}\right)+\left(\frac{1-e^{-2\mu_4 t}}{\mu_4}\right)\right\}+\left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)-\left(\frac{1-e^{-\mu_4 t}}{\mu_4}\right)\right\}\right]$$

(116)

Coefficient of variation of number of customers in **FOURTH** queue $CV_4(t) = \frac{\sqrt{V_4(t)}}{L_4(t)} x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2\sum_{m=1}^{A}\binom{m}{2}\left(\frac{q^{m-1}p}{1-q^A}\right)\left\{\left(\frac{1-e^{-2\mu_1 t}}{\mu_1}\right)-4\left(\frac{1-e^{-(\mu_4+\mu_1)t}}{\mu_4+\mu_1}\right)+\left(\frac{1-e^{-2\mu_4 t}}{\mu_4}\right)\right\}+\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left\{\left(\frac{1-e^{-\mu_1 t}}{\mu_1}\right)-\left(\frac{1-e^{-\mu_4 t}}{\mu_4}\right)\right\}\right]}}{\left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left[1-\left(\frac{\mu_4 e^{-\mu_1 t}-\mu_1 e^{-\mu_4 t}}{\mu_4-\mu_1}\right)\right]\right]} x100$$

(117)

V. NUMERICAL ILLUSTRATION

The transient behaviour of the model is studied by considering Geometric batch size arrival distribution and the performance measures are calculated by varying system parameters as

$t = 0.1, 0.2, 0.3, 0.4, 0.5;\ \lambda = 10, 11, 12, 13, 14;\ \mu_i = 10, 11, 12, 13, 14;\ i = 1,2,3,4;\ \theta_j = 0.1, 0.2, 0.3, 0.4, 0.5; j = 1,2,\ p = 0.1, 0.2, 0.3, 0.4, 0.5\ and\ A = 10, 15, 20, 25, 30.$

The mean number of customers in each buffer L_1, L_2, L_3, L_4 is calculated along with mean number of customers $L(t)$ in the entire system by varying the parameters $t, \lambda, \mu_1, \mu_2, \mu_3, \mu_4, \theta_1, \theta_2, \theta_3$ one at atime keeping all other fixed and the calculations are recorded in Table1.The probability of emptiness of each server and also the utilization of servers are calculated correspondingly for each value of parameters as above and the values are tabulated in Table2.The throughputs of four servers $Thp_1, Thp_2, Thp_3, Thp_4$ along with average waiting times of customers in four buffers W_1, W_2, W_3, W_4 are also computed and tabulated in Table3.The Variance of the number of customers V_1, V_2, V_3, V_4 along with coefficient of variation of the number of customers in each queue are calculated and the values are tabulated in Table 4.

From Table1. It is observed that as time t increasing from 0.1 to 0.5 the mean number of customers in each buffer is also increasing. The same phenomenon is reflected in mean number of customers in the entire system. Also as the service rate of first server μ_1 is increasing from 10 to 14 keeping μ_2, μ_3, μ_4 unaltered the mean number of customers in first server $L_1(t)$ is decreasing , the mean number of customers in the remaining queues are increasing and mean number of customers in the entire system $L(t)$ is decreasing. Similarly when μ_2 is increasing L_2(t) decreasing , μ_3 is increasing L_3 (t) decreasing with no change in the other queues measures. The same phenomenon can be observed with the fourth queue. Thus the improvement in performance of first server improves the performance of entire system. In the same pattern when the probability $\theta_1 (or\ \theta_2)$ that the customers from first server join second (or third server) increases the buffer size at second server $L_2(t) (or$ at third server $L_3(t)$) is increasing correspondingly where as at fourth server it is decreasing. As the batch size distribution parameters 'A' is increasing then $L_1(t), L_2(t), L_3(t)\ and\ L_4(t)$ are increasing where as for the distribution parameter 'p' the phenomenon goes inversely..

Table 2. indicates that the probability of emptiness has shown decrease with respect to increase in time. In particular it has sudden decrease when t moves from 0.1 to 0.2 and decreasing normally thereafter when t=0.2,0.3,0.4,0.5. Similarly with increase in mean arrival rate λ the probability of emptiness at each server decreases while the utilizations of servers U_1, U_2, U_3, U_4 increase. This clearly indicates that the system performs in accordance with time. As the service rate μ_1 increases from $10\ to\ 14$ the probability of emptiness at first service station increases while utilization of first server decreases where as the probability of emptiness at other service stations decrease and utilization increase. The probability of emptiness of the system increase as the service rates $\mu_1, \mu_2, \mu_3, \mu_4$ increase. Similarly the probability of emptiness decreases as the probability of customers joining a particular server increases while it's utilization gets increased. Thus as θ_1 increases from 0.1 to 0.5 system emptiness decreases marginally from 0.2281 to 0.2277 and probability of emptiness of second server decreases from 0.8991 to 0.6454.This has an impact on the fourth server since the joining probability of fourth queue is directly dependent on $\theta_3 and\ \theta_2$ ($\theta_3 = 1 - \theta_1 - \theta_2$) .Therefore the probability of emptiness at fourth server increases from 0.5831 to 0.7601 and it's utilization decreases from 0.4169 to 0.2399.Similarly as θ_2 increase from 0.1 to 0.5 the probability of emptiness of third server decreases from 0.9018 to 0.6518 and it's utilization increases from 0.0982 to 0.3482.

Also the probability of emptiness of fourth server increases from 0.5530 to 0.7049 and utilization decreases from 0.4470 to 0.2951. It is observed that as the batch size distribution parameter 'A' increases the probability of emptiness of the system increases where as at servers it decrease. As the other batch size distribution parameter 'p' is increasing the probability of emptiness of the system and servers increase and utilization of each server decrease as well.

From Table.3 it is observed that the throughputs $Thp_1, Thp_2, Thp_3, Thp_4$ and mean waiting times of customer at each of the queues W_1, W_2, W_3, W_4 have shown increase with increase in time. Similarly an increase in λ led to an increase in throughputs as well as mean waiting times. Further it is observed that the increase in service rates at second, third and fourth servers led to increase in throughputs and waiting times except at first server. It is also observed that the increase in μ_1 leads to increase in $Thp_1, Thp_2, Thp_3, Thp_4$ and decrease in W_1, W_2, W_3, W_4. As the probability of joining second queue increases from θ_1=0.1 to 0.5 the throughput Thp_2 increases correspondingly from 0.7063 to 2.4822,this in turn increase the waiting time W_2 from 0.1587 to 0.2259.As this influences on $\theta_3$which decreases from 0.7 to 0.3, the throughput Thp_4 decreases from 3.7521 to 2.1951 while mean waiting time W_4 decreases from 0.1968 to 0.1466. Similar phenomenon is observed with variation in θ_2, the probability of joining third queue after being served at first queue .

From Table.4.it is observed that with increase in time the variance of the number of customers in first, second , third and fourth queues increase and the coefficient of variation decrease. Similarly the increase in λ leads to increase in variance V_1, V_2, V_3, V_4 in each of the four queues and decrease in coefficient of variation CV_1, CV_2, CV_3, CV_4 at the four queues. Further we observe that the increase in service rates μ_2, μ_3, μ_4 led to decrease in variance of each queue and increase in coefficient of variation On the other hand the increase in μ_1 leads to decrease in V_1 and increase in V_2, V_3, V_4 .As the probability of joining the second or third queue, θ_1 or θ_2 increases the variances in second or third queues increases where as the variance in fourth queue decreases. It is also observed as the batch size parameter 'A' increases the variances V_1, V_2, V_3, V_4 also increase but the increase in probability 'p' leads to decrease in variances V_1, V_2, V_3, V_4.

VI. SENSITIVITY ANALYSIS

In this section the values of parameters as t=0.1, λ=15, μ_1=12, μ_2=14, μ_3=11, μ_4=13, θ_1=0.3, θ_2=0.2, $p = 0.5$ *and* $A = 20$ is considered and sensitivity of the model is analysed.. The effect of varying the parameters on performance measures $L_1, L_2, L_3, L_4, L, W_1, W_2, W_3$ *and* W_4 with change of $\mp$15%, $\mp$10% and $\mp$5% was computed and are presented in Table 5.

From Table 5. It is observed that as time t increases the values of $L_1, L_2, L_3, L_4, L, W_1, W_2, W_3$ and W_4 also increase. The same phenomenon is observed with variation in arrival rate λ .It is also observed that as μ_1 increases L_1, L, W_1 decrease whereas L_2, L_3, L_4, W_2, W_3 and W_4 increase. It is observed that as μ_2 increases L_2, L and W_2 decrease whereas L_3, L_4, W_3 and W_4 remain constant , When μ_3 increases L_3, L, W_3 decrease whereas L_1, L_2, L_4, W_1, W_2 and W_4 remain constant .When μ_3 increases L_3, L, W_3 decrease whereas L_1, L_2, L_4, W_1, W_2 and W_4 remain constant .Similar phenomenon is observed with μ_4 .We also observed that with increase in θ_1 the performance measures L_2, L, W_2 are increasing , L_4, W_4 are decreasing whereas L_1, L_3, W_1 and W_3 remain constant .Similarly with increase in θ_2 the performance measures L_3, L, W_3 are increasing , L_4 and W_4 are decreasing whereas L_1, L_2, W_1 and W_2 remain constant. We also observed that with increase in batch size distribution parameters ' A' the performance measures $L_1, L_2, L_3, L_4, L, W_1, W_2, W_3$ and W_4 increase. and decrease with increase in probability 'p' .

VII. STEADY STATE ANALYSIS

In this section we study the steady-state analysis of queuing model .The Joint Probability generating function of number of customers in first, second,….kth queues respectively in steady state is

$$\lim_{t\to\infty} P(Z_1, Z_2, \dots Z_k; t) = P(Z_1, Z_2, \dots Z_k) =$$
$$exp\left[\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\dots\sum_{r_k=0}^{r_{k-1}}(-1)^{r_2}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\dots\binom{r_{k-1}}{r_k}C_m\right.$$
$$\left\{(z_1-1)+\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}+\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}+\dots+\frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_1-r_2}\left\{\frac{\theta_1\mu_1(z_2-1)}{\mu_2-\mu_1}\right\}^{r_2-r_3}$$
$$\left\{\frac{\theta_2\mu_1(z_3-1)}{\mu_3-\mu_1}\right\}^{r_3-r_4}\dots\left\{\frac{\theta_{k-2}\mu_1(z_{k-1}-1)}{\mu_{k-1}-\mu_1}\right\}^{r_{k-1}-r_k}\left\{\frac{\theta_{k-1}\mu_1(z_k-1)}{\mu_k-\mu_1}\right\}^{r_k}$$
$$\left.\left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\dots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right] \quad (118)$$

