Julia Set with Trigonometric Function

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Abstract*: Julia sets are generated by initializing a complex number $z = x + yi$ where z is then iterated using the iteration function $f_c(z) = z_n^2 + c$, where n indicates the number of iteration and c is a constant complex number. Recently, study of cubic Julia sets was introduced in Noor Orbit (NO) with improved escape criterions for cubic polynomials. In this paper, we investigate the complex dynamics of different functions and apply the iteration function to generate an entire new class of Julia sets. Here, we introduce different types of orbits on cubic Julia sets with trigonometric functions. The two functions to investigate from Julia sets are sine and cosine functions.

***Index Terms*: Julia Sets, Cubic Julia Set, Trigonometric Functions.**

I. INTRODUCTION

A fractal is an abstract object used to describe and to simulate naturally occurring objects. Julia sets can be simple like acircle, or extremely complicated like a fractal. Interesting facts about Julia sets and related mathematics began in the 1920's with Gaston Julia. His extraordinary talents were recognized from an early age and altough in every subject. Julia sets are defined by iterating a function of a complex number. Julia sets have been studied for quadratic [4,5,8,9], cubic [4,5,7] and also for higher degree polynomials. Pick appoint in the complex plane (i.e., a complex number; this can be represented as a point $z = (x,y)$ in the plane). Iterate the function starting at this. We can create complicated and nice fractals using $f_{n+1}(z) = z_n^2 + c$. What happens if we use trigonometry function instead of just complex number z? The two obvious functions to investigate in this research are sine and cosine functions.

Kodri and Titaley [2] investigate a combination of simple complex number fractals and Batik Minahasa motives, which are motives of traditional clothes of people of Minahasa, Indonesia. Their result shows that by selecting a complex number$c = a + bi$, where $-2 \leq a \leq 2$ and $-2 \leq b \leq 2$ gives interesting shapes of Julia sets. Ashish et.al [7] generated study of cubic Julia sets in Noor Orbit (NO) and took the shapes of Christmas tree, Sikh Mythological symbol Khanda, and wall decorative pictures. We recall the definitions of Noor Orbit, Julia Set, and different fixed points as defined by Ashish et al. [7].

We recall the definitions of Noor Orbit, Julia Set, and different fixed points as defined by Ashish et al. [7].

Definition 1 (Noor Orbit). Let us consider a sequences $\{x_n\}$ of iterates for initial point $x_0 \in X$ such that,

$$NO(T, x_0, \alpha_n, \beta_n, \gamma_n) = \{x_{n+1} : x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n;$$
$$y_n = (1-\beta_n)x_n + \beta_n T z_n ;$$
$$z_n = (1-\gamma_n)x_n + \gamma_n T x_n ;\ n = 0,1,2,\ldots\},$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers, which is called as Noor-orbit (NO). The above sequences of iterates is called as Noor iterates [7].

Definition 2. (Prison set and Escape set [7]) Let $z_{n+1} = z_n^2 + c$, $n = 1,2,3$ and form an orbitz_0, z_1, ... with starting point $z_0$thenthe orbit is anescape set for c if $|z_n| \to \infty$, for$n \to \infty$.$E_c = \{z_0 : |z_n| \to \infty \text{ If } n \to \infty\}$is prisoner set for$c$ *parameter*, $P_c = \{z_0 : z_0 \notin E_c\}$.

Definition 3 (Julia Sets[7]) The filled in Julia set of the function f is defined as $K(f) = \{z \in C : f(z)$ does not tend to ∞, where C is complex space. The Julia set of the function A is defined to be the boundary of $K(f)$ i.e.$J(f) = \partial K(f)$

where $J(f)$ denotes the Julia set. The set of points whose orbits are bounded under the orbit for the following cubic polynomials

$$f_{m,n} = z^3 + az + c.$$

A Julia set thus, satisfies the following properties:

(i) Closed;
(ii) Nonempty;
(iii) Forward invariant t (If $z \in J(f)$, then $f(z) \in J(f)$, where f is the function);
(iv) Backward invariant;
(v) Equal to the closure of the set of repelling cycles of f.

In trigonometric function $f(z) = \sin(z)$, 0 is defined as fixed point for f. If $z_0 \in R$, then either $f(z_0) = 0$ or $f(z_0) \to 0$. Also, we have the points lying on the imaginary axis have their orbits that tend to infinity.

