International Journal of Recent Technology and Engineering (IJRTE)
ISSN: 2277-3878, Volume-8 Issue-3, September 2019

Oscillation of First Order Linear Delay Differential Equation with Variable Coefficients

P. Sowmiya, G.K. Revathi, N. Preethi

ABSTRACT: **In this article the authors established sufficient condition for the oscillation of the first order linear delay differential equation**

$$x'(t) + p(t)x(t-\tau) = 0 \qquad (*)$$

where $p(t) \in c([t_\circ, \infty), R^+), R^+ = [0, \infty), \tau$ **is a positive constant, with several variable delays. Some interesting examples are provided to illustrate the results.**

KEYWORDS: **Oscillation, delay differential equation and variable coefficients.** ***AMS Subject Classification 2010: 39A10 and 39A12***

I. INTRODUCTION

The major role of differential equations are identifying solutions to many engineering problems. They also have unique applications in Science and Technology. One can easily identify the solutions to various niche problems by means of differential equations. But many fields both in nature and man made machines like biology, Medicines, Chemistry, Physics, Economics, Engineering, Technology etc involves time delays. Whether we like it or not time delays occur very often in almost every situation. Hence they are unavoidable in reality. A simple example in nature is reforestation. A cut forest, after replanting, will take atleast twenty-five years before reaching any kind of maturity. Hence any model of forest harvesting and regeneration clearly must have time delays built into it.The study of the oscillation behavior of solutions of delay differential equation is extremely important in applications. For example, if two or more classicalcharged particles are moving in space, each particle's motion is influenced by the electromagnetic fields of the others. Assuming that those fields propagate with finite speed, any equations attempting to describe this motion via action at a distance will involve time delays which depend on the (unknown) trajectories. If one assumes further that the basic laws of physics are symmetric with respect to time reversal, then the existence of these delays implies that there should also be advanced terms in the equations.

Manuscript published on 30 September 2019

* Correspondence Author

P. Sowmiya, Assistant professor, ShriSakthikailassh Women's college, Salem-636 003. Tamil nadu, India .

G.K.Revathi, Assistant Professor , Division of Mathematics, School of advanced sciences, Vellore Institute of Technology,Chennai 127 Tamil nadu, India .

N. Preethi, Assistant Professor, Sri Krishna College of Engineering and Technology Kuniamuthur, oimbatore 108 Tamilnadu, India

.Also, most of the work in the theory of oscillation of ordinary differential equations are centre around second and higher order differential equations because of the fact that first order ordinary differential equations in general do not posses oscillatory solutions. But for a delay differential equations one can easily identify a solution. The idea of delayed-advanced interactions appeared atleast as early in the works of [1-6], [8-15]. These observations are motivated the authors to study of oscillatory behavior of solutions of delay differential equations

$$x'(t) + p(t)x(t-\tau) = 0 \qquad (1)$$

Where $p(t) \in c([t_\circ, \infty), R^+)$, $R^+ = [0, \infty), \tau$ is a positive constant.

II. OSCILLATION RESULTS

THEOREM: 1

Let $p \in c([t_\circ, \infty), R^+), R^+)$ and let τ be a positive constant and $\lim_{t\to\infty} inf \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$ (2)

then every solution of (1) oscillates.

PROOF: Assume on the contrary that (1) has an eventually positive solution (t). Then there exist a $t^* \geq t_\circ + \tau$ such that for $t \geq t^*, x(t) > 0, x(t-\tau) > 0,\ x'(t) \leq 0$ and $x(t-\tau \geq xt$.

Also from (2) it follows that there exists a constant $c > 0$ and $t_1 \geq t^*$ such that

$$\int_{t-\tau}^{t} p(s)ds \geq c > \frac{1}{e},\ t \geq t_1 \qquad (3)$$

Then

$$x'(t) + p(t)x(t) \leq 0, \qquad t \geq t_1$$

or

$$\frac{x'(t)}{x(t)} + p(t) \leq 0, \qquad t \geq t_1$$

By integrating both sides from $t-\tau$ to t and by using (3) it is found that

$$ln\frac{x(t)}{x(t-\tau)} + c \leq 0, \qquad t \geq t_1 + \tau$$

or $e^c x(t) \leq x(t-\tau), \qquad t \geq t_1 + \tau$

It can easily show that

$e^c \geq ec$ for all c ∈ R.

And so

$$(ec)x(t) \leq x(t-\tau), \quad t \geq t_1 + \tau.$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$(ec)^k x(t) \leq x(t-\tau),\ t \geq t_1 + k\tau$ (4)

Choose k such that

$(2/c)^2 \leq (ec)^k$ (5)

Which is possible because by (3), $ec > 1$.

Now fix a $\tilde{t} \geq t_1 + k\tau$. Then because of (3), then there exist a $\varepsilon\ \epsilon\ (\tilde{t} \geq \tilde{t} + \tau)$ such that

$\int_{\tilde{t}}^{\varepsilon} p(s)ds \geq \frac{c}{2}$ and $\int_{\varepsilon}^{\tilde{t}+\tau} p(s)ds \geq \frac{c}{2}$ (6)

By integrating (1) over the intervals $[\tilde{t}, \varepsilon]$ and $[\varepsilon, \tilde{t}+t]$, we find

$x(\varepsilon) - x(\tilde{t}) + \int_{\tilde{t}}^{\varepsilon} p(s)\, x(s-\tau)ds = 0$(7)

and

$x(\tilde{t}+\varepsilon) - x(\varepsilon) + \int_{\varepsilon}^{\tilde{t}+\tau} p(s)x(s-\tau)ds = 0$ (8)

By omitting the first terms in (7) and (8) and by using the decreasing nature of $x(t)$ and (6) we find

$$-x(\tilde{t}) + x(\varepsilon-\tau)\frac{c}{2} < 0$$

and $-x(\varepsilon) + x(\tilde{t})\frac{c}{2} < 0,$

or $x(\varepsilon) > \frac{c}{2}x(\tilde{t}) > \left(\frac{c}{2}\right)^2 x(\varepsilon-\tau)$ (9)

(9) and (4) imply that

$$(ec)^k \leq \frac{x(\varepsilon-\tau)}{x(\varepsilon)} < \left(\frac{2}{c}\right)^2$$

Which contradicts (5) and completes the proof of the theorem.