VII.A.CHARACTERISTICS OF THE MODEL UNDER EQUILIBRIUM

Putting $z_1 = 0, z_2 = 0, \dots. z_k = 0$ in (118) and expanding we get the probability that the k-server system is empty in steady state as $P(0,0,..0) =$

$$exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{r_2}\dots\sum_{r_k=0}^{r_{k-1}}(-1)^{r_2+r_1}\binom{m}{r_1}\binom{r_1}{r_2}\binom{r_2}{r_3}\dots\binom{r_{k-1}}{r_k}C_m\right.$$
$$\left(1+\frac{\theta_1\mu_1}{\mu_2-\mu_1}+\frac{\theta_2\mu_1}{\mu_3-\mu_1}+\dots+\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_1-r_2}\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^{r_2-r_3}\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^{r_3-r_4}\dots$$
$$\left.\left(\frac{\theta_{k-2}\mu_1}{\mu_{k-1}-\mu_1}\right)^{r_{k-1}-r_k}\left(\frac{\theta_{k-1}\mu_1}{\mu_k-\mu_1}\right)^{r_k}\left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2(r_2-r_3)+\dots+\mu_{k-1}(r_{k-1}-r_k)+\mu_k r_k}\right\}\right] \quad (119)$$

VII.B.PERFORMANCE ANALYSIS OF FIRST QUEUE

Putting $z_2 = 1, z_3 = 1, \dots. z_k = 1$ in (118) we get probability generating function of first queue size as

$$P(Z_1) = exp\left[\lambda\sum_{m=1}^{\infty}\sum_{r_1=1}^{m}C_m\binom{m}{r_1}(z_1-1)^{r_1}\left\{\frac{1}{\mu_1 r_1}\right\}\right] \quad (120)$$

Mean number of customers in first queue is

$$E(N_1) = L_1 = \frac{\lambda}{\mu_1}E(X) \quad (121)$$

Where E(X) is the mean of batch size arrivals to first queue and is given by $E(X) = \sum_{m=1}^{\infty} m.C_m$

Putting $Z_1 = 0$ in (120) we get the probability that the first queue is empty as

$$P(0,.,.,.,.) = exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_1=1}^{m} C_m \binom{m}{r_1} \left\{\frac{(-1)^{r_1}}{\mu_1 r_1}\right\}\right] \quad (122)$$

Utilization of first server is

$$U_1 = 1 - P(0,.,.,.,.) = 1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_1=1}^{m} C_m \binom{m}{r_1} \left\{\frac{(-1)^{r_1}}{\mu_1 r_1}\right\}\right] \quad (123)$$

Throughput of first server is

$$Thp1 = \mu_1 . U_1 = \mu_1 \left\{1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_1=1}^{m} C_m \binom{m}{r_1} \left\{\frac{(-1)^{r_1}}{\mu_1 r_1}\right\}\right]\right\} \quad (124)$$

Average waiting time customers of a customer in first queue is

$$W_1 = \frac{L_1}{Thp_1} = \frac{\left(\frac{\lambda}{\mu_1^2}\right) E(X)}{\left\{1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_1=1}^{m} C_m \binom{m}{r_1} \left\{\frac{(-1)^{r_1}}{\mu_1 r_1}\right\}\right]\right\}} \quad (125)$$

Variance of the number of customers in first queue is

$$V(Z_1) = V_1 = \left(\frac{2\lambda}{\mu_1}\right) \sum_{m=2}^{\infty} C_m \binom{m}{2} \quad (126)$$

Coefficient of variation of the number of customers in first queue

$$CV_1 = \frac{\sqrt{V_1}}{L_1} x100 = \frac{\sqrt{\left(\frac{2\lambda}{\mu_1}\right) \sum_{m=2}^{\infty} C_m \binom{m}{2}}}{\frac{\lambda}{\mu_1} E(X)} x100 \quad (127)$$

VII.C. PERFORMANCE ANALYSIS OF ith QUEUE FOR i =2,3,..k

Putting $z_1 = 1, z_2 = 1, z_3 = 1, .. z_{i-1} = 1$ in (118) we get probability generating function of ith queue size distribution as

$$P(Z_i) = exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_i=1}^{m} \sum_{r_i=0}^{r_i} (-1)^{r_i} C_m \binom{m}{r_i} \binom{r_i}{r_i} \left\{\frac{\theta_{i-1}\mu_1 (z_i - 1)}{\mu_i - \mu_i}\right\}^{r_i} \left\{\frac{1}{\mu_i (r_i - r_i) + \mu_i r_i}\right\}\right] \quad (128)$$

Mean number of customers in ith queue is

$$E(N_i) = L_i = \left(\frac{\lambda \theta_{i-1}}{\mu_i}\right) E(X) \quad (129)$$

Where $E(X)$ is the mean of batch size arrivals at ith queue and $E(X) = \sum_m m.C_m$

Putting $n_i = 0$ in we get the probability that the ith buffer is empty as

$$P(.,.,0,..) = exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_i=1}^{m} \sum_{r_i=0}^{r_i} (-1)^{r_i + r_i} C_m \binom{m}{r_i} \binom{r_i}{r_i} \left\{\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right\}^{r_i} \left\{\frac{1}{\mu_i (r_i - r_i) + \mu_i r_i}\right\}\right] \quad (130)$$

Utilization of ith server is $U_i = 1 - P(.,.,.,.,0;t) =$

$$1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_i=1}^{m} \sum_{r_i=0}^{r_i} (-1)^{r_i + r_i} C_m \binom{m}{r_i} \binom{r_i}{r_i} \left\{\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right\}^{r_i} \left\{\frac{1}{\mu_i (r_i - r_i) + \mu_i r_i}\right\}\right] \quad (131)$$

Throughput of ith server is $Thp_i = \mu_i . U_i =$

$$\mu_i \left[1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_i=1}^{m} \sum_{r_i=0}^{r_i} (-1)^{r_i + r_i} C_m \binom{m}{r_i} \binom{r_i}{r_i} \left\{\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right\}^{r_i} \left\{\frac{1}{\mu_i (r_i - r_i) + \mu_i r_i}\right\}\right]\right] \quad (132)$$

Average waiting time of a customer in ith queue is $W_i = \frac{L_i}{Thp_i} =$

$$\frac{\left(\frac{\lambda \theta_{i-1}}{\mu_i^2}\right) E(X)}{\left[1 - exp\left[\lambda \sum_{m=1}^{\infty} \sum_{r_i=1}^{m} \sum_{r_i=0}^{r_i} (-1)^{r_i + r_i} C_m \binom{m}{r_i} \binom{r_i}{r_i} \left\{\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right\}^{r_i} \left\{\frac{1}{\mu_i (r_i - r_i) + \mu_i r_i}\right\}\right]\right]} \quad (133)$$

Variation of the number of customers in ith queue is $V(z_i) = V_i =$

$$\lambda \left[\left(\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right)^2 \sum_{m=1}^{\infty} \binom{m}{2} . C_m \left\{\left(\frac{1}{\mu_i}\right) - \left(\frac{4}{\mu_i + \mu_i}\right) + \left(\frac{1}{\mu_i}\right)\right\} + \left(\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right) \left\{\left(\frac{1}{\mu_i}\right) + \left(\frac{1}{\mu_i}\right)\right\} E(X)\right] \quad (134)$$

Coefficient of variation in number of the number of customers in ith queue $CV_i = \frac{\sqrt{V_i}}{L_i} x100 =$

$$\frac{\sqrt{\lambda \left[\left(\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right)^2 \sum_{m=1}^{\infty} \binom{m}{2} C_m \left\{\left(\frac{1}{\mu_i}\right) - \left(\frac{4}{\mu_i + \mu_i}\right) + \left(\frac{1}{\mu_i}\right)\right\} + \left(\frac{\theta_{i-1}\mu_1}{\mu_i - \mu_i}\right) \left\{\left(\frac{1}{\mu_i}\right) + \left(\frac{1}{\mu_i}\right)\right\} E(X)\right]}}{\left(\frac{\lambda \theta_{i-1}}{\mu_i}\right) E(X)} x100 \quad (135)$$

VII.D.PERFORMANCE MEASURES OF THE STAEDY STATE 4-SERVER MODEL WHEN BATCH SIZE DISTRIBUTION IS GEOMETRIC

The Joint Probability generating function of number of customers in first, second, third and fourth queues respectively in steady state is $P(Z_1, Z_2, Z_3, Z_4) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \sum_{r_4=0}^{r_3} (-1)^{r_2} \binom{m}{r_1} \binom{r_1}{r_2} \binom{r_2}{r_3} \binom{r_3}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{(z_1 - 1) + \frac{\theta_1\mu_1(z_2 - 1)}{\mu_2 - \mu_1} + \frac{\theta_2\mu_1(z_3 - 1)}{\mu_3 - \mu_1} + \frac{\theta_3\mu_1(z_4 - 1)}{\mu_4 - \mu_1}\right\}^{r_1 - r_2} \left\{\frac{\theta_1\mu_1(z_2 - 1)}{\mu_2 - \mu_1}\right\}^{r_2 - r_3} \left\{\frac{\theta_2\mu_1(z_3 - 1)}{\mu_3 - \mu_1}\right\}^{r_3 - r_4} \left\{\frac{\theta_3\mu_4(z_4 - 1)}{\mu_4 - \mu_1}\right\}^{r_4} \left\{\frac{1}{\mu_1(r_1 - r_2) + \mu_2(r_2 - r_3) + \mu_3(r_3 - r_4) + \mu_4 r_4}\right\}\right] \quad (136)$$

VII.E.CHARACTERISTICS OF THE MODEL

Putting $z_1 = 0, z_2 = 0, ... z_k = 0$ in (135) and expanding we get the probability that the 4-server system is empty in steady state $P(0,0,0,0)$=

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \sum_{r_4=0}^{r_3} (-1)^{r_2 + r_1} \binom{m}{r_1} \binom{r_1}{r_2} \binom{r_2}{r_3} \binom{r_3}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left(1 + \frac{\theta_1\mu_1}{\mu_2 - \mu_1} + \frac{\theta_2\mu_1}{\mu_3 - \mu_1} + .. + \frac{\theta_{k-1}\mu_1}{\mu_k - \mu_1}\right)^{r_1 - r_2} \left(\frac{\theta_1\mu_1}{\mu_2 - \mu_1}\right)^{r_2 - r_3} \left(\frac{\theta_2\mu_1}{\mu_3 - \mu_1}\right)^{r_3 - r_4} \left(\frac{\theta_3\mu_1}{\mu_4 - \mu_1}\right)^4 \left\{\frac{1}{\mu_1(r_1 - r_2) + \mu_2(r_2 - r_3) + \mu_3(r_3 - r_4) + \mu_4 r_4}\right\}\right] \quad (137)$$

VII.F.PERFORMANCE ANALYSIS OF FIRST QUEUE

Putting $z_2 = 1, z_3 = 1, z_4 = 1$ in (136) we get probability generating function of first queue size as

$$P(Z_1) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \left(\frac{q^{m-1}p}{1-q^A}\right) \binom{m}{r_1} (z_1 - 1)^{r_1} \left\{\frac{1}{\mu_1 r_1}\right\}\right] \quad (138)$$

Mean number of customers in first buffer is

$$L_1 = \left[\frac{\lambda}{\mu_1}\right] \left\{\frac{1 - q^A - \mu A q^A}{p(1-q^A)}\right\} \quad (139)$$

Putting $Z_1 = 0$ in (138) we get the probability that the first queue is empty as

$$P(0,.,.,.,.) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1 r_1}\right\}\right] \quad (140)$$

Utilization of first server is

$$U_1 = 1 - P(0,.,.,.,.) = 1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1 r_1}\right\}\right] \quad (141)$$

Throughput of first server is

$$\text{Thp1} = \mu_1 . U_1 = \mu_1 . \left\{1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1 r_1}\right\}\right]\right\} \quad (142)$$