A fixed point $z_0$is said to be an *attracting fixed point* for f if there is a neighborhood D of z_0 such that if $z \in D$, then $f(z) \in D$ for all $n > 0$, and in fact

$$f(z) \to z_0 \quad \text{as} \quad n \to \infty.$$

A fixed point z_0 is said to be a *repelling fixed point* for f if there is a deleted neighborhood D of z_0 such that if $z \in D$, then $f(z) \notin D$ for some $n > 0$. This means that an orbit with an initial condition starting even very close to z_0 will eventually need to move away from z_0. If $|f'(z_0)| = 1$, the fixed point at z_0 is said to be*neutral*. It maybe that $z_0$is attracting, repelling, or neutral.

Orbits will escape for Cosine function, if $[C^n_\gamma(z) \to \infty$ as $n \to \infty$. Here, the imaginary component will increase, while the real component will remain. Sine andCosine functions

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are thus declared as topologically complete [4]. We introduce in this paper trigonometric functions of the type $\{\sin(z^*) + z\}$ and $\{\cos(z^*) + z\}$ with $z^* \to z^3 + mz + n$ and applied iterated function system to develop an entirely new class of Julia set which gives the escape criterion for cubic polynomial.

II. METHODS

The process of generating Julia set images $z \to \sin(z^*) + c$ and $z \to \cos(z^*) + c$ is similar to the one employed for the self-squared function. This process consists of iterating the function up to N times. Starting from a value z_0, we obtain $z_1, z_2, z_3 \ldots$ by applying the transformations $z \to \sin(z^*) + c$ and $z \to \cos(z^*) + c$ one by one respectively.
The research method can be seen in the flowchart in Figure. 1

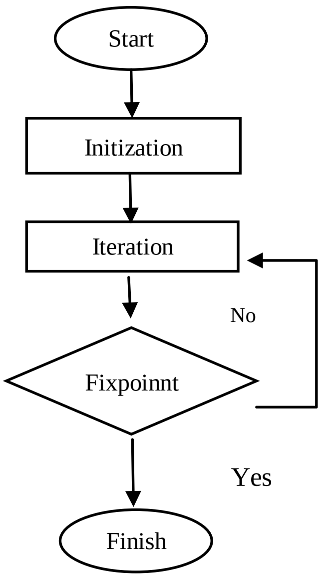

Fig. 1. Flowchart of Julia sets

Algorithm J

a. Determine the numbers of points (n) of each orbit;
b. Determine the color of points inside Julia set with color other than black and the color of points outside Julia set with black;
c. Put the complex number a and c;
d. Choose points on the plane to be examined;
e. For every chosen point on the plane, computre from initial point of orbit using the following function $f_c(z) = z^3 + az + c$;
f. If there is a point of an orbit outside a circle with radius 2, stop iteration; this is the fixpoint;
g. If all point of orbit inside circle of radius 2, then color the Julia set with black;
h. If all points on the plane have been computed, the process is finished;

III. RESULTS AND DISCUSSION

Using computational work in Matlab, we generated Julia sets for cubic polynomials with trigonometric function. We iterated the cubic polynomial $z^* \to z^3 + az + c$ where a and c are complex number with iterative function system. Which gives the Escape criterion for the following cubic polynomials :

$$F_{a,c}(z) = (z^3 + az + c), \qquad (1)$$

where m and n are complex numbers.

Suppose $f(z)$ has a fixed point at z_0, then z_0 is

- Attracting if $0 < |f'(z_0)| < 1$,
- Repelling if $|f'(z_0)| > 1$,
- Neutral if $|f'(z_0)| = 1$.

For quadratic complex polynomials $f_c(z) = z^2 + az + c$, we will choose $T_c z = z^2 + c$ and $z' = az$ where a and c complex numbers.

Theorem 1. Assume that $|z| \geq |c| > 2(1+|a|)/\alpha$, $|z| \geq |c| > 2(1+|a|)/\beta$, and $|z| \geq |c| > 2(1+|a|)/\gamma$, where $0<\alpha,\beta,\gamma<1$ and a, c are a complex numbers. Then there exist $\lambda > 0$, such that $|z_1| > (1 + \lambda)|z|$ and $|z_n| > (1 + \lambda)^n|z|$, for $n > 1$.