NOTE: 1

If $\lim_{t\to\infty} sup \int_{t-\tau}^{t} p(s)ds < \frac{1}{e}$ then equation (1) has a nonoscillatory solution.

If $\int_{t-\tau}^{t} p(s)ds \leq \frac{1}{e}$, then (1) has a positive solution.

If $\lim_{t\to\infty} sup \int_{t-\tau}^{t} p(s)ds > 1$

Then every solution of (1) is oscillatory.

EXAMPLE: 1

Consider

$$x'(t) + \left(\left(\sqrt{2}+\frac{1}{e}\right)\left(\frac{2}{\pi}\right) + \cos t\right)\left(x\left(t-\frac{\pi}{2}\right)\right) = 0 \quad (10)$$

Where $p(t) = \left(\sqrt{2}+\frac{1}{e}\right)\left(\frac{2}{\pi}\right) + \cos t > 0$ for $t \in R_+$.And

$$\int_{t-\frac{\pi}{2}}^{t} p(s)ds = \int_{t-\frac{\pi}{2}}^{t} \left(\left(\sqrt{2}+\frac{1}{e}\right)\left(\frac{2}{\pi}\right) + \cos s\right) ds$$

$$= \sqrt{2} + \frac{1}{e} + \sin t + \cos t$$

Hence

$$\lim_{t\to\infty} inf \int_{t-\frac{\pi}{2}}^{t} p(s)ds = \frac{1}{e},$$

which does not satisfy (2).

However

$$\lim_{t\to\infty} sup \int_{t-\frac{\pi}{2}}^{t} p(s)ds = 2\sqrt{2} + \frac{1}{e} > 1.$$

Consequently (9) satisfies condition $\lim_{t\to\infty} sup \int_{t-\frac{\pi}{2}}^{t} p(s)ds > 1$.

Therefore every solution of (9) is oscillatory.

REMARK: 1 The differential inequality $x'(t) + p(t)x(t-\tau) \leq 0, t \geq t_0$ has no eventually positive solution if (2) holds.

III. IN THEOREM 2, A NEW SUFFICIENT CONDITION FOR OSCILLATION OF (1) IS OBTAINED.

THEOREM: 2

Let $p(t) \in C\left[[t_\circ, \infty), R^+\right]$ and let τ be a positive constant. Suppose that there exist a $\tilde{t} > t_\circ + \tau$ such that

$\int_{t-\tau}^{t} p(s)ds \geq \frac{1}{e}, \quad t \geq \tilde{t}$ (11)

And

$\int_{t_\circ+\tau}^{\infty} p(t)\left[exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) - 1\right]dt = \infty$ (12)

Then every solution of (1) oscillates.

PROOF: Assume on the contrary, that (1) has an eventually positive solution $x(t)$. Then there exists a $t_1 \geq \tilde{t}$ such that for $t \geq t_1, x(t) > 0, x(t-\tau) > 0,$ $x'(t) \leq 0$ and $x(t-\tau) \geq x(t)$.

Let us define

$w(t) = \frac{x(t-\tau)}{x(t)}$ for $t \geq t_1$ (13)

Then

$w(t) \geq 1, \quad t \geq t_1$ (14)

Dividing both sides by (1) by $x(t)$, for $t \geq t_1$, we obtain

$\frac{x'(t)}{x(t)} + p(t)w(t) = 0,\ t \geq t_1$ (15)

Integrating both sides of (15) from $t-\tau$ to t, for $t \geq t_1 + \tau$ yields,

Retrieval Number C4201098319/2019©BEIESP
DOI: 10.35940/ijrte.C4201.098319
Journal Website: www.ijrte.org

Published By:
Blue Eyes Intelligence Engineering
& Sciences Publication

$$\log x(t) - \log x(t-\tau) + \int_{t-\tau}^{t} p(s)w(s)ds = 0\,,$$

$t \geq t_1 + \tau,$ or

$$w(t) = exp\left(\int_{t-\tau}^{t} p(s)\,w(s)ds\right),$$

$t \geq t_1 + \tau.$

By (11) for $t \geq t_1 + \tau$, there exists $\gamma(t)$ with $0 < \gamma(t) \leq \tau$, such that

$\int_{t-\gamma(t)}^{t} p(s)ds = \frac{1}{e}\,, t \geq t_1 + \tau$ (16)

It follows that for $\geq t_1 + \tau$,

$$w(t) = exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t-\gamma(t)} p(s)w(s)ds\right)$$

$$\geq exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t-\gamma(t)} p(s)ds\right)$$

$$= exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right)$$

$$= exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right) exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right)$$

It can be easily shown that $e^c > ec$ for all $c \geq 0$ and so

$w(t) \geq e\int_{t-\gamma(t)}^{t} p(s)w(s)ds\ exp\left(\int_{t-\tau}^{t} p(s)ds - 1e, \quad t \geq t1+\tau\right.$ (17)

$$p(t)w(t) \geq e\,p(t)\int_{t-\gamma(t)}^{t} p(s)w(s)ds\ exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right), t \geq t_1 + \tau$$

or by (11), (14) and (16)

$$p(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right)$$

$$\geq e\,p(t)\int_{t-\gamma(t)}^{t} p(s)w(s)ds\left[exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) - 1\right]$$

$$\geq p(t)\left[exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) - 1\right],$$

