Solutions to Non-Linear Diophantine Equation $p^x + (p+5)^y = z^2$ with p is Mersenne Prime

Agus Sugandha, Agung Prabowo, Agustini Tripena

***Abstract**: This research seeks for a solution (if any) to non-linear Diophantine equation $p^x + (p+5)^y = z^2$ with p is Mersenne prime. There are 3 possibilities of solution to the non-linear Diophantine equation, which are single solution, many solutions, or no solution. The research methodology is conducted in two stages, which are using simulation to seek for solution (if any) to non-linear Diophantine equation $p^x + (p+5)^y = z^2$ with p is Mersenne prime and using Catalan's conjecture and characteristics of congruency theory. it is proven that the non-linear Diophantine equation has no solution for $p \neq 3$.*

***Index Terms**:* **Non-linear Diophantine equation, Solution, Catalan's conjecture**

I. INTRODUCTION

Non-linear Diophantine equation is introduced for the first time by Fermat in the Last Fermat Theorem. Fermat introduces non-linear Diophantine equation in the form of $x^n + y^n = z^n$. Specifically, for $n = 2$, all of the solutions are referred to as the Pythagorean triple. However, for $n > 2$, there is no positive integer of x, y and z which satisfies equation $x^n + y^n = z^n$. This is known as Fermat last theorem, which is the foundation of theories of Non-linear Diophantine equation.

Studies of Non-Linear Diophantine Equation have developed very much. Some of the methods to determine solution to non-linear Diophantine equation include using Catalan's conjecture and Congruence theory. Some of the studies are given below [1] studies Diophantine equation $p^x + (p+1)^y = z^2$, with x, y, and z are non-negative integer and p is Mersenne prime, resulting in two solutions of $(p, x, y, z) = (7,0,1,3)$ and $(3,2,2,5)$. In 2012, Sroysang studies non-linear Diophantine equation [2] $3^x + 5^y = z^2$, resulting in single solution of $(x, y, z) = (1,0,2)$.

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In 2012, Sroysang studies non-linear Diophantine equation $31^x + 32^y = z^2$, resulting in no solution [3]. In 2014, Simtrakankul studies non-linear Diophantine equation $\left(2^k - 1\right)^x + \left(2^k\right)^y = z^2$, with k is positive even integer, $k = 2$, resulting in two solutions of $(x, y, z) = (1,0,2)$ and $(2,2,5)$, while for $k \geq 4$, resulting in single solution of $(x, y, z) = (1,0,2^{\frac{k}{2}})$. [4]. In 2016, Rabago [5] studies non-linear Diophantine equation $2^x + 17^y = z^2$, resulting in five solutions of $(x, y, z) = (3,1,5)$, $(5,1,7)$, $(6,1,9)$, $(7,3,71)$, and $(9,1,23)$.

Based on the previous researches, the author is interested in conducting further research. The further research is intended to develop the research conducted by Chotchaisthit in determining solution to non-linear Diophantine equation $p^x + (p+5)^y = z^2$ with p is Mersenne prime.

II. RESULTS AND DISCUSISON

This research employs Catalan's Conjecture, stating that solution for integers $a > 1, b > 1, x > 1, y > 1$,1, b > 1, x > 1, y > 1,to equation $a^x - b^y = 1$ is $(a, x, b, y) = (3,2,2,3)$ (3, 2, 2, 3). For example, Diophantine equation $p^x + (p+5)^y = z^2$, with p is Mersenne prime.

Theorem 1. Diophantine equation $p^x + (p+5)^y = z^2$, with p is Mersenne prime, has only two solutions in the form of non-negative integer, which are $(p, x, y, z) = (3,0,1,3)$ and $(p, x, y, z) = (3,1,0,2)$, 2) 0, 2).