Average waiting time of a customer in first queue is

$$W_1 = \frac{L_1}{Thp_1} = \frac{\left[\frac{\lambda}{\mu_1^2}\right]\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]}{\left\{1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} (-1)^{r_1} \binom{m}{r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \left[\frac{1}{\mu_1 r_1}\right]\right]\right\}} \quad (143)$$

Variance of number of customers in first queue is

$$V_1 = \lambda \sum_{m=1}^{A} \left[\binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left(\frac{2}{\mu_1}\right)\right] \quad (144)$$

Coefficient of variation of number of customers in first queue is

$$CV_1 = \frac{\sqrt{V_1}}{L_1} x100 = \frac{\sqrt{\lambda \sum_{m=1}^{A} \left[\binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left(\frac{2}{\mu_1}\right)\right]}}{\left[\frac{\lambda}{\mu_1}\right]\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]} x100 \quad (145)$$

VII.G..PERFORMANCE ANALYSIS OF SECOND QUEUE

Putting $z_1 = 1, z_3 = 1, z_4 = 1$ in (136)we get probability generating function of **SECOND** queue size distribution as

$$P(Z_2) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2} \binom{m}{r_1} \binom{r_1}{r_2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_1 \mu_1 (z_2-1)}{\mu_2-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (146)$$

Mean number of customers in **SECOND** queue is

$$L_2 = \left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\right] \quad (147)$$

Probability that the **SECOND** queue is empty is $P(.,0,..,.) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1} \binom{m}{r_1} \binom{r_1}{r_2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (148)$$

Utilization of **SECOND** server is $U_2 = 1 - P(.,0,..,.)$

$$= 1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1} \binom{m}{r_1} \binom{r_1}{r_2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right] \quad (149)$$

Throughput of **SECOND** server is $Thp_2 = \mu_2 . U_2 =$

$$\mu_2 \left[1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1} \binom{m}{r_1} \binom{r_1}{r_2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_2)+\mu_2 r_2}\right\}\right]\right] \quad (150)$$

Average waiting time of a customer in **SECOND** queue

$$W_2 = \frac{L_2}{Thp_2} = \frac{\left[\left(\frac{\lambda\theta_1}{\mu_2^2}\right)\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]\right]}{\left[1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_2=0}^{r_1} (-1)^{r_2+r_1} \binom{m}{r_1} \binom{r_1}{r_2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left[\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right]^{r_1} \left[\frac{1}{\mu_1(r_1-r_2)+\mu_2 r_2}\right]\right]\right]} \quad (151)$$

Variance of number of customers in **SECOND** queue is $V_2 =$

$$\lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) . \left\{\frac{1}{\mu_1} - \frac{4}{\mu_2+\mu_1} + \frac{1}{\mu_2}\right\} + \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_2}\right\}\right] \quad (152)$$

Coefficient of variation of number of customers in **SECOND** queue is

$$CV_2 = \frac{\sqrt{V_2}}{L_2} x100 = \frac{\sqrt{\lambda\left[\left(\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1} - \frac{4}{\mu_2+\mu_1} + \frac{1}{\mu_2}\right\} + \left\{\frac{\theta_1\mu_1}{\mu_2-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_2}\right\}\right]}}{\left[\left(\frac{\lambda\theta_1}{\mu_2}\right)\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]\right]} x100 \quad (153)$$

VII.H.PERFORMANCE ANALYSIS OF THIRD QUEUE

Putting $z_1 = 1, z_2 = 1, z_4 = 1$ in (136)we get probability generating function of **THIRD** queue size distribution as

$$P(Z_3) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3} \binom{m}{r_1} \binom{r_1}{r_3} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_2\mu_1 (z_3-1)}{\mu_3-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (154)$$

Mean number of customers in **THIRD** queue is

$$L_3 = \left[\left(\frac{\lambda\theta_2}{\mu_3}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\right] \quad (155)$$

Probability that the **THIRD** queue is empty is

$$P(.,.,0,.) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1} \binom{m}{r_1} \binom{r_1}{r_3} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (156)$$

Utilization of **THIRD** server is $U_3 = 1 - P(.,.,0,.)$

$$= 1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \binom{m}{r_1} \binom{r_1}{r_3} \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right] \quad (157)$$

Throughput of **THIRD** server is $Thp_3 = \mu_3 . U_3 =$

$$\mu_3 . \left[1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1} \binom{m}{r_1} \binom{r_1}{r_3} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_3)+\mu_3 r_3}\right\}\right]\right] \quad (158)$$

Average waiting time of a customer in **THIRD** queue is $W_3 = \frac{L_3}{Thp_3} =$

$$\frac{\left[\left(\frac{\lambda\theta_2}{\mu_3^2}\right)\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]\right]}{\left[1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_3=0}^{r_1} (-1)^{r_3+r_1} \binom{m}{r_1} \binom{r_1}{r_3} \left(\frac{q^{m-1}p}{1-q^A}\right) \left[\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right]^{r_1} \left[\frac{1}{\mu_1(r_1-r_3)+\mu_3 r_3}\right]\right]\right]} \quad (159)$$

Variance of number of customers in **THIRD** queue is $V_3(t) =$

$$\lambda\left[\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) . \left\{\frac{1}{\mu_1} - \frac{4}{\mu_3+\mu_1} + \frac{1}{\mu_3}\right\} + \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_3}\right\}\right] \quad (160)$$

Coefficient of variation of number of customers in **THIRD** queue is

$$CV_3 = \frac{\sqrt{V_3}}{L_3} x100 = \frac{\sqrt{\lambda\left[\left(\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1} - \frac{4}{\mu_3+\mu_1} + \frac{1}{\mu_3}\right\} + \left\{\frac{\theta_2\mu_1}{\mu_3-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_3}\right\}\right]}}{\left[\left(\frac{\lambda\theta_2}{\mu_3}\right)\left[\frac{1-q^A-pAq^A}{p(1-q^A)}\right]\right]} x100 \quad (161)$$

VII.I.PERFORMANCE ANALYSIS OF FOURTH QUEUE

Putting $z_1 = 1, z_2 = 1, z_3 = 1$ in (136)we get probability generating function of **FOURTH** queue size distribution as

$$P(Z_4) = exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_4=0}^{r_1} (-1)^{r_4} \binom{m}{r_1} \binom{r_1}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_3\mu_1 (z_4-1)}{\mu_4-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right] \quad (162)$$

Mean number of customers in **FOURTH** queue is

$$L_4 = \left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\right] \quad (163)$$

Probability that the **FOURTH** queue is empty is $P(.,.,.,.,0) =$

$$exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_4=0}^{r_1} (-1)^{r_4+r_1} \binom{m}{r_1} \binom{r_1}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right] \quad (164)$$

Utilization of **FOURTH** server is $U_4 = 1 - P(.,.,.,.,0) =$

$$1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_4=0}^{r_1} (-1)^{r_4+r_1} \binom{m}{r_1} \binom{r_1}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right] \quad (165)$$

Throughput of **FOURTH** server is $Thp_4 = \mu_4 . U_4 =$

$$\mu_4 . \left[1 - exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_4=0}^{r_1} (-1)^{r_4+r_1} \left(\frac{q^{m-1}p}{1-q^A}\right) \binom{m}{r_1} \binom{r_1}{r_4} \left\{\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right\}^{r_1} \left\{\frac{1}{\mu_1(r_1-r_4)+\mu_4 r_4}\right\}\right]\right] \quad (166)$$

Average waiting time of a customer in **FOURTH** queue is $W_4 = \frac{L_4}{Thp_4} =$

$$\frac{\left[\left(\frac{\lambda\theta_3}{\mu_4^2}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\right]}{\left[1-exp\left[\lambda \sum_{m=1}^{A} \sum_{r_1=1}^{m} \sum_{r_4=0}^{r_1} (-1)^{r_4+r_1} \binom{m}{r_1} \binom{r_1}{r_4} \left(\frac{q^{m-1}p}{1-q^A}\right) \left(\frac{\theta_3\mu_4}{\mu_4-\mu_1}\right)^{r_1} \left(\frac{1}{\mu_1(r_1-r_4)+\mu_4 r_4}\right)\right]\right]} \quad (167)$$

Variance of number of customers in **FOURTH** queue is $V_4 =$

$$\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) . \left\{\frac{1}{\mu_1} - \frac{4}{\mu_4+\mu_1} + \frac{1}{\mu_4}\right\} + \left\{\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right\}\left\{\frac{1-q^A-pAq^A}{p(1-q^A)}\right\}\left\{\frac{1}{\mu_1} - \frac{1}{\mu_4}\right\}\right] \quad (168)$$

Coefficient of variation of number of customers in **FOURTH** queue is $CV_4 = \frac{\sqrt{V_4}}{L_4} x100 =$

$$\frac{\sqrt{\lambda\left[\left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)^2 \sum_{m=1}^{A} \binom{m}{2} \left(\frac{q^{m-1}p}{1-q^A}\right) \left\{\frac{1}{\mu_1} - \frac{4}{\mu_4+\mu_1} + \frac{1}{\mu_4}\right\} + \left(\frac{\theta_3\mu_1}{\mu_4-\mu_1}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\left\{\frac{1}{\mu_1} - \frac{1}{\mu_4}\right\}\right]}}{\left[\left(\frac{\lambda\theta_3}{\mu_4}\right)\left(\frac{1-q^A-pAq^A}{p(1-q^A)}\right)\right]} x100 \quad (169)$$

Table 1.Values of Mean Number of Customers in the Queue in Transient State.