$t \geq t_1 + \tau.$

By integrating both sides from $t_2 = t_1 + 2\tau$ to $T > t_2$,it is found that

$$\int_{t_2}^{T} p(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right)dt$$

$$\geq \int_{t_2}^{T} p(t)\left[exp\left(\int_{t-\pi}^{t} p(s)ds - \frac{1}{e}\right) - 1\right]dt$$

(18)

Define a function $N(t) \in c'\left[[t_1, \infty), (0, \infty)\right]$ Such that

$N'(t) = \max_{t_1 \leq s \leq t+\tau} p\,(\,s\,)$ (19)

Where $t \geq t_1$ from (11) and (19), we have

$N'(t) \geq \frac{1}{e}\,, \quad t \geq t_1$ (20)

Thus $N(t)$ is increasing on $[t_1, \infty)$and $\lim_{t\to\infty} N(t) = \infty$. One can easily show that

$\int_{t_1}^{\infty} exp\,(-N(t))dt < \infty$ (21)

And

$\int_{t_1+\tau}^{\infty} p(t)exp\,(-N(t-\tau))dt < \infty$ (22)

Set

$Q(t) = p(t) + \exp\big(-N(t)\big)\,, \ t \geq t_1$ (23)

In view of (17)

$$w(t) - e\int_{t-\gamma(t)}^{t} p(s)w(s)ds \geq 0,$$

$t \geq t_1 + \tau.$

It follows that

$$\int_{t_2}^{T} p(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} p(s)w\,(s)ds\right)dt$$

$$\leq \int_{t_2}^{T} Q(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} Q(s)w(s)ds\right)dt$$

$$+e\int_{t_2}^{T} Q(t)\left(\int_{t-\gamma(t)}^{t} \exp(-N(s))w(s)ds\right)dt.$$

(24)

We claim that

$\lim_{t\to\infty} \sup w(t) = \infty$ (25)

Otherwise, there exists an M>0 such that

$w(t) \leq M, \quad t \geq t_1.$ (26)

Then by using decreasing nature of $\exp(-N(t))$, we have

$$\int_{t_2}^{T} Q(t)\left(\int_{t-\gamma(t)}^{t} \exp(-N(s))w(s)ds\right)dt$$
$$\leq M\tau \int_{t_2}^{T} Q(t)\exp(-N(t-\tau))\,dt$$

From this, (12), (18), (19), (21), (22) and (24), we have

$$\lim_{T\to\infty}\int_{t_2}^{T} Q(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} Q(s)w(s)ds\right)dt = \infty \quad (27)$$

Set

$$u = \sigma(t) = \int_{t_2-\tau}^{t} Q(s)ds\,, t \geq t_2 - \tau \quad (28)$$

Then $\sigma(t) \to \infty$ as $t \to \infty$, $\sigma(t)$ is strictly increasing and, thus σ^{-1} exists.

Set $\quad v(u) = w(\sigma^{-1}(u)) \quad (29)$

Then,

$$\int_{t_2}^{T} Q(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} Q(s)w(s)ds\right)dt$$
$$= \int_{\sigma(t_2)}^{\sigma(T)}\left(w(\sigma^{-1}(u)) - e\int_{\sigma^{-1}(u)-\gamma(\sigma^{-1}(u))}^{\sigma^{-1}(u)} Q(s)w(s)ds\right)du$$
$$= \int_{\sigma(t_2)}^{\sigma(T)}\left(v(u) - e\int_{\sigma(\sigma^{-1}(u)-\gamma(\sigma^{-1}(u)))}^{u} v(\varepsilon)d\varepsilon\right)du$$

It follows that,

$$\int_{t_2}^{T} Q(t)\left(w(t) - e\int_{t-\gamma(t)}^{t} Q(s)w(s)ds\right)dt$$
$$= \int_{\sigma(t_2)}^{\sigma(T)}\left(v(u) - e\int_{u-\frac{1}{e}}^{u} v(s)ds\right)du \quad (30)$$

Because,

$$\sigma\left(\sigma^{-1}(u) - \gamma(\sigma^{-1}(u))\right) = \sigma(t-\gamma(t))$$
$$= \int_{t_2-\tau}^{t-\gamma(t)} Q(s)ds$$
$$= u - \frac{1}{e}$$

From (27), (28) and (30), we obtain

$$\lim_{A\to\infty}\int_{\sigma(t_2)}^{A}\left(v(u) - e\int_{u-\frac{1}{e}}^{u} v(s)ds\right)du = \infty \quad (31)$$

For A > $\sigma(t_2)$,

$$\int_{\sigma(t_2)}^{A} e\left(\int_{u-\frac{1}{e}}^{u} v(s)ds\right)du$$
$$= \int_{\sigma(t_2)-\frac{1}{e}}^{\sigma(t_2)} [es + 1 - e\sigma(t_2)]v(s)ds$$
$$+ \int_{\sigma(t_2)}^{A-\frac{1}{e}} v(s)ds + \int_{A-\frac{1}{e}}^{A} e(A-s)v(s)ds. \quad (32)$$

From (31) and (32), we obtain $\lim_{A\to\infty}\int_{A-\frac{1}{e}}^{A} v(s)ds = +\infty$. Which results in $\lim_{u\to\infty}\sup v(u) = +\infty$ and thus, $\lim_{t\to\infty}\sup w(t) = +\infty$

Which violates (26). Hence (25) holds.

Because of (20), for any $t \geq t_1 + \tau$, there exists a $\varepsilon \in (t-\tau, t)$ such that

$$\int_{\varepsilon}^{t} p(s)ds \geq \frac{1}{2e}, \qquad \int_{t}^{\varepsilon+\tau} p(s)ds \geq \frac{1}{2e} \quad (33)$$

By integrating (1) over the intervals $[\varepsilon, t]$and $[t, \varepsilon+\tau]$, we find that

$$x(t) - x(\varepsilon) + \int_{\varepsilon}^{t} p(s)x(s-\tau)ds = 0 \quad (34)$$

and

$$x(\varepsilon+\tau) - x(t) + \int_{t}^{\varepsilon+\tau} p(s)x(s-\tau)ds = 0 \quad (35)$$

Now, $w(t) < (2e)^2, \;\; t \geq t_1 + \tau$

This contradicts (25) and completes the proof.