Proof. If p is Mersenne prime and $p = 2^q - 1$ = 2∧q-1 for q of prime while z must be odd and $p \equiv 3 \; (mod\; 4$ 3 (mod 4ga $z^2 \equiv 1 \; (mod\; 4)$ and $p + 5 \equiv 0 \; (mod\; 4)$ ≡ 0 (mod 4)

Diophantine equation $p^x + (p+5)^y = z^2$ is divided into the following cases:

Case 1 : For $x = 0$ and $q = 2$

Diophantine equation $p^x + (p+5)^y = z^2$ the following solution :

$$(2^q + 4)^y = (p+5)^y = z^2 - 1 = (z+1)(z-1)$$

Taking random non-negative integer α and β will result in $8^\alpha = z + 1, 8^\beta = (z - 1)$,where $\alpha > \beta$ and $\alpha + \beta = y$y. Therefore, note that:

$$8^\beta \left(8^{\alpha-\beta} - 1\right) = 8^\alpha - 8^\beta = (z+1) - (z-1) = 2$$

This means $\beta = \frac{1}{3}$ and $8^{\alpha-\beta} - 1 = 1$,

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thus it results in $\alpha = \frac{2}{3}$. Since q is prime, then

take $q = 2$, thus it results in $y = 1$, $p = 3$, and $z = 3$

Therefore , $(p, x, y, z) = (3, 0, 1, 3)$ 3) 3) is the solution to Diophantine equation $p^x + (p+5)^y = z^2$ in this case.

Case 2 : For $x = 0$ and $q \neq 2$

Diophantine equation $p^x + (p+5)^y = z^2$ has the following solution:

$$(2^q + 4)^y = (p+5)^y = z^2 - 1 = (z+1)(z-1)$$

Taking random non-negative integer α and β results in $12^\alpha = z + 1, 12^\beta = (z - 1)$, where $\alpha > \beta$ and $\alpha + \beta = y$

Therefore, note that:

$$12^\beta\left(12^{\alpha-\beta} - 1\right) = 12^\alpha - 12^\beta = (z+1) - (z-1) = 2$$

This means $\beta = {}^{12}\log 2$ and $12^{\alpha-\beta} - 1 = 1$

Thus it results in $\alpha = {}^{12}\log 4 = {}^{12}\log 4$

Since q is prime other than 2, then take $q = 3$, and it results in ${}^{12}\log 8$.. Since it results in non-integer y value, there is no solution to Diophantine equation $p^x + (p+5)^y = z^2$ in this case.

Case 3 : For $x \geq 1$ and $q = 2$

We have $(p+5)^y \equiv 0 \pmod 4$ and $z^2 \equiv 1 \pmod 4$, then $p^x \equiv 1 \pmod 4$. If $x = 2k$ 2k for integer $k \geq 1$, it results in:

$$(2^q + 4)^y = (p+5)^y = z^2 - 1 = (z + p^k)(z - p^k)$$

Taking random non-negative integer α and β results in $8^\alpha = z + p^k, 8^\beta = (z - p^k)$,where $\alpha > \beta$ and $\alpha + \beta = y$

Therefore, note that:

$$8^\beta\left(8^{\alpha-\beta} - 1\right) = 8^\alpha - 8^\beta = (z + p^k) - (z - p^k) = 2p^k$$

This means $= \frac{1}{3}$, $\alpha > \beta = \frac{1}{3}$ and $8^{\alpha-\beta} - 1 = p^k$

Therefore, $z = p^k + 8^\beta$.

With Catalan's conjecture, $8^{\alpha-\beta} - p^k = 1$ has no solution only if $\alpha - 1 > 1$ and $k > 1$- and k>1. Therefore, $\alpha - 1 \leq 1$. We know that $\alpha > 1$, $\alpha + \frac{1}{3} = y$, +1/3=y, while $k \geq 1$.

In case $k = 1$, it results in:

$$8^{y-\frac{2}{3}} - 1 = 8^{\alpha - \frac{1}{3}} - 1 = p = 2^q - 1$$

$$2^{3y-2} - 1 = 2^{3\alpha-1} - 1 = p = 2^q - 1$$

This results in $3y - 2 = q$,y-2=q,a q is prime, then take $q = 2$ and $y = \frac{4}{3}$. Since it results in non-integer y value, there is no solution to Diophantine equation $p^x + (p+5)^y = z^2$ in this case.