t	λ	μ_1	μ_2	μ_3	μ_4	θ_1	θ_2	θ_3	p	A	$L_1(t)$	$L_2(t)$	$L_3(t)$	$L_4(t)$	L(t)
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	10	21.4151	0.1121	0.2175	0.7385	22.4832
0.2	15	6	7	8	9	0.1	0.2	0.7	0.2	10	33.1679	0.3021	0.5709	1.8917	35.9326
0.3	15	6	7	8	9	0.1	0.2	0.7	0.2	10	39.6180	0.4700	0.8700	2.8286	43.7866
0.4	15	6	7	8	9	0.1	0.2	0.7	0.2	10	43.1579	0.5938	1.0813	3.4664	48.2994
0.5	15	6	7	8	9	0.1	0.2	0.7	0.2	10	45.1006	0.6775	1.2186	3.8667	50.8634
0.1	**10**	6	7	8	9	0.1	0.2	0.7	0.2	10	14.2767	0.0748	0.1450	0.4923	14.9888
0.1	**11**	6	7	8	9	0.1	0.2	0.7	0.2	10	15.7044	0.0822	0.1595	0.5416	16.4877
0.1	**12**	6	7	8	9	0.1	0.2	0.7	0.2	10	17.1321	0.0897	0.1740	0.5908	17.9866
0.1	**13**	6	7	8	9	0.1	0.2	0.7	0.2	10	18.5597	0.0972	0.1885	0.6400	19.4854
0.1	**14**	6	7	8	9	0.1	0.2	0.7	0.2	10	19.9874	0.1047	0.2030	0.6893	20.9844
0.1	15	**10**	7	8	9	0.1	0.2	0.7	0.2	10	18.0017	0.1653	0.3202	1.0863	19.5735
0.1	15	**11**	7	8	9	0.1	0.2	0.7	0.2	10	17.2715	0.1765	0.3419	1.1597	18.9496
0.1	15	**12**	7	8	9	0.1	0.2	0.7	0.2	10	16.5840	0.1870	0.3622	1.2284	18.3616
0.1	15	**13**	7	8	9	0.1	0.2	0.7	0.2	10	15.9362	0.1969	0.3813	1.2929	17.8073
0.1	15	**14**	7	8	9	0.1	0.2	0.7	0.2	10	15.3254	0.2062	0.3992	1.3533	17.2841
0.1	15	6	**10**	8	9	0.1	0.2	0.7	0.2	10	**21.4151**	0.1024	**0.2175**	**0.7385**	22.4735
0.1	15	6	**11**	8	9	0.1	0.2	0.7	0.2	10	**21.4151**	0.0994	**0.2175**	**0.7385**	22.4705
0.1	15	6	**12**	8	9	0.1	0.2	0.7	0.2	10	**21.4151**	0.0966	**0.2175**	**0.7385**	22.4677
0.1	15	6	**13**	8	9	0.1	0.2	0.7	0.2	10	**21.4151**	0.0939	**0.2175**	**0.7385**	22.4650
0.1	15	6	**14**	8	9	0.1	0.2	0.7	0.2	10	**21.4151**	0.0913	**0.2175**	**0.7385**	22.4624
0.1	15	6	7	**10**	9	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	0.2048	**0.7385**	22.4705
0.1	15	6	7	**11**	9	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	0.1989	**0.7385**	22.4646
0.1	15	6	7	**12**	9	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	0.1932	**0.7385**	22.4589
0.1	15	6	7	**13**	9	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	0.1878	**0.7385**	22.4535
0.1	15	6	7	**14**	9	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	0.1827	**0.7385**	22.4484
0.1	15	6	7	8	**10**	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	**0.2175**	0.7168	22.4615
0.1	15	6	7	8	**11**	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	**0.2175**	0.6961	22.4408
0.1	15	6	7	8	**12**	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	**0.2175**	0.6764	22.4211
0.1	15	6	7	8	**13**	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	**0.2175**	0.6575	22.4022
0.1	15	6	7	8	**14**	0.1	0.2	0.7	0.2	10	**21.4151**	**0.1121**	**0.2175**	0.6394	22.3841
0.1	15	6	7	8	9	**0.1**	0.2	0.7	0.2	10	**21.4151**	0.1121	**0.2175**	0.7385	22.4832
0.1	15	6	7	8	9	**0.2**	0.2	0.6	0.2	10	**21.4151**	0.2243	**0.2175**	0.6330	22.4899
0.1	15	6	7	8	9	**0.3**	0.2	0.5	0.2	10	**21.4151**	0.3364	**0.2175**	0.5275	22.4965
0.1	15	6	7	8	9	**0.4**	0.2	0.4	0.2	10	**21.4151**	0.4486	**0.2175**	0.4220	22.5032
0.1	15	6	7	8	9	**0.5**	0.2	0.3	0.2	10	**21.4151**	0.5607	**0.2175**	0.3165	22.5098
0.1	15	6	7	8	9	0.1	**0.1**	0.8	0.2	10	**21.4151**	**0.1121**	0.1087	0.8440	22.4799
0.1	15	6	7	8	9	0.1	**0.2**	0.7	0.2	10	**21.4151**	**0.1121**	0.2175	0.7385	22.4832
0.1	15	6	7	8	9	0.1	**0.3**	0.6	0.2	10	**21.4151**	**0.1121**	0.3262	0.6330	22.4864
0.1	15	6	7	8	9	0.1	**0.4**	0.5	0.2	10	**21.4151**	**0.1121**	0.4350	0.5275	22.4897
0.1	15	6	7	8	9	0.1	**0.5**	0.4	0.2	10	**21.4151**	**0.1121**	0.5437	0.4220	22.4929

0.1	15	6	7	8	9	0.1	0.2	**0.7**	0.2	10	**21.4151**	**0.1121**	**0.2175**	**0.7385**	**22.4832**
0.1	15	6	7	8	9	0.1	0.2	**0.7**	0.2	10	**21.4151**	**0.1121**	**0.2175**	**0.7385**	**22.4832**
0.1	15	6	7	8	9	0.1	0.2	**0.7**	0.2	10	**21.4151**	**0.1121**	**0.2175**	**0.7385**	**22.4832**
0.1	15	6	7	8	9	0.1	0.2	**0.7**	0.2	10	**21.4151**	**0.1121**	**0.2175**	**0.7385**	**22.4832**
0.1	15	6	7	8	9	0.1	0.2	**0.7**	0.2	10	**21.4151**	**0.1121**	**0.2175**	**0.7385**	**22.4832**
0.1	15	6	7	8	9	0.1	0.2	0.7	**0.1**	10	52.4123	0.1372	0.2661	0.9037	53.7193
0.1	15	6	7	8	9	0.1	0.2	0.7	**0.2**	10	21.4151	0.1121	0.2175	0.7385	22.4832
0.1	15	6	7	8	9	0.1	0.2	0.7	**0.3**	10	11.4401	0.0899	0.1743	0.5918	12.2961
0.1	15	6	7	8	9	0.1	0.2	0.7	**0.4**	10	6.8783	0.0720	0.1397	0.4744	7.5644
0.1	15	6	7	8	9	0.1	0.2	0.7	**0.5**	10	4.4898	0.0588	0.1140	0.3871	5.0497
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	**10**	21.4151	0.1121	0.2175	0.7385	22.4832
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	**15**	25.1142	0.1315	0.2551	0.8661	26.3669
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	**20**	26.8836	0.1448	0.2730	0.9271	28.2285
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	**25**	27.6646	0.1449	0.2810	0.9540	29.0445
0.1	15	6	7	8	9	0.1	0.2	0.7	0.2	**30**	27.9896	0.1446	0.2843	0.9652	29.3857

Table 2.Probability of Emptiness and Utilization of Servers and System in Transient State.

t	λ	μ_1	μ_2	μ_3	μ_4	θ_1	θ_2	θ_3	p	A	$P_{0000}t$	$P_{0..}t$	$P_{.0.}t$	$P_{..0}t$	$P_{...0}t$	$U_1(t)$	$U_2(t)$	$U_3(t)$	$U_4(t)$
0.	1	6	7	8	9	0.	0.	0.	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	1	2	7	2	10	1	8	1	4	1	2	9	6	9
0.	1	6	7	8	9	0.	0.	0.	0.	10	0.058	0.081	0.753	0.603	0.270	0.918	0.246	0.396	0.729
2	5	6	7	8	9	1	2	7	2	10	0	6	8	3	6	4	2	7	4
0.	1	6	7	8	9	0.	0.	0.	0.	10	0.017	0.036	0.644	0.462	0.138	0.964	0.356	0.537	0.861
0.	**1**	6	7	8	9	0.	0.	0.	0.	10	0.373	0.396	0.931	0.877	0.698	0.603	0.068	0.122	0.302
1	**0**	6	7	8	9	1	2	7	2	10	4	6	5	1	0.673	4	5	9	0
0.	**1**	6	7	8	9	0.	0.	0.	0.	10	0.338	0.361	0.925	0.865	3	0.638	0.075	0.134	0.326
1	**1**	6	7	8	9	1	2	7	2	10	3	6	0	7	0.649	4	0	3	7
0.	**1**	6	7	8	9	0.	0.	0.	0.	10	0.306	0.329	0.918	0.854	6	0.670	0.081	0.145	0.350
0.	1	**1**	7	8	9	0.	0.	0.	0.	10	0.231	0.272	0.857	0.757	0.494	0.727	0.142	0.242	0.505
1	5	**0**	7	8	9	1	2	7	2	10	1	6	9	5	2	4	1	5	8
0.	1	**1**	7	8	9	0.	0.	0.	0.	10	0.231	0.279	0.849	0.745	0.479	0.721	0.150	0.254	0.520
1	5	**1**	7	8	9	1	2	7	2	10	8	0	7	2	4	0	3	8	6
0.	1	**1**	7	8	9	0.	0.	0.	0.	10	0.232	0.285	0.842	0.734	0.466	0.714	0.157	0.265	0.533
0.	1	6	**1**	8	9	0.	0.	0.	0.	10	0.228	0.249	0.907	0.821	0.583	0.750	0.093	0.178	0.416
1	5	6	**0**	8	9	1	2	7	2	10	3	8	0	4	1	2	0	6	9
0.	1	6	**1**	8	9	0.	0.	0.	0.	10	0.228	0.249	0.909	0.821	0.583	0.750	0.090	0.178	0.416
1	5	6	**1**	8	9	1	2	7	2	10	3	8	4	4	1	2	6	6	9
0.	1	6	**1**	8	9	0.	0.	0.	0.	10	0.228	0.249	0.911	0.821	0.583	0.750	0.088	0.178	0.416
0.	1	6	7	**1**	9	0.	0.	0.	0.	10	0.228	0.249	0.899	0.829	0.583	0.750	0.100	0.170	0.416
1	5	6	7	**0**	9	1	2	7	2	10	3	8	1	9	1	2	9	1	9
0.	1	6	7	**1**	9	0.	0.	0.	0.	10	0.228	0.249	0.899	0.833	0.583	0.750	0.100	0.166	0.416
1	5	6	7	**1**	9	1	2	7	2	10	4	8	1	4	1	2	9	6	9
0.	1	6	7	**1**	9	0.	0.	0.	0.	10	0.228	0.249	0.899	0.837	0.583	0.750	0.100	0.162	0.416
0.	1	6	7	8	**1**	0.	0.	0.	0.	10	0.228	0.249	0.899	0.821	0.589	0.750	0.100	0.178	0.410
1	5	6	7	8	**0**	1	2	7	2	10	5	8	1	4	4	2	9	6	6
0.	1	6	7	8	**1**	0.	0.	0.	0.	10	0.228	0.249	0.899	0.821	0.595	0.750	0.100	0.178	0.404
1	5	6	7	8	**1**	1	2	7	2	10	8	8	1	4	6	2	9	6	4
0.	1	6	7	8	**1**	0.	0.	0.	0.	10	0.229	0.249	0.899	0.821	0.601	0.750	0.100	0.178	0.398
0.	1	6	7	8	9	**0.**	0.	0.	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	**1**	2	7	2	10	1	8	1	4	1	2	9	6	9
0.	1	6	7	8	9	**0.**	0.	0.	0.	10	0.228	0.249	0.817	0.821	0.617	0.750	0.183	0.178	0.382
1	5	6	7	8	9	**2**	2	7	2	10	0	8	0	4	8	2	0	6	2
0.	1	6	7	8	9	**0.**	0.	0.	0.	10	0.227	0.249	0.749	0.821	0.658	0.750	0.250	0.178	0.342
0.	1	6	7	8	9	0.	**0.**	0.	0.	10	0.228	0.249	0.899	0.901	0.553	0.750	0.100	0.098	0.447
1	5	6	7	8	9	1	**1**	8	2	10	2	8	1	8	0	2	9	2	0
0.	1	6	7	8	9	0.	**0.**	0.	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	1	**2**	7	2	10	1	8	1	5	1	2	9	5	9
0.	1	6	7	8	9	0.	**0.**	0.	0.	10	0.228	0.249	0.899	0.754	0.617	0.750	0.100	0.245	0.382