COROLLARY: 1

Let $p(t) \in C[[t_\circ, \infty), R^+]$, and let τ be a positive constant. If (11) holds, and $\int_{t_\circ+\tau}^{\infty} p(t)\left[\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right]dt = \infty$, then every solution of (1) oscillates.

EXAMPLE: 2

Consider the delay differential equation

$$x'(t) + \left(\frac{1}{1+t} + \frac{1}{e}\right)x(t-1) = 0, \quad (36)$$

where $t \epsilon [0, \infty)$. Clearly, for $t \geq 1$.

$$\int_{t-1}^{t}\left(\frac{1}{1+t} + \frac{1}{e}\right)dt$$
$$= \log\frac{1+t}{t} + \frac{1}{e} > \frac{1}{e}$$

And $\lim_{t\to\infty}\int_{t-1}^{t}\left(\frac{1}{1+t}+\frac{1}{e}\right)dt=\frac{1}{e}$.

Hence (2) is not satisfied. But for any $T>1$.

$$\int_1^T\left(\frac{1}{1+t}+\frac{1}{e}\right)\log\frac{1+t}{t}dt \geq \frac{1}{e}\int_1^T\log\frac{1+t}{t}dt\to\infty \; as \; T\to\infty$$

Then, by corollary (1), every solution of (36) oscillates.

DEFINITION: 1

Let $p(t)\in C\left[[t_\circ,\infty),R^+\right]$ and define the following sequence of functions as

$$p_1(t)=\int_{t-\tau}^{t}p(s)ds, \qquad t\geq t_\circ+\tau$$

$$p_{k+1}(t)=\int_{t-\tau}^{t}p(s)p_k(s)ds,$$

$$t\geq t_\circ+(k+1)\tau \qquad (37)$$

$$\overline{p_1}(t)=\int_{t}^{t-\tau}p(s)ds, \qquad t\geq t_\circ$$

$$\bar{p}_{k+1}(t)=\int_{t}^{t+\tau}p(s)\bar{p}_k(s)ds, \qquad t\geq t_\circ$$

$k=1,2,3,\cdots$ now by using different methods new sufficient conditions are obtained for the oscillation (1) which improves conditions (11) and (12).

THEOREM: 3

Let $p(t)\in C\left[[t_\circ,\infty),R^+\right]$ and let τ be a positive constant. Suppose that there exist a $t_1>t_\circ+\tau$ and a positive integer n such that

$$p_n(t)\geq\frac{1}{e^n}\;,\overline{p_n}(t)\geq\frac{1}{e^n}\;,t\geq t_1 \qquad (38)$$

and

$$\int_{t_\circ+n\pi}^{\infty}p(t)\left[exp\left(e^{n-1}p_n(t)-\frac{1}{e}\right)-1\right]dt=\infty \qquad (39)$$

Where $p_n(t)$ and $\overline{p_n}(t)$ are defined by (37). Then every solution of (1) oscillates.

PROOF: Assume, on the contrary, that (1) has no eventually positive solution $x(t)$.

Then there exist a $t_2\geq t_1$, such that

$$x(t-\tau)\geq x(t)>0\,,x'(t)\leq 0\,, \qquad t\geq t_2$$

Set $w(t)=\frac{x(t-\tau)}{x(t)}\;,\;t\geq t_2$

Then $w(t)\geq 1\;,\;t\geq t_2 \qquad (40)$

Dividing by both sides of (1) by $x(t)$, for $t\geq t_2$, we obtain

$$\frac{x'(t)}{x(t)}+\frac{p(t)x(t-\tau)}{x(t)}=0\;, \qquad t\geq t_2$$

ie., $\frac{x'(t)}{x(t)}+p(t)w(t)=0\;,\;t\geq t_2 \quad (41)$

$$\log x(t)-\log x(t-\tau)+\int_{t-\tau}^{t}p(s)w(s)ds=0\;, \qquad t\geq t_2+\tau,$$

$$\log\frac{x(t)}{x(t-\tau)}+\int_{t-\tau}^{t}p(s)w(s)ds=0\,, \qquad t\geq t_2+\tau$$

or $w(t)=exp\left(\int_{t-\tau}^{t}p(s)w(s)ds\right)$,

$$t\geq t_2+\tau \qquad (42)$$

It is easy to show that $e^c\geq ec$ for all $c\geq 0$, and so

$$w(t)=e\int_{t-\tau}^{t}p(s)w(s)ds,$$

$$t\geq t_2+\tau \qquad (43)$$

Set

$$w_1(t)=\int_{t-\tau}^{t}p(s)w(s)ds, \qquad t\geq t_2+\tau$$

$$w_2(t)=\int_{t-\tau}^{t}p(s)w_1(s)ds\;, \quad t\geq t_2+2\tau$$

$$\vdots$$

$$w_n(t)=\int_{t-\tau}^{t}p(s)w_{n-1}(s)ds\;,$$

$$t\geq t_2+n\pi \qquad (44)$$

And $v(t)=w(t)-1\;,\;t\geq t_2$

$$v_1(t)=\int_{t-\tau}^{t}p(s)v(s)ds\;, \quad t\geq t_2+\tau$$

$$v_2(t)=\int_{t-\tau}^{t}p(s)v(s)ds\;, \quad t\geq t_2+2\tau$$

$$\vdots$$

$$v_n(t)=\int_{t-\tau}^{t}p(s)v_{n-1}(s)ds\,,$$

$$t\geq t_2+n\tau \qquad (45)$$

By (40),

$$v(t)\geq 0,\;\; t\geq t_2\,,v_i(t)\geq 0,$$

$$t\geq t_2+i\tau,\quad i=1,2,\cdots,n \qquad (46)$$

From (42) and (43), we easily obtain

$$w(t) \geq e^{n-1}w_{n-1}(t), t \geq t_2 + (n-1)\tau,$$

And $w(t) \geq \exp\left(e^{n-1}\int_{t-\tau}^{t} p(s)w_{n-1}(s)ds\right)$,

$$t \geq t_2 + n\pi \qquad (47)$$

In view of (37),(45) and (46),(47) turned as

$$w(t) \geq \exp\left(e^{n-1}\int_{t-\tau}^{t} p(s)v_{n-1}(s)ds + e^{n-1}p_n(t)\right)$$