Case 4 : For $x = 1$ and $q \neq 2$

We have $(p+5)^y \equiv 0 \pmod 4$ and $z^2 \equiv 1 \pmod 4$, then $p \equiv 1 \pmod 4$. If $x = 1$, it results in :

$$(2^q + 4)^y = (p+5)^y = z^2 - 1 = (z + p^{\frac{1}{2}})(z - p^{\frac{1}{2}})$$

Taking random non-negative integer α and β results in $12^\alpha = z + p^{\frac{1}{2}}, 12^\beta = (z - p^{\frac{1}{2}})$,where $\alpha > \beta$ and $\alpha + \beta = y$. Therefore, note that:

$$12^\beta\left(12^{\alpha-\beta} - 1\right) = 12^\alpha - 12^\beta = \left(z + p^{\frac{1}{2}}\right) - \left(z - p^{\frac{1}{2}}\right) = 2p^{\frac{1}{2}}$$

This means ${}^{12}\log 2$, $\alpha > \beta = {}^{12}\log 2$ and $12^{\alpha-\beta} - 1 = p^{\frac{1}{2}}$. Therefore, $z = p^{\frac{1}{2}} + 12^\beta$

With Catalan's conjecture, $12^{\alpha-\beta} - p^{\frac{1}{2}} = 1$ has no solution only if $\alpha - 1 > 1$. Therefore, $\alpha - 1 \leq 1$. We know that $\alpha > 1$, $\alpha + {}^{12}\log 2 = y$.

Note that

$$12^{y-\frac{2}{3}} - 1 = 12^{\alpha-\frac{1}{3}} - 1 = p^{\frac{1}{2}} = (2^q - 1)^{\frac{1}{2}}$$

This results in $12^{y-\frac{2}{3}} - 1 = (2^q - 1)^{\frac{1}{2}}$, since q is prime with $q \neq 2$, then take $q = 3$ and $y = {}^{12}\log 4(1 + \sqrt{3})\ (12)^{\frac{3}{2}}$.

Since it results in non-integer y value, there is no solution to Diophantine equation $p^x + (p+5)^y = z^2$ in this case.

Case 5 : For $x = 1$ and $q = 2$

We have $(p+5)^y \equiv 0 \pmod 4$ and $z^2 \equiv 1 \pmod 4$, then $p \equiv 1 \pmod 4$. If $x = 1$, it results in :

$$(2^q + 4)^y = (p+5)^y = z^2 - 1 = (z + p^{\frac{1}{2}})(z - p^{\frac{1}{2}})$$

Taking random non-negative integer α and β results in $8^\alpha = z + p^{\frac{1}{2}}, 8^\beta = (z - p^{\frac{1}{2}})$, where $\alpha > \beta >$ β $\alpha + \beta = y$. Therefore, note that:

$$8^\beta\left(8^{\alpha-\beta} - 1\right) = 8^\alpha - 8^\beta = \left(z + p^{\frac{1}{2}}\right) - \left(z - p^{\frac{1}{2}}\right) = 2p^{\frac{1}{2}}$$

This means $= \frac{1}{3}$, $\alpha > \beta = \frac{1}{3}$ and $8^{\alpha-\beta} - 1 = p^{\frac{1}{2}}$

Therefore, $z = p^{\frac{1}{2}} + 8^\beta$

With Catalan's conjecture, $8^{\alpha-\beta} - p^{\frac{1}{2}} = 1$ has no solution only if $\alpha - 1 > 1$. Therefore, $\alpha - 1 \leq 1$. We know that $\alpha > 1$, $\alpha + \frac{1}{3} = y$.

Note that

$$8^{y-\frac{2}{3}} - 1 = 8^{\alpha-\frac{1}{3}} - 1 = p^{\frac{1}{2}} = (2^q - 1)^{\frac{1}{2}}$$

$$2^{3y-2} - 1 = 2^{3\alpha-1} - 1 = p = (2^q - 1)^{\frac{1}{2}}$$

This results in $2^{3y-2} - 1 = (2^q - 1)^{\frac{1}{2}}$, since q is prime, then take $q = 2$ and $3 + 8^y = z^2$

$$\sqrt{3 + 8^y} = z \qquad (*)$$

Integer satisfies equation (*) only if $z = 2$ and $y = 0$. It, therefore, results in $y = 0, p = 3$, and $z = 2$ z=2 thus $(p, x, y, z) = (3, 1, 0, 2)$ 2) 2) is the solution to Diophantine equation $p^x + (p+5)^y = z^2$ in this case.

III. CONCLUSION

If p is part of *Mersenne* prime, we may prove that Diophantine equation $p^x + (p+5)^y = z^2$ has no solution for $p \neq 3$.

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