0.	1	6	7	8	9	0.	0.	**0.**	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	1	2	**7**	2	10	1	8	1	4	1	2	9	6	9
0.	1	6	7	8	9	0.	0.	**0.**	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	1	2	**7**	2	10	1	8	1	4	1	2	9	6	9
0.	1	6	7	8	9	0.	0.	**0.**	0.	10	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
0.	1	6	7	8	9	0.	0.	0.	**0.**	10	0.226	0.242	0.944	0.789	0.714	0.758	0.055	0.210	0.285
1	5	6	7	8	9	1	2	7	**1**	10	6	0	2	4	2	0	8	6	8
0.	1	6	7	8	9	0.	0.	0.	**0.**	10	0.228	0.249	0.952	0.821	0.745	0.750	0.047	0.178	0.254
1	5	6	7	8	9	1	2	7	**2**	10	1	8	7	4	3	2	3	6	7
0.	1	6	7	8	9	0.	0.	0.	**0.**	10	0.230	0.258	0.959	0.851	0.770	0.741	0.040	0.148	0.229
0.	1	6	7	8	9	0.	0.	0.	0.	**10**	0.228	0.249	0.899	0.821	0.583	0.750	0.100	0.178	0.416
1	5	6	7	8	9	1	2	7	2	**15**	1	8	1	4	1	2	9	6	9
0.	1	6	7	8	9	0.	0.	0.	0.	**20**	0.227	0.247	0.884	0.799	0.557	0.752	0.115	0.200	0.442
1	5	6	7	8	9	1	2	7	2	**25**	8	7	6	6	8	3	4	4	2
0.	1	6	7	8	9	0.	0.	0.	0.	**30**	0.227	0.247	0.878	0.790	0.549	0.752	0.121	0.209	0.451

Table 3.Values of Throughput and Waiting Time of Customers in Queues in Transient State.

t	λ	μ_1	μ_2	μ_3	μ_4	θ_1	θ_2	θ_3	p	A	$Thp_1(t)$	$Thp_2(t)$	$Thp_3(t)$	$Thp_4(t)$	$W_1(t)$	$W_2(t)$	$W_3(t)$	$W_4(t)$
0.	15	6	7	8	9	0.	0.	0.	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	1	2	7	2	10	2	3	8	1	6	7	2	8
0.	15	6	7	8	9	0.	0.	0.	0.	10	5.510	1.723	3.173	6.564	6.019	0.175	0.179	0.288
2	15	6	7	8	9	1	2	7	2	10	4	4	6	6	1	3	9	2
0.	15	6	7	8	9	0.	0.	0.	0.	10	5.784	2.492	4.302	7.749	6.849	0.188	0.202	0.365
0.	**10**	6	7	8	9	0.	0.	0.	0.	10	3.620	0.479	0.983	2.718	3.943	0.156	0.147	0.181
1	**11**	6	7	8	9	1	2	7	2	10	4	5	2	0	4	0	5	1
0.	**12**	6	7	8	9	0.	0.	0.	0.	10	3.830	0.525	1.074	2.940	4.099	0.156	0.148	0.184
1	**13**	6	7	8	9	1	2	7	2	10	4	0	4	3	9	6	5	2
0.	**14**	6	7	8	9	0.	0.	0.	0.	10	4.022	0.571	1.164	3.153	4.259	0.157	0.149	0.187
0.	15	**10**	7	8	9	0.	0.	0.	0.	10	7.274	0.994	1.940	4.552	2.474	0.166	0.165	0.238
1	15	**11**	7	8	9	1	2	7	2	10	0	7	0	2	8	2	1	6
0.	15	**12**	7	8	9	0.	0.	0.	0.	10	7.931	1.052	2.038	4.685	2.177	0.167	0.167	0.247
1	15	**13**	7	8	9	1	2	7	2	10	0	1	4	4	7	8	7	5
0.	15	**14**	7	8	9	0.	0.	0.	0.	10	8.572	1.105	2.127	4.802	1.934	0.169	0.170	0.255
0.	15	6	**10**	8	9	0.	0.	0.	0.	10	4.501	0.930	1.428	3.752	4.757	0.110	0.152	0.196
1	15	6	**11**	8	9	1	2	7	2	10	2	0	8	1	6	1	2	8
0.	15	6	**12**	8	9	0.	0.	0.	0.	10	4.501	0.996	1.428	3.752	4.757	0.099	0.152	0.196
1	15	6	**13**	8	9	1	2	7	2	10	2	6	8	1	6	7	2	8
0.	15	6	**14**	8	9	0.	0.	0.	0.	10	4.501	1.059	1.428	3.752	4.757	0.091	0.152	0.196
0.	15	6	7	**10**	9	0.	0.	0.	0.	10	4.501	0.706	1.701	3.752	4.757	0.158	0.120	0.196
1	15	6	7	**11**	9	1	2	7	2	10	2	3	0	1	6	7	4	8
0.	15	6	7	**12**	9	0.	0.	0.	0.	10	4.501	0.706	1.832	3.752	4.757	0.158	0.108	0.196
1	15	6	7	**13**	9	1	2	7	2	10	2	3	6	1	6	7	5	8
0.	15	6	7	**14**	9	0.	0.	0.	0.	10	4.501	0.706	1.947	3.752	4.757	0.158	0.099	0.196
0.	15	6	7	8	**10**	0.	0.	0.	0.	10	4.501	0.706	1.428	4.106	4.757	0.158	0.152	0.174
1	15	6	7	8	**11**	1	2	7	2	10	2	3	8	0	6	7	2	6
0.	15	6	7	8	**12**	0.	0.	0.	0.	10	4.501	0.706	1.428	4.448	4.757	0.158	0.152	0.156
1	15	6	7	8	**13**	1	2	7	2	10	2	3	8	4	6	7	2	5
0.	15	6	7	8	**14**	0.	0.	0.	0.	10	4.501	0.706	1.428	4.778	4.757	0.158	0.152	0.141
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	**1**	2	7	2	10	2	3	8	1	6	7	2	8
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	4.501	1.281	1.428	3.439	4.757	0.175	0.152	0.184
1	15	6	7	8	9	**2**	2	6	2	10	2	0	8	8	6	1	2	0
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	4.501	1.754	1.428	3.078	4.757	0.191	0.152	0.171
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	4.501	0.706	0.785	4.023	4.757	0.158	0.138	0.209
1	15	6	7	8	9	1	**1**	8	2	10	2	3	6	0	6	7	4	8
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	1	**2**	7	2	10	2	3	0	1	6	7	3	8
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	4.501	0.706	1.961	3.439	4.757	0.158	0.166	0.184

0.	15	6	7	8	9	0.	0.	**0.**	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	1	2	**7**	2	10	2	3	8	1	6	7	2	8
0.	15	6	7	8	9	0.	0.	**0.**	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	1	2	**7**	2	10	2	3	8	1	6	7	2	8
0.	15	6	7	8	9	0.	0.	**0.**	0.	10	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	4.548	0.390	1.684	2.572	11.52	0.351	0.157	0.351
1	15	6	7	8	9	1	2	7	**1**	10	0	6	8	2	4	3	9	3
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	4.501	0.331	1.428	2.293	4.757	0.338	0.152	0.322
1	15	6	7	8	9	1	2	7	**2**	10	2	1	8	3	6	6	2	2
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	4.446	0.286	1.188	2.064	2.572	0.314	0.146	0.286
0.	15	6	7	8	9	0.	0.	0.	0.	**10**	4.501	0.706	1.428	3.752	4.757	0.158	0.152	0.196
1	15	6	7	8	9	1	2	7	2	**15**	2	3	8	1	6	7	2	8
0.	15	6	7	8	9	0.	0.	0.	0.	**20**	4.513	0.807	1.603	3.979	5.563	0.162	0.159	0.217
1	15	6	7	8	9	1	2	7	2	**25**	8	8	2	8	9	8	1	6
0.	15	6	7	8	9	0.	0.	0.	0.	**30**	4.517	0.853	1.676	4.059	5.951	0.169	0.162	0.228

Table 4.Values of Variances and coefficients of Variation of Customers in Queues in Transient State

t	λ	μ_1	μ_2	μ_3	μ_4	θ_1	θ_2	θ_3	p	A	$V_1(t)$	$V_2(t)$	$V_3(t)$	$V_4(t)$	$CV_1(t)$	$CV_2(t)$	$CV_3(t)$	$CV_4(t)$
0.	15	6	7	8	9	0.	0.	0.	0.	10	24.64	1.132	1.300	3.505	23.18	949.1	524.2	253.5
1	15	6	7	8	9	1	2	7	2	10	5	2	2	6	1	9	5	3
0.	15	6	7	8	9	0.	0.	0.	0.	10	34.46	1.408	2.044	31.89	17.69	392.8	250.4	298.5
2	15	6	7	8	9	1	2	7	2	10	3	7	0	0	9	7	2	2
0.	15	6	7	8	9	0.	0.	0.	0.	10	38.73	1.704	2.960	164.2	15.70	277.7	197.7	453.0
0.	**10**	6	7	8	9	0.	0.	0.	0.	10	16.43	1.086	1.191	2.307	29.77	1393.	752.7	308.5
1	**11**	6	7	8	9	1	2	7	2	10	0418.	3	3	7	7	3	3	7
0.	**12**	6	7	8	9	0.	0.	0.	0.	10	07	1.095	1.212	2.509	28.27	1273.	690.3	292.4
1	**13**	6	7	8	9	1	2	7	2	10	19.71	4	3	0	4427.	2	1	6
0.	**14**	6	7	8	9	0.	0.	0.	0.	10	6	1.104	1.233	2.727	97	1171.	638.3	279.5
0.	15	**10**	7	8	9	0.	0.	0.	0.	10	19.24	1.210	1.515	8.805	23.69	665.5	384.4	273.1
1	15	**11**	7	8	9	1	2	7	2	10	6	2	0	8	424.0	1262	0136	7
0.	15	**12**	7	8	9	0.	0.	0.	0.	10	18.19	1.228	1.568	10.95	34	7.8	6.2	285.4
1	15	**13**	7	8	9	1	2	7	2	10	2	1	1	9	24.38	596.7	351.4	6
0.	15	**14**	7	8	9	0.	0.	0.	0.	10	17.23	1.245	1.620	13.55	3	5	7	299.6
0.	15	6	**10**	8	9	0.	0.	0.	0.	10	24.64	1.118	1.300	3.505	23.18	1032.	524.2	253.5
1	15	6	**11**	8	9	1	2	7	2	10	5	8	2	6	1	9	5	3
0.	15	6	**12**	8	9	0.	0.	0.	0.	10	24.64	1.114	1.300	3.505	23.18	1062.	524.2	253.5
1	15	6	**13**	8	9	1	2	7	2	10	5	8	2	6	1	2	5	3
0.	15	6	**14**	8	9	0.	0.	0.	0.	10	24.64	1.111	1.300	3.505	23.18	1091.	524.2	253.5
0.	15	6	7	**10**	9	0.	0.	0.	0.	10	24.64	1.132	1.276	3.505	23.18	949.2	551.7	253.5
1	15	6	7	**11**	9	1	2	7	2	10	5	2	7	6	2	0	1	3
0.	15	6	7	**12**	9	0.	0.	0.	0.	10	24.64	1.132	1.266	3.505	23.18	949.2	565.6	253.5
1	15	6	7	**13**	9	1	2	7	2	10	5	2	0	6	2	0	9	3
0.	15	6	7	**14**	9	0.	0.	0.	0.	10	24.64	1.132	1.256	3.505	23.18	949.2	580.1	253.5
0.	15	6	7	8	**10**	0.	0.	0.	0.	10	24.64	1.132	1.300	3.320	23.18	949.2	524.2	254.2
1	15	6	7	8	**11**	1	2	7	2	10	6	2	2	1	2	0	6	0
0.	15	6	7	8	**12**	0.	0.	0.	0.	10	24.64	1.132	1.300	3.155	23.18	949.2	524.2	255.2
1	15	6	7	8	**13**	1	2	7	2	10	6	2	2	8	2	0	6	0
0.	15	6	7	8	**14**	0.	0.	0.	0.	10	24.64	1.132	1.300	3.009	23.18	949.2	524.2	256.4
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	**1**	2	7	2	10	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	24.64	1.313	1.300	2.751	23.18	510.9	524.2	262.0
1	15	6	7	8	9	**2**	2	6	2	10	6	2	2	1	2	0	6	3
0.	15	6	7	8	9	**0.**	0.	0.	0.	10	24.64	1.560	1.300	2.204	23.18	371.3	524.2	281.5
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	24.64	1.132	1.127	4.562	23.18	949.2	976.8	253.0
1	15	6	7	8	9	1	**1**	8	2	10	6	2	5	2	2	0	5	7
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	1	**2**	7	2	10	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	0.	**0.**	0.	0.	10	24.64	1.132	1.533	2.751	23.18	949.2	379.6	262.0