$$= \exp\left(e^{n-1}\int_{t-\tau}^{t} p(s)v_{n-1}(s)ds + \frac{1}{e}\right)\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right), t \geq t_2 + n\tau$$

And so

$$w(t) \geq \left(e^{n}\int_{t-\tau}^{t} p(s)v_{n-1}(s)ds + 1\right)\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right), t \geq t_2 + n\tau$$

By (38) and (46)

$$p(t)\left[w(t) - \left(e^{n}\int_{t-\tau}^{t} p(s)v_{n-1}(s)ds + 1\right)\right] \geq p(t)\left(e^{n}\int_{t-\tau}^{t} p(s)v_{n-1}(s)ds + 1\right)\left(\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right) - 1\right),$$

$$\geq p(t)\left(\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right) - 1\right), \quad t \geq t_2 + n\tau$$

Or

$$p(t)[v(t) - e^{n}v_n(t)] \geq p(t)\left(\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right) - 1\right),$$

$$t \geq t_2 + n\tau \qquad (48)$$

By integrating both sides of (48) from $t_3 = t_2 + n\tau$ to $T \geq t_3 + n\tau$,

$$\int_{t_3}^{T} p(t)[v(t) - e^{n}v_n(t)]dt \geq \int_{t_3}^{T} p(t)\left(\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right) - 1\right)dt \qquad (49)$$

From (49) and (39) we have

$$\lim_{T\to\infty}\int_{t_3}^{T} p(t)[v(t) - e^{n}v_n(t)]dt = \infty, \qquad (50)$$

Since

$$e^{n}\int_{t_3}^{T} p(t)v_n(t)dt$$

$$= e^{n}\int_{t_3}^{T} p(t)dt \int_{t-\tau}^{t} p(s)v_{n-1}(s)ds$$

$$\geq e^{n}\int_{t_3}^{T-\tau} p(s)v_{n-1}(s)ds \int_{s}^{s+\tau} p(t)dt$$

$$= e^{n}\int_{t_3}^{T-\tau} p(t)\overline{p_1}(t)dt \int_{t-\tau}^{t} p(s)v_{n-2}(s)ds$$

$$\geq \int_{t_3}^{T-2\tau} p(s)v_{n-2}(s)ds \int_{s}^{s+\tau} p(t)\overline{p_1}(t)dt$$

$$= e^{n}\int_{t_3}^{T-2\tau} p(t)\overline{p_2}(t)v_{n-2}dt$$

We have

$$e^{n}\int_{t_3}^{T} p(t)v_n(t)dt$$

$$\geq e^{n}\int_{t_3}^{T-n\tau} p(t)v(t)\overline{p_n}(t)dt$$

$$\geq \int_{t_3}^{T-n\tau} p(t)v(t)\,dt.$$

Thus,

$$\int_{t_3}^{T} p(t)[v(t) - e^{n}v_n(t)]dt \leq \int_{t_3}^{T} p(t)\,v(t)\,dt - \int_{t_3}^{T-n\tau} p(t)v(t)\,dt$$

$$= \int_{T-n\tau}^{T} p(t)v(t)\,dt.$$

In view of (50) we have $\lim_{T\to\infty}\int_{T-n\tau}^{T} p(t)v(t)\,dt = \infty$ (51)

This shows that either $\lim_{T\to\infty}\int_{T-n\tau}^{T} p(t)\,dt = \infty$ (52)

or $\lim_{t\to\infty}\sup v(t) = \infty$ (53)

If (52) holds, then $\lim_{t\to\infty}\sup\int_{t-\tau}^{t} p(s)\,ds = \infty$.

By a known result in [7], every solution of (1) oscillates. If (53) holds, then

$\lim_{t\to\infty}\sup w(t) = \infty$. (54)

On the other hand, integrating both sides of (1) from $t-\tau$ to t we have

$$x(t) - x(t-\tau) + \int_{t-\tau}^{t} p(s)\, x(s-\tau)\, ds = 0.$$

$$t \geq t_2,$$

and so

$$x(t-\tau) > \int_{t-\tau}^{t} p(s)\, x(s-\tau)\, ds, \quad t \geq t_2 \qquad (55)$$

From this, by successively substituting (n−2) times and using the decreasing nature of x(t), it follows that

$$x(t-\tau) > \int_{t-\tau}^{t} p(s) p_{n-2}(s)\, x(s-\tau)\, ds$$

$$> x(t-\tau) \int_{t-\tau}^{t} p(s) p_{n-2}(s)\, ds,$$

And so

$$x(t-\tau) > x(t-\tau) p_{n-1}(t),$$

$$t \geq t_2 + (n-2)\tau. \qquad (56)$$

By (38), for any $t \geq t_1 + \tau$ there exists a $\varepsilon \in (t-\tau, t)$ such that

$$\int_{\varepsilon}^{t} p(s) p_{n-1}(s)\, ds \geq \frac{1}{2e^n}\int_{t}^{\varepsilon+t} p(s) p_{n-1}(s)\, ds \geq \frac{1}{2e^n} \qquad (57)$$

By integrating the both sides of (1) over $[\varepsilon, t]$ and $[t, \varepsilon+\tau]$, we have

$$x(t) - x(\varepsilon) + \int_{\varepsilon}^{t} p(s)\, x(s-\tau)\, ds = 0,$$

$$t \geq t_2 + (n-1)\tau, \qquad (58)$$

and

$$x(\varepsilon + t) - x(t) + \int_{t}^{\varepsilon+t} p(s)\, x(s-\tau)\, ds = 0,$$

$$t \geq t_2 + (n-1)\tau, \qquad (59)$$

substituting (56) into (58) and (59), omitting the first terms in (58) and (59) and by considering the originality of x(t) and (57) , it is clear that

$$-x(\varepsilon) + \frac{1}{2e^n} x(t-\tau) < 0,$$

$$-x(t) + \frac{1}{2e^n} x(\varepsilon) < 0.$$

or $\quad x(t) > \frac{1}{2e^n} x(\varepsilon) > \frac{1}{4e^{2n}} x(t-\tau)$

or $\quad w(t) < 4e^{2n}, \ t \geq t_2 + (n-1)\tau.$

This contradicts (55) and completes the proof of the theorem.

COROLLARY: 2

Let $p(t) \in C[[t_0, \infty), R^+]$ and let τ be a positive constant. Suppose that, for some positive integer n,

$$\operatorname{Lim}_{t\to\infty} \inf p_n(t) \geq \frac{1}{e^n}$$

And $\quad \lim_{t\to\infty} \overline{p_n}(t) > \frac{1}{e^n} \qquad (60)$

Where $p_n(t)$, $\overline{p_n}(t)$ are defined by (1). Then every solution of (1) oscillates.

COROLLARY: 3

Let $p(t) \in C[[t_0, \infty), R^+]$ and let τ be a positive constant. If (10) holds, for some positive integer n,

$$\int_{t_0+n\tau}^{\infty} p(t)\left(e^{n-1} p_n(t) - \frac{1}{e}\right) dt = \infty \qquad (61)$$

Where $p_n(t)$ is defined by (37), then every solution of (1) oscillates.

EXAMPLE: 3

Consider the delay differential equation

$$x'(t) + \frac{1}{2e}(1+\cos t)\, x(t-\pi) = 0, \qquad t \geq 0$$

Clearly for $t \geq \pi$,

$$p_1(t) = \int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\, ds$$

$$= \frac{1}{2e}(\pi + 2\sin t)$$

$$\lim_{t\to\infty} \inf \int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\, ds = \frac{1}{2e}(\pi - 2) < \frac{1}{e}.$$

This shows that (2) and (10) do not hold.

$$p_2(t) = \int_{t-\pi}^{t} p(s)\, p_1(s)\, ds$$

$$= \frac{1}{4e^2}\int_{t-\pi}^{t} (1+\cos s)(\pi + 2\sin s)\, ds$$

$$= \frac{1}{4e^2}(\pi^2 + 2\pi \sin t - 4\cos t)$$

$$p_3(t) = \int_{t-\pi}^{t} p(s)\, p_2(s)\, ds$$

$$= \frac{1}{8e^3}\int_{t-\pi}^{t} (1+\cos s)(\pi^2 + 2\pi \sin s - 4\cos s)\, ds$$

$$= \frac{1}{8e^3}(\pi^3 - 2\pi + (2\pi^2 - 8)\sin t - 4\pi \cos t)$$

$$p_4(t) = \int_{t-\pi}^{t} p(s)\, p_3(s)\, ds$$

$$= \frac{1}{16e^4}\int_{t-\pi}^{t} (1+\cos s)(\pi^3 - 2\pi + (2\pi^2 - 8)\sin s - 4\pi \cos s)\, ds$$

$$= \frac{1}{16e^4}(\pi^4 - 4\pi^2 + 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4)\cos t),$$

$\lim_{t\to\infty} \inf p_4(t) = \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 - \right.$ 2 π3−6 π2+π2−4 2>2216e4

And $\quad \overline{p_1}(t) = \int_{t}^{t+\pi} \frac{1}{2e}(1+\cos s)\, ds$

$$= \frac{1}{2e}(\pi - 2\sin t)$$

$$\overline{p_2}(t) = \int_{t}^{t+\pi} p(s)\, \overline{p_1}(s)\, ds$$

$$= \frac{1}{4e^2}\int_{t}^{t+\pi} (1+\cos s)(\pi - 2\sin s)\, ds$$

$$= \frac{1}{4e^2}(\pi^2 - 2\pi \sin t + 4\cos t)$$

$$\overline{p_3}(t) = \int_{t}^{t+\pi} p(s)\, \overline{p_2}(s)\, ds$$

$= \frac{1}{8e^3}\int_t^{t+\pi}(1+\cos s)(\pi^2 - 2\pi\sin s - 4\cos s)\,ds$

$= \frac{1}{8e^3}(\pi^3 - 2\pi + -(2\pi^2 - 8)\sin t - 4\pi\cos t)$

$\overline{p_4}(t) = \int_t^{t+\pi} p(s)\overline{p_3}(s)ds$

$= \frac{1}{16e^4}\int_t^{t+\pi}(1+\cos s)(\pi^3 - 2\pi - (2\pi^2 - 8)\sin s - 4\pi\cos s)\,ds$

$= \frac{1}{16e^4}(\pi^4 - 4\pi^2 - 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4\cos t,$

$\lim_{t\to\infty}\inf \overline{p_4}(t) = \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 - 2\pi 3 - 6\pi 2 + 4\pi 2 - 4\, 2, > 2216e4,\right.$

Then by corollary (7), every solution of the equation oscillates.