0.	15	6	7	8	9	0.	0.	**0.**	0.	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	1	2	**7**	2	10	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	0.	0.	**0.**	0.	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	1	2	**7**	2	10	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	0.	0.	**0.**	0.	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	35.58	1.167	1.392	5.199	11.38	787.4	443.4	252.3
1	15	6	7	8	9	1	2	7	**1**	10	0	2	6	0	1	4	7	1
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	1	2	7	**2**	10	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	0.	0.	0.	**0.**	10	15.81	1.102	1.225	2.516	34.76	1168.	635.0	268.0
0.	15	6	7	8	9	0.	0.	0.	0.	**10**	24.64	1.132	1.300	3.505	23.18	949.2	524.2	253.5
1	15	6	7	8	9	1	2	7	2	**15**	6	2	2	6	2	0	6	3
0.	15	6	7	8	9	0.	0.	0.	0.	**20**	38.62	1.162	1.384	5.336	24.74	819.8	461.3	266.7
1	15	6	7	8	9	1	2	7	2	**25**	8	3	9	7	8	5	2	3
0.	15	6	7	8	9	0.	0.	0.	0.	**30**	47.77	1.178	1.433	6.869	25.71	749.6	438.6	282.7

Table 5.Values of $L_1, L_2, L_3, L_4, L, W_1, W_2, W_3$ and W_4 for different Values of $t, \lambda, \mu_1, \mu_2, \mu_3, \mu_4, p, A, \theta_1, \theta_2$ and θ_3.(SENSITIVITY ANALYSIS)

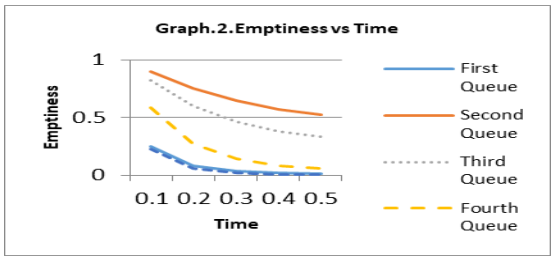
Variation Parameter	Performance Measure	Percentage Change in Parameter						
		-15%	-10%	-5%	0	5%	10%	15%
t = 0.1	$L_1(t)$	18.9620	19.8043	20.6218	21.4151	22.1849	22.9320	23.6571
	$L_2(t)$	0.0862	0.0946	0.1033	0.1121	0.1211	0.1303	0.1395
	$L_3(t)$	0.1678	0.1841	0.2006	0.2175	0.2346	0.2520	0.2695
	$L_4(t)$	0.5723	0.6267	0.6822	0.7385	0.7955	0.8532	0.9113
	$L(t)$	**19.7883**	**20.7097**	**21.6079**	**22.4832**	**23.3361**	**24.1675**	**24.9774**
	$W_1(t)$	4.5329	4.6087	4.6838	4.7576	4.8308	4.9031	4.9739
	$W_2(t)$	0.1563	0.1570	0.1578	0.1587	0.1596	0.1605	0.1612
	$W_3(t)$	0.1478	0.1493	0.1508	0.1522	0.1537	0.1552	0.1566
	$W_4(t)$	0.1829	0.1875	0.1922	0.1968	0.2015	0.2062	0.2108
λ=15	$L_1(t)$	18.2028	19.2736	20.3443	21.4151	22.4858	23.5566	24.6273
	$L_2(t)$	0.0953	0.1009	0.1065	0.1121	0.1178	0.1234	0.1290
	$L_3(t)$	0.1849	0.1957	0.2066	0.2175	0.2284	0.2392	0.2501
	$L_4(t)$	0.6277	0.6646	0.7016	0.7385	0.7754	0.8123	0.8493
	$L(t)$	**19.1107**	**20.2348**	**21.359**	**22.4832**	**23.6074**	**24.7315**	**25.8557**
	$W_1(t)$	4.3633	4.4870	4.6132	4.7406	4.8696	5.0001	5.1326
	$W_2(t)$	0.1576	0.1579	0.1583	0.1587	0.1592	0.1597	0.1601
	$W_3(t)$	0.1501	0.1507	0.1515	0.1522	0.1530	0.1536	0.1544
	$W_4(t)$	0.1897	0.1921	0.1945	0.1968	0.1992	0.2017	0.2042
μ_1=12	$L_1(t)$	17.8521	17.4140	16.9915	16.5840	16.1907	15.8111	15.4447
	$L_2(t)$	0.1676	0.1743	0.1808	0.1870	0.1930	0.1988	0.2044
	$L_3(t)$	0.3247	0.3377	0.3502	0.3622	0.3738	0.3850	0.3957
	$L_4(t)$	1.1014	1.1454	1.1877	1.2284	1.2676	1.3053	1.3415
	$L(t)$	**19.4458**	**19.0714**	**18.7102**	**18.3616**	**18.0251**	**17.7002**	**17.3863**
	$W_1(t)$	2.0489	2.0091	1.9710	1.9345	1.8995	1.8660	1.8339
	$W_2(t)$	0.1665	0.1675	0.1683	0.1692	0.1700	0.1708	0.1716
	$W_3(t)$	0.1656	0.1672	0.1688	0.1703	0.1717	0.1731	0.1744
	$W_4(t)$	0.2405	0.2458	0.2509	0.2657	0.2605	0.2651	0.2694
μ_2=14	$L_1(t)$	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**
	$L_2(t)$	0.0969	0.095	0.0931	0.0913	0.0896	0.0879	0.0863
	$L_3(t)$	**0.3622**	**0.3622**	**0.3622**	**0.3622**	**0.3622**	**0.3622**	**0.3622**
	$L_4(t)$	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**
	$L(t)$	**23.1026**	**23.1007**	**23.0988**	**23.097**	**23.0953**	**23.0936**	**23.092**
	$W_1(t)$	**4.7576**	**4.7576**	**4.7576**	**4.7576**	**4.7576**	**4.7576**	**4.7576**
	$W_2(t)$	0.0782	0.0781	0.0779	0.0777	0.0777	0.0775	0.0774
	$W_3(t)$	**0.2535**	**0.2535**	**0.2535**	**0.2535**	**0.2535**	**0.2535**	**0.2535**
	$W_4(t)$	**0.3274**	**0.3274**	**0.3274**	**0.3274**	**0.3274**	**0.3274**	**0.3274**
μ_3=11	$L_1(t)$	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**	**21.4151**
	$L_2(t)$	**0.3364**	**0.3364**	**0.3364**	**0.3364**	**0.3364**	**0.3364**	**0.3364**
	$L_3(t)$	0.2088	0.2054	0.2021	0.1989	0.1958	0.1927	0.1897
	$L_4(t)$	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**	**1.2284**
	$L(t)$	**23.1887**	**23.1853**	**23.182**	**23.1788**	**23.1757**	**23.1726**	**23.1696**

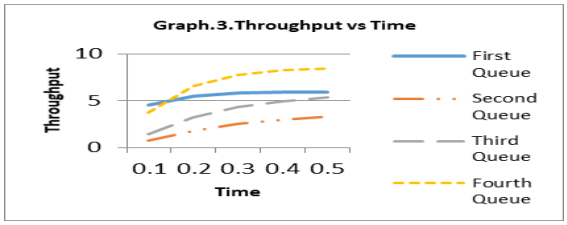
	$W_1(t)$	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576
	$W_2(t)$	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763
	$W_3(t)$	0.1098	0.1095	0.1092	0.1089	0.1085	0.1082	0.1079
	$W_4(t)$	0.3274	0.3274	0.3274	0.3274	0.3274	0.3274	0.3274
μ_4=13	$L_1(t)$	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151
	$L_2(t)$	0.3364	0.3364	0.3364	0.3364	0.3364	0.3364	0.3364
	$L_3(t)$	0.3622	0.3622	0.3622	0.3622	0.3622	0.3622	0.3622
	$L_4(t)$	0.6951	0.6872	0.6696	0.6575	0.6456	0.6342	0.6230
	$L(t)$	22.8088	22.8009	22.7833	22.7712	22.7593	22.7479	22.7367
	$W_1(t)$	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576
	$W_2(t)$	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763
	$W_3(t)$	0.2535	0.2535	0.2535	0.2535	0.2535	0.2535	0.2535
	$W_4(t)$	0.1323	0.1321	0.1300	0.1290	0.1279	0.1269	0.1259
θ_1=0.3	$L_1(t)$	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151
	$L_2(t)$	0.2860	0.3028	0.3196	0.3364	0.3533	0.3701	0.3869
	$L_3(t)$	0.3622	0.3622	0.3622	0.3622	0.3622	0.3622	0.3622
	$L_4(t)$	0.5750	0.5591	0.5433	0.5275	0.5112	0.4958	0.4800
	$L(t)$	22.6383	22.6392	22.6402	22.6412	22.6418	22.6432	22.6442
	$W_1(t)$	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576
	$W_2(t)$	0.1842	0.1867	0.1892	0.1917	0.1943	0.1968	0.1993
	$W_3(t)$	0.2535	0.2535	0.2535	0.2535	0.2535	0.2535	0.2535
	$W_4(t)$	0.1771	0.1751	0.1733	0.1714	0.1694	0.1676	0.1657
θ_2=0.2	$L_1(t)$	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151	21.4151
	$L_2(t)$	0.3364	0.3364	0.3364	0.3364	0.3364	0.3364	0.3364
	$L_3(t)$	0.1849	0.1957	0.2066	0.2175	0.2284	0.2392	0.2501
	$L_4(t)$	0.6646	0.6541	0.6435	0.633	0.6224	0.6119	0.6013
	$L(t)$	22.601	22.6013	22.6016	22.602	22.6023	22.6026	22.6029
	$W_1(t)$	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576	4.7576
	$W_2(t)$	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763	0.4763
	$W_3(t)$	0.1481	0.1494	0.1508	0.1522	0.1537	0.1550	0.1564
	$W_4(t)$	0.1732	0.1717	0.1702	0.1687	0.1672	0.1657	0.1642
p=0.5	$L_1(t)$	6.1396	5.5066	4.9615	4.4898	4.0798	3.7218	3.4079
	$L_2(t)$	0.0683	0.0649	0.0617	0.0588	0.0561	0.0536	0.0513
	$L_3(t)$	0.1325	0.1258	0.1197	0.114	0.1088	0.1039	0.0995
	$L_4(t)$	0.4499	0.4273	0.4064	0.3871	0.3693	0.353	0.3379
	$L(t)$	6.7903	6.1246	5.5493	5.0497	4.614	4.2323	3.8966
	$W_1(t)$	1.4040	1.2634	1.1423	1.0373	0.9457	0.8657	0.7954
	$W_2(t)$	0.1520	0.1515	0.1509	0.1505	0.1501	0.1496	0.1490
	$W_3(t)$	0.1407	0.1397	0.1388	0.1378	0.1371	0.1361	0.1355
	$W_4(t)$	0.1594	0.1562	0.1533	0.1505	0.1479	0.1456	0.1434
A=20	$L_1(t)$	25.9906	26.3369	26.6324	26.8836	27.0967	27.2769	27.429
	$L_2(t)$	0.1361	0.1379	0.1395	0.1408	0.1419	0.1428	0.1436
	$L_3(t)$	0.264	0.2675	0.2705	0.273	0.2752	0.277	0.2786
	$L_4(t)$	0.8963	0.9082	0.9184	0.9271	0.9344	0.9406	0.9459
	$L(t)$	27.287	27.6505	27.9608	28.2245	28.4482	28.6373	28.7971
	$W_1(t)$	5.7238	5.8123	5.8877	5.9511	6.0047	6.0494	6.0872
	$W_2(t)$	0.1578	0.1607	0.1631	0.1650	0.1667	0.1680	0.1693
	$W_3(t)$	0.1559	0.1586	0.1609	0.1628	0.1645	0.1658	0.1670
	$W_4(t)$	0.2190	0.2227	0.2258	0.2284	0.2306	0.2325	0.2340

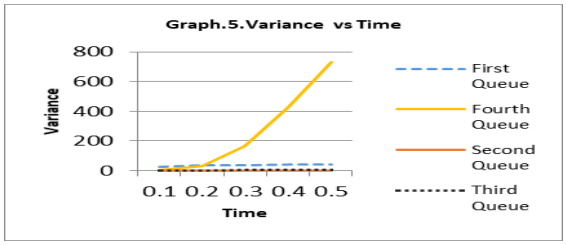
VIII. COMPARITIVE STUDY

A comparative study between transient and steady state of the system with values of time t = 0.1, 0.2, 0.3, 0.4,0.5,1,2,3,4 and 5 is carried. The difference and percentage of variation in all performance measures are calculated and given in Table.6. FromTable.6 it is observed that there is high significant difference between transient behaviour and steady state behaviour of the model. At t=0.1 the variation and percentage of variation in performance measures are highly significant which can be observed in last two columns of Table.6.and as we move towards 0.5 the variation narrows down. At t=1the percentage of variation is reduced further and some of the measures even differ very closely. This explains that as t increases the difference between transient and steady state behaviour become negligible and from t=3 onwards we find no difference between them. This indicates that the system attains equilibrium after time t=3 units.

IX.GRAPHS







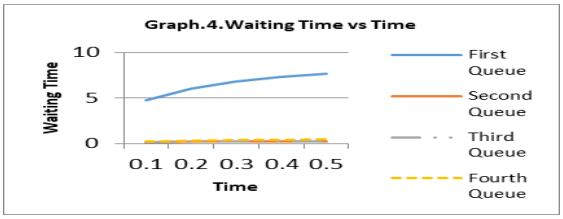


Table 6.Comparative tables of performance measures for values of t=0.1,0.2,0.3,0.4,0.5,1,2,3,4 and 5
($\lambda = 15, \mu_1 = 6, \mu_2 = 7, \mu_3 = 8, \mu_4 = 9, \theta_1 = 0.1, \theta_2 = 0.2, p = 0.2, A = 10$)

Time t	Performance	Transient	Steady State	Difference between	% of variation
t=0.1	L_1	**21.4151**	**47.4637**	-26.049	-54.881
	L_2	**0.1121**	**0.8137**	-0.702	-86.223
	L_3	**0.2175**	**1.4239**	-1.206	-84.725
	L_4	**0.7385**	**4.4299**	-3.691	-83.329
	L	22.4832	54.1312	-31.648	-58.465
	P_{0000}	**0.2281**	**0.0010**	0.227	22710.0
	$P_{0...}$	**0.2498**	**0.0096**	0.240	2502.083
	$P_{.0..}$	**0.8991**	**0.4616**	0.438	94.779
	$P_{..0.}$	**0.8214**	**0.2733**	0.548	200.549
	$P_{...0}$	**0.5831**	**0.0351**	0.548	1561.254
	U_1	0.7502	0.9904	-0.240	-24.253
	U_2	0.1009	0.5384	-0.438	-81.259
	U_3	0.1786	0.7267	-0.548	-75.423
	U_4	0.4169	0.9649	-0.548	-56.793
	Thp_1	4.5012	5.9424	-1.441	-24.253
	Thp_2	0.7063	3.7688	-3.063	-81.259
	Thp_3	1.4288	5.8136	-4.385	-75.423
	Thp_4	3.7521	8.6841	-4.932	-56.793
	W_1	4.7576	7.9873	-3.230	-40.435
	W_2	0.1587	0.2159	-0.057	-26.489
	W_3	0.1522	0.2449	-0.093	-37.848
	W_4	0.1968	0.5101	-0.313	-61.416

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=0.2	L_1	33.1679	**47.4637**	-14.296	-43.101
	L_2	0.3021	**0.8137**	-0.512	-169.348
	L_3	0.5709	**1.4239**	-0.853	-149.413
	L_4	1.8917	**4.4299**	-2.538	-134.176
	L	35.9326	54.1312	-18.199	-50.646
	P_{0000}	0.0580	**0.0010**	0.057	98.276

	$P_{0...}$	0.0816	**0.0096**	0.072	88.235
	$P_{.0..}$	0.7537	**0.4616**	0.292	38.755
	$P_{..0.}$	0.6033	**0.2733**	0.330	54.699
	$P_{...0}$	0.2706	**0.0351**	0.236	87.029
	U_1	0.9184	0.9904	-0.072	-7.840
	U_2	0.2463	0.5384	-0.292	-118.595
	U_3	0.3967	0.7267	-0.330	-83.186
	U_4	0.7294	0.9649	-0.236	-32.287
	Thp_1	5.5104	5.9424	-0.432	-7.840
	Thp_2	1.7241	3.7688	-2.045	-118.595
	Thp_3	3.1736	5.8136	-2.640	-83.186
	Thp_4	6.5646	8.6841	-2.120	-32.287
	W_1	6.0191	7.9873	-1.968	-32.698
	W_2	0.1752	0.2159	-0.041	-23.218
	W_3	0.1799	0.2449	-0.065	-36.153
	W_4	0.2882	0.5101	-0.222	-77.021
Time t	**Performance**	**Transient State**	**Steady State**	**Difference between Transient and Steady state**	**% of variation**
t=0.3	L_1	39.6180	**47.4637**	-7.846	-19.803
	L_2	0.4700	**0.8137**	-0.344	-73.128
	L_3	0.8700	**1.4239**	-0.554	-63.667
	L_4	2.8286	**4.4299**	-1.601	-56.611
	L	43.7866	54.1312	-10.345	-23.625
	P_{0000}	0.0177	**0.0010**	0.017	94.350
	$P_{0...}$	0.0360	**0.0096**	0.026	73.333
	$P_{.0..}$	0.6440	**0.4616**	0.182	28.323
	$P_{..0.}$	0.4622	**0.2733**	0.189	40.870
	$P_{...0}$	0.1389	**0.0351**	0.104	74.730
	U_1	0.9640	0.9904	-0.026	-2.739
	U_2	0.3560	0.5384	-0.182	-51.236
	U_3	0.5378	0.7267	-0.189	-35.125
	U_4	0.8611	0.9649	-0.104	-12.054
	Thp_1	5.7840	5.9424	-0.158	-2.739
	Thp_2	2.4920	3.7688	-1.277	-51.236
	Thp_3	4.3024	5.8136	-1.511	-35.125
	Thp_4	7.7499	8.6841	-0.934	-12.054
	W_1	6.8496	7.9873	-1.138	-16.610
	W_2	0.1886	0.2159	-0.027	-14.475
	W_3	0.2022	0.2449	-0.043	-21.123
	W_4	0.3650	0.5101	-0.145	-39.763
Time t	**Performance**	**Transient State**	**Steady State**	**Difference between Transient and Steady state**	**% of variation**
t=0.4	L_1	43.1579	**47.4637**	-4.306	-9.977
	L_2	0.5938	**0.8137**	-0.220	-37.033
	L_3	1.0813	**1.4239**	-0.343	-31.684
	L_4	3.4664	**4.4299**	-0.964	-27.795

	L	48.2994	54.1312	-5.832	-12.074
	P_{0000}	0.0068	**0.0010**	0.006	85.294
	$P_{0...}$	0.0210	**0.0096**	0.011	54.286
	$P_{.0..}$	0.5724	**0.4616**	0.111	19.357
	$P_{..0.}$	0.3805	**0.2733**	0.107	28.173
	$P_{...0}$	0.0841	**0.0351**	0.049	58.264
	U_1	0.9790	0.9904	-0.011	-1.164
	U_2	0.4276	0.5384	-0.111	-25.912
	U_3	0.6195	0.7267	-0.107	-17.304
	U_4	0.9159	0.9649	-0.049	-5.350
	Thp_1	5.8740	5.9424	-0.068	-1.164
	Thp_2	2.9932	3.7688	-0.776	-25.912
	Thp_3	4.9560	5.8136	-0.858	-17.304
	Thp_4	8.2431	8.6841	-0.441	-5.350
	W_1	7.3473	7.9873	-0.640	-8.711
	W_2	0.1984	0.2159	-0.018	-8.832
	W_3	0.2182	0.2449	-0.027	-12.259
	W_4	0.4205	0.5101	-0.090	-21.306