IV. IF THE DELAY τ IS ALSO VARIABLE CONSIDER THE LINEAR EQUATION OF THE FORM

$x'(t) + p(t)\,x(t-\tau)) = 0$ (62)

Where p, $\tau \in C\,[[t_0, \infty), R^+]$, $\tau(t) \le t$ and $\lim_{t\to\infty}(t - \tau(t)) = \infty$.

Set $T_0 = \inf\{t - \tau(t)\}$. $t \ge t_0$

The following result gives the condition for (62) to have a positive solution.

THEOREM: 4

Equation (62) has a positive solution with respect to t_0 if and only if there exists a continuous function $\lambda_0(t)$ on $[T_0, \infty)$ such that $\lambda_0(t) > 0$ for $t \ge t_0$

And

$\lambda_0(t) \ge p(t)\exp\int_{t-\tau(t)}^{t}\lambda_0(s)\,ds$, $t \ge t_0$

COROLLARY: 4

If $\int_{t-\tau(t)}^{t} p(s)ds \le \frac{1}{e}$, $t \ge t_0$

Then (62) has a positive solution with respect to t_0.

NOW THE AUTHORS ESTABLISH SEVERAL SUFFICIENT CONDITIONS FOR THE OSCILLATION OF ALL SOLUTIONS OF THE LINEAR DELAY DIFFERENTIAL EQUATIONS WITH SEVERAL VARIABLE DELAYS OF THE FORM.

$x'(t) + \sum_{i=1}^{n} p_i(t)\,x(t - \tau_i(t)) = 0$

$t \ge 0$ (63)

Where $p_i, \tau_i \in C\,([t_0, \infty), R^+]$, for i = 1,2,⋯,n.

The following result is an extension of theorem (1) to the case of several variable delays. To prove our result we need a lemma.

LEMMA: 1

Suppose that $p \in C\,[[T, \infty), (0, \infty)]$, $\tau \in C\,[[T, \infty), (0, \infty)]$,

$\lim_{t\to\infty}\inf(t - \tau(t)) = \infty$.

And $\lim_{t\to\infty}\inf\int_{t-\tau(t)}^{t} p(s)\,ds > 0$.

Set $T_{-1} = \inf_{t\ge T}\{t - \tau(t)\}$.

Assume that $\alpha \in C\,[[T_{-1}, \infty), (-\infty, 0]]$ satisfies the inequality

$\alpha(t) + p(t)\exp\left(-\int_{t-\tau(t)}^{t}\alpha(s)\,ds\right) \le 0$, $t \ge T$.

Then $\lim_{t\to\infty}\inf\left(-\int_{t-\tau(t)}^{t}\alpha(s)\,ds\right) < \infty$.

THEOREM: 5

Assume that $p_i, \tau_i \in C\,([t_0, \infty), R^+]$, for i = 1,2…n. holds.

Set,

$\tau(t) = \max_{1\le i\le n}\{\tau_i(t)\}\ t \ge 0$ (64)

And suppose that

$\lim_{t\to\infty}(t - \tau(t)) = \infty$. (65)

Then

$\lim_{t\to\infty}\inf\int_{t-\tau(t)}^{t}\sum_{i=1}^{n} p_i(s) > \frac{1}{e}$, (66)

is a sufficient condition for the oscillation of all solutions of equation (63).

PROOF: Assume on the contrary, that (63) has an eventually positive solution x(t). Then there exists T ≥ 0 such that

$x(t) \ge 0$ for T_{-1},

where

$T_{-1} = \min_{1\le i\le n}[\inf_{t\ge T}\{t - \tau_i(t)\}]$.

Therefore, there exists a continuous function $\alpha \in C\,([T_{-1}, \infty], R)$, Such that $\alpha(t) + \sum_{i=1}^{n} p_i(t)\,exp\left(-\int_{t-\tau_i(t)}^{t}\alpha(s)\,ds\right) = 0$, $t \ge T$.

Hence

$\alpha(t) + \left(\sum_{i=1}^{n} p_i(t)\right)exp\left(-\int_{t-\tau(t)}^{t}\alpha(s)\,ds\right) \le 0$, $t \ge T$. (67)

By lemma (1)

$m \equiv \lim_{t\to\infty}\inf\left(-\int_{t-\tau(t)}^{t}\alpha(s)\,ds\right) < \infty$. (68)

By rearranging the terms in (67) and then by integrating from $t - \tau(t)$ to t, it is found that

$\int_{t-\tau(t)}^{t}\alpha(s)ds \ge \int_{t-\tau(t)}^{t}\left(\sum_{i=1}^{n} p_i(u)\right) exp\left(-\int_{u-\tau(u)}^{u}\alpha(s)\,ds\right)du$

Therefore by (68) and (67) we are lead to a contradiction that

$$m \geq \lim_{t\to\infty} \inf\left(\int_{t-\tau(t)}^{t} \sum_{i=1}^{n} p_i(u)\, du \right) e^m$$

$> \frac{1}{e}\, em$ = m

This completes the proof.

NOW WE CONSIDER THE LINEAR DELAY DIFFERENTIAL EQUATION WITH NONPOSITIVE AND NONNEGATIVE COEFFICIENTS OF THE FORM

$x'(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0$ (69)

Where

p,q$\in C([t_0, \infty), R^+), \tau, \sigma \in (0, \infty)$. (70)

The following theorem is an oscillation criterion for (69).

THEOREM: 6

Assume that $\tau \geq \sigma \geq 0$ and

$$\bar{p}(t) = p(t) - q(t+\sigma-\tau)\tau \geq 0\ (\not\equiv 0),\quad t \geq t_1 \geq t_0$$

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \bar{p}(s)ds > 0,$$

And for every $\lambda > 0$

$$\lim_{t\to\infty} \inf\left(\frac{1}{\lambda} \exp\left(\lambda \int_{t-\tau}^{t} \bar{p}(s)ds \right) + \int_{t-\tau}^{t-\sigma} q(s+\sigma-\tau) \exp\left(\lambda \int_{s}^{t} \bar{p}(u)du \right) ds \right) > 1$$

(71)

Then every solution of (61) is oscillatory.