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=0.5	L_1	45.1006	**47.4637**	-2.3631	-5.2396
	L_2	0.6775	**0.8137**	-0.1362	-20.1033
	L_3	1.2186	**1.4239**	-0.2053	-16.8472
	L_4	3.8667	**4.4299**	-0.5632	-14.5654
	L	50.8634	54.1312	-3.2678	-6.4247
	P_{0000}	0.0034	**0.0010**	0.0024	70.5882
	$P_{0...}$	0.0150	**0.0096**	0.0054	36.0000
	$P_{.0..}$	0.5279	**0.4616**	0.0663	12.5592
	$P_{..0.}$	0.3341	**0.2733**	0.0608	18.1981
	$P_{...0}$	0.0597	**0.0351**	0.0246	41.2060
	U_1	0.9850	0.9904	-0.0054	-0.5482
	U_2	0.4721	0.5384	-0.0663	-14.0436
	U_3	0.6659	0.7267	-0.0608	-9.1305
	U_4	0.9403	0.9649	-0.0246	-2.6162
	Thp_1	5.9100	5.9424	-0.0324	-0.5482
	Thp_2	3.3047	3.7688	-0.4641	-14.0436
	Thp_3	5.3272	5.8136	-0.4864	-9.1305
	Thp_4	8.4627	8.6841	-0.2214	-2.6162
	W_1	7.6312	7.9873	-0.3561	-4.6658
	W_2	0.2050	0.2159	-0.0109	-5.3135
	W_3	0.2288	0.2449	-0.0162	-7.0711
	W_4	0.4569	0.5101	-0.0532	-11.6446

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=1	L_1	47.3461	**47.4637**	-0.1176	-0.2484
	L_2	0.8040	**0.8137**	-0.0097	-1.2065
	L_3	1.4112	**1.4239**	-0.0127	-0.8999
	L_4	4.3981	**4.4299**	-0.0318	-0.7230
	L	53.9594	54.1312	-0.1718	-0.3184
	P_{0000}	0.0011	**0.0010**	0.0001	9.0909
	$P_{0...}$	0.0098	**0.0096**	0.0002	2.0408
	$P_{.0..}$	0.4660	**0.4616**	0.0044	0.9442
	$P_{..0.}$	0.2768	**0.2733**	0.0035	1.2645
	$P_{...0}$	0.0363	**0.0351**	0.0012	3.3058
	U_1	0.9902	0.9904	-0.0002	-0.0202
	U_2	0.5340	0.5384	-0.0044	-0.8240
	U_3	0.7232	0.7267	-0.0035	-0.4840
	U_4	0.9637	0.9649	-0.0012	-0.1245
	Thp_1	5.9412	5.9424	-0.0012	-0.0202
	Thp_2	3.7380	3.7688	-0.0308	-0.8240
	Thp_3	5.7856	5.8136	-0.0280	-0.4840
	Thp_4	8.6733	8.6841	-0.0108	-0.1245
	W_1	7.9691	7.9873	-0.0182	-0.2281
	W_2	0.2151	0.2159	-0.0008	-0.3794
	W_3	0.2439	0.2449	-0.0010	-0.4140
	W_4	0.5071	0.5101	-0.0030	-0.5978

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=2	L_1	47.4634	**47.4637**	-0.0003	-0.0006
	L_2	0.8136	**0.8137**	-0.0001	-0.0123
	L_3	1.4239	**1.4239**	0.0000	0.0000
	L_4	4.4299	**4.4299**	0.0000	0.0000
	L	54.1308	54.1312	-0.0004	-0.0007
	P_{0000}	0.0010	**0.0010**	0.0000	0.0000
	$P_{0...}$	0.0096	**0.0096**	0.0000	0.0000
	$P_{.0..}$	0.4616	**0.4616**	0.0000	0.0000
	$P_{..0.}$	0.2733	**0.2733**	0.0000	0.0000
	$P_{...0}$	0.0351	**0.0351**	0.0000	0.0000
	U_1	0.9904	0.9904	0.0000	0.0000
	U_2	0.5384	0.5384	0.0000	0.0000
	U_3	0.7267	0.7267	0.0000	0.0000
	U_4	0.9649	0.9649	0.0000	0.0000
	Thp_1	5.9424	5.9424	0.0000	0.0000
	Thp_2	3.7688	3.7688	0.0000	0.0000
	Thp_3	5.8136	5.8136	0.0000	0.0000
	Thp_4	8.6841	8.6841	0.0000	0.0000
	W_1	7.9872	7.9873	-0.0001	-0.0006

	W_2	0.2159	0.2159	0.0000	-0.0123
	W_3	0.2449	0.2449	0.0000	0.0000
	W_4	0.5101	0.5101	0.0000	0.0000

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=3	L_1	47.4637	**47.4637**	0.000	0.000
	L_2	0.8137	**0.8137**	0.000	0.000
	L_3	1.4239	**1.4239**	0.000	0.000
	L_4	4.4299	**4.4299**	0.000	0.000
	L	54.1312	54.1312	0.000	0.000
	P_{0000}	0.0010	**0.0010**	0.000	0.000
	$P_{0...}$	0.0096	**0.0096**	0.000	0.000
	$P_{.0..}$	0.4616	**0.4616**	0.000	0.000
	$P_{..0.}$	0.2733	**0.2733**	0.000	0.000
	$P_{...0}$	0.0351	**0.0351**	0.000	0.000
	U_1	0.9904	0.9904	0.000	0.000
	U_2	0.5384	0.5384	0.000	0.000
	U_3	0.7267	0.7267	0.000	0.000
	U_4	0.9649	0.9649	0.000	0.000
	Thp_1	5.9424	5.9424	0.000	0.000
	Thp_2	3.7688	3.7688	0.000	0.000
	Thp_3	5.8136	5.8136	0.000	0.000
	Thp_4	8.6841	8.6841	0.000	0.000
	W_1	7.9873	7.9873	0.000	0.000
	W_2	0.2159	0.2159	0.000	0.000
	W_3	0.2449	0.2449	0.000	0.000
	W_4	0.5101	0.5101	0.000	0.000

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=4	L_1	47.4637	**47.4637**	0.0000	0.0000
	L_2	0.8137	**0.8137**	0.0000	0.0000
	L_3	1.4239	**1.4239**	0.0000	0.0000
	L_4	4.4299	**4.4299**	0.0000	0.0000
	L	54.1312	54.1312	0.0000	0.0000
	P_{0000}	0.0010	**0.0010**	0.0000	0.0000
	$P_{0...}$	0.0096	**0.0096**	0.0000	0.0000
	$P_{.0..}$	0.4616	**0.4616**	0.0000	0.0000
	$P_{..0.}$	0.2733	**0.2733**	0.0000	0.0000
	$P_{...0}$	0.0351	**0.0351**	0.0000	0.0000
	U_1	0.9904	0.9904	0.0000	0.0000
	U_2	0.5384	0.5384	0.0000	0.0000
	U_3	0.7267	0.7267	0.0000	0.0000
	U_4	0.9649	0.9649	0.0000	0.0000

	Thp_1	5.9424	5.9424	0.0000	0.0000
	Thp_2	3.7688	3.7688	0.0000	0.0000
	Thp_3	5.8136	5.8136	0.0000	0.0000
	Thp_4	8.6841	8.6841	0.0000	0.0000
	W_1	7.9873	7.9873	0.0000	0.0000
	W_2	0.2159	0.2159	0.0000	0.0000
	W_3	0.2449	0.2449	0.0000	0.0000
	W_4	0.5101	0.5101	0.0000	0.0000

Time t	Performance	Transient State	Steady State	Difference between Transient and Steady state	% of variation
t=5	L_1	47.4637	**47.4637**	0.000	0.000
	L_2	0.8137	**0.8137**	0.000	0.000
	L_3	1.4239	**1.4239**	0.000	0.000
	L_4	4.4299	**4.4299**	0.000	0.000
	L	54.1312	54.1312	0.000	0.000
	P_{0000}	0.0010	**0.0010**	0.000	0.000
	$P_{0...}$	0.0096	**0.0096**	0.000	0.000
	$P_{.0..}$	0.4616	**0.4616**	0.000	0.000
	$P_{..0.}$	0.2733	**0.2733**	0.000	0.000
	$P_{...0}$	0.0351	**0.0351**	0.000	0.000
	U_1	0.9904	0.9904	0.000	0.000
	U_2	0.5384	0.5384	0.000	0.000
	U_3	0.7267	0.7267	0.000	0.000
	U_4	0.9649	0.9649	0.000	0.000
	Thp_1	5.9424	5.9424	0.000	0.000
	Thp_2	3.7688	3.7688	0.000	0.000
	Thp_3	5.8136	5.8136	0.000	0.000
	Thp_4	8.6841	8.6841	0.000	0.000
	W_1	7.9873	7.9873	0.000	0.000
	W_2	0.2159	0.2159	0.000	0.000
	W_3	0.2449	0.2449	0.000	0.000
	W_4	0.5101	0.5101	0.000	0.000

performance	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	Steady State
L_1	21.4151	33.1679	39.6180	43.1579	45.1006	**47.4637**
L_2	0.1121	0.3021	0.4700	0.5938	0.6775	**0.8137**
L_3	0.2175	0.5709	0.8700	1.0813	1.2186	**1.4239**
L_4	0.7385	1.8917	2.8286	3.4664	3.8667	**4.4299**
L	22.4832	35.9326	43.7866	48.2994	50.8634	54.1312
P_{0000}	0.2281	0.0580	0.0177	0.0068	0.0034	**0.0010**
$P_{0...}$	0.2498	0.0816	0.0360	0.0210	0.0150	**0.0096**
$P_{.0..}$	0.8991	0.7537	0.6440	0.5724	0.5279	**0.4616**
$P_{..0.}$	0.8214	0.6033	0.4622	0.3805	0.3341	**0.2733**
$P_{...0}$	0.5831	0.2706	0.1389	0.0841	0.0597	**0.0351**
U_1	0.7502	0.9184	0.9640	0.9790	0.9850	0.9904

U_2	0.1009	0.2462	0.3560	0.4276	0.4721	0.5384
U_3	0.1786	0.3967	0.5378	0.6195	0.6659	0.7267
U_4	0.4169	0.7294	0.8611	0.9159	0.9403	0.9649
Thp_1	4.5012	5.5104	5.7840	5.8740	5.9100	5.9424
Thp_2	0.7063	1.7234	2.4920	2.9932	3.3047	3.7688
Thp_3	1.4288	3.1736	4.3024	4.4560	5.3272	5.8136
Thp_4	3.7521	6.5646	7.7499	8.2431	8.4627	8.6841
W_1	4.7576	6.0191	6.8496	7.3473	7.6312	7.9873
W_2	0.1587	0.1753	0.1886	0.1984	0.2050	0.2159
W_3	0.1522	0.1799	0.2022	0.2182	0.2268	0.2449
W_4	0.1968	0.2882	0.3650	0.4205	0.4569	0.5101

IX. CONCLUSION

This paper addresses the development and analysis of a K-node tandem Queuing model with bulk arrivals and state dependent service rates. It is assumed that the customers arrive in batches following Zero truncated Geometric distribution. The service rate of each service station is dependent on the content of the buffers connected to it. The explicit expressions for system characteristics such as average number of customers in the Queue, probability of idleness of each service station, throughput of the nodes, average waiting time customers in each Queue, utilisation of each server are derived. The sensitivity of the model with respect to the changes in parameters is explained. It is noticed that the bulk size distribution parameters have significant influence on system performance measures. It is observed that the congestion in the queues and the mean waiting time or delay in service may be reduced by regulating the bulk size distribution parameters. A comparative study of the model between transient and steady state revealed that the time t has significant influence on the performance measures. The proposed model is very useful for understanding the population models , econometrics and return of investment. The present results are new to researchers working on forked queuing network models. Thus the present study enriches literature in this direction.

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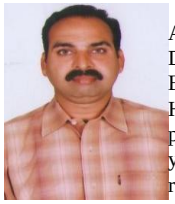

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