Now we give a sufficient condition for the oscillation of all solutions of (69). The following lemma will be useful in the proof of main result.

LEMMA: 2

Assume that $p, q \in C[(t_0, \infty), R^+]$ and $\tau, \sigma \in [0, \infty)$,

$\tau \geq \sigma$ (72)

$p(t) \geq q(t+\sigma-\tau)$ and

$p(t) \not\equiv q(t+\sigma-\tau)$, for $t \geq t_0 + \tau - \sigma$ (73)

And

$\int_{t-\tau}^{t-\sigma} q(s)ds \leq 1$, For $t \geq t_0 + \tau$ (74)

Let $x(t)$ be an eventually positive solution of (69) and set

$$z(t) = x(t) - \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s)ds$$

For $t \geq t_0 + \tau - \sigma$ (75)

Then eventually $z(t)$is a non-increasing and positive function.

THEOREM: 7

Assume that (70), (72), (73) and (74) hold and that

$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} [p(s) - q(s+\sigma-\tau)]\, ds > \frac{1}{e}$ (76)

Then every solution of (69) oscillates.

PROOF: Assume on the contrary that (69) has an eventually positive solution $x(t)$. By lemma 2, it follows that the function $z(t)$, which is defined by (75), is an eventually positive function.

Also by

$$z'(t) = -[p(t) - q(t+\sigma-\tau)]x(t-\tau) \leq 0$$

And the fact that eventually $0 < z(t) \leq x(t)$, we see that eventually,

$$z'(t) + [p(t) - q(t+\sigma-\tau)]z(t-\tau) \leq 0 \quad (77)$$

But in view of (76) and Remark (1) it follows that the inequality (77) cannot have an eventually positive solution. This contradicts the fact that $z(t)$ is eventually positive and completes the proof.

REMARK: 2

If $q(t) \equiv 0$ in condition (70) of theorem (5), then (70) reduces to the condition

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$$

given in theorem (1).

If $q(t) \equiv 0$ in condition (76) of theorem (6), then (76) reduces to the condition

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$$

given in theorem (1).

V. FINDINGS:

The oscillatory behavior of solutions of first order linear delay differential equation (1) with variable coefficients have been discussed by taking delay τ as a positive constant. Also, sufficient condition for linear delay differential equations with several variable delays were evaluated. Also linear delay differential equation with non positive and non negative coefficients cases are also discussed.

VI. CONCLUSION:

Delay differential equations are differential equations that not only depends on the current state but also depends on the past history. Hence they have tremendous real time applications. In general the first order differential equations do not posses oscillatory solutions. But in this article the authors have found the oscillatory solutions to the first order delay differential equation by giving importance to variable delays. Because of this unique beautiful and flexible nature they can be extensively used in many engineering problems. These findings can be extended further to the study of neutral delay differential equations and advanced delay differential equations.

REFERENCES

1. Bingtuan Li, Oscillations of Delay Differential Equations with Variable Coefficients, **J. Math. Anal. Appl. 192**(1995), 312-321.
2. Chuan Jun Tian and B.G Zhang, New Oscillation Criteria for a Class of Linear Delay Differential Equations, **Bull London Math. Soc. 31**(1999), 196-206.
3. S.G. Deo, V. Lakshmikantham and V. Ragavendra. Text Book of Ordinary Differential Equations, **Tata McGraw Hill Pub. Co. New Delhi,**1997.
4. A. Elbert and I.P. Stavroulakis, Oscillation and Non Oscillation Criteria for Delay Differential Equations, **Proc. Ameri. Math. Soc. 123**(1995), 1503-1510.
5. A. Elbert, Comparison Theorems for First Order Nonlinear Differential with Delay, **Studia Sci. Math. Hungan.11**(1976), 259-267.
6. L.H. Erbe, Qingkai Kong, B.G. Zhang, Oscillation Theory for Functional Differential Equations.
7. L.H. Erbe and B.G. Zhang, Oscillation of First Order Linear Differential Equations with Deviating Arguments, Differential and Integral Equation **I**(1988), 305-314.
8. I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, **Oxford Sci. Publ. Clarendon Press**, 1999.
9. KondalsamyGopalsamy, Stability and oscillations in delay differential Equations of population dynamics, **Springer science and Business media**, 2013
10. G.S. Ladde, V.Lakshmikantham, B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Dekker, New York, 1987.
11. B. Li, Oscillation of First Order Delay Differential Equations, **Proc.** Amer. Math. Soc. 124(1996),3729-3737.
12. X.H Tang and J.S. Yu, Oscillation of First Order Delay Differential Equations, J.Math. Anal. Appl. 248(2000),247-259.
13. Xianhua Tang Jianhua Shan, Oscillations of Delay Differential Equations with Variable Coefficients, J.Math. Anal. Appl.217(1998), 43-71.
14. T.Xianhua, Oscillation of first order delay differential equations with oscillating coefficients, Applied Mathematics-A Journal of Chinese Universities, Springer, 15(3), 252-258, 2000
15. J.S Yu and Jurang Yan, Oscillation in First Order Neutral Differential Equations with "Integrally Small" Coefficient, J.Math. Anal. Appl. 187(1994), 361-370.

AUTHORS PROFILE

P. Sowmiya, Assistant professor, ShriSakthikailassh Women's college, Salem-636 003. Tamilnadu, India . psowmiyababu@gmail.com

G.K.Revathi, Assistant Professor , Division of Mathematics, School of advanced sciences, Vellore Institute of Technology,Chennai 127 Tamilnadu, India . gk_revathi@yahoo.co.in

N. Preethi, Assistant Professor, Sri Krishna College of Engineering and Technology Kuniamuthur, oimbatore 108 Tamilnadu, India .preethi.mphil@gmail.com