Proof. Define from (1)

$$z_{n+1} = (1-\alpha)z_n' + \alpha T w_n;$$

$$w_n = (1-\beta)z_n' + \beta T u_n;$$

$$u_n = (1-\gamma)z_n' + \gamma T z_n\,; \quad n = 0,1,2,\ldots \qquad (2)$$

$f_c(z) = z^2 + az + c$, we will choose $T_c z = z^2 + c$ and $z' = az$

(i) For $z_0 = z;\ w_0 = w;\ u_0 = u$

$$u = (1-\gamma)z_n' + \gamma T z_n$$
$$= (1 - \gamma)\, az + \gamma(z^2 + c)$$
$$= (1 - \gamma)\, az + (1-(1-\gamma)(z^2 + c)$$
$$|u| = |(1 - \gamma)\, az + (1-(1-\gamma)(z^2 + c)| \qquad (3)$$
$$\geq (1 - (1-\gamma)\ |(z^2 + c)| - |(1-\gamma)\, az| \qquad (4)$$
$$\geq \gamma\, |z^2| - \gamma|c| - |az| + |\gamma az|$$
$$\geq \gamma\, |z^2| - \gamma|z| - |az| + |\gamma az| \quad \ldots |z| \geq c \qquad (5)$$
$$\geq \gamma\, |z^2| - \gamma|z| - |az| \quad \ldots |a| \geq 0 \qquad (6)$$
$$\geq \gamma\, |z^2| - |z| - |az| \quad \ldots \gamma < 1 \qquad (7)$$
$$\geq \gamma\, |z^2| - |az| - |z|$$
$$= \gamma\, |z^2| - (1 + |a|)\, |z|$$
$$= |z|\, (\gamma\, |z| - (1 + |a|)$$
$$|u| \geq |z|\, (\gamma\, |z| - (1 + |a|) \qquad (8)$$
$$|u| \geq |z|\, \left(\frac{\gamma|z|}{(1+|a|}-1\right) \qquad (9)$$

(ii) For $z_0 = z;\ w_0 = w;\ u_0 = u$

$$w_n = (1-\beta)z_n' + \beta T u_n$$
$$w = (1 - \beta)\, az + \beta(u^2 + c) \qquad (10)$$
$$= az - \beta az + uw^2 + \beta c$$
$$|w| = |az - \beta az + \beta u^2 + \beta c \qquad (11)$$
$$\geq \beta\, |u^2| - \beta|c| - |az| + |\beta az|$$
$$\geq \beta\, |u^2| - \beta|z| - |az| + |\beta az| \ \ldots.\ |z| \geq c \qquad (12)$$
$$\geq \beta\, |u^2| - |z| - |az|$$

$$|w| \geq \beta\, |u^2| - (1 + |a|)\, |z| \qquad (13)$$

we know that $|u| \geq |z| \left(\frac{\gamma|z|}{(1+|a|} - 1\right)$ so

$$|u|^2 \geq |z|^2 \left(\frac{\gamma|z|}{(1+|a|} - 1\right)^2 > |z|^2 > \gamma\, |z|^2 \text{ this implies}$$

$$|w| \geq \beta\, |u^2| - (1 + |a|)\, |z|$$

$$\geq \beta\gamma|z|^2 - (1 + |a|)|z| \quad (14)$$

$$= |z|(\,\beta\gamma|z| - (1 + |a|)) \quad (15)$$

$$|w| \geq |z| \left(\frac{\beta\gamma|z|}{(1+|a|} - 1\right) \quad (16)$$

(iii) For $z_0 = z;\ w_0 = w;$

$$z_n = (1-\alpha)z_{n-1}' + \alpha T w_{n-1} \quad (17)$$

$$|z_n| = |(1-\alpha)z_{n-1}' + \alpha T w_{n-1}|$$

$$|z_1| = |(1 - \alpha)z' + \alpha T w| \quad (18)$$

$$= |(1 - \alpha)az + \alpha(w^2 + c)| \quad (19)$$

$$= |az - \alpha az + \alpha\, w^2 + \alpha c|$$

$$\geq (1 - (1 - \alpha))\, |w^2 + c| - |(1 - \alpha)az| \quad (20)$$

$$\geq \alpha\, |w^2| - \alpha|z| - |az| + |\alpha az| \quad (21)$$

$$\geq \alpha\, |w^2| - \alpha|z| - |az| \quad (22)$$

$$\geq \alpha\, |w^2| - |az| - |z| \quad (23)$$

$$|z_1| \geq \alpha\, |w^2| - (1 + |a|)|z| \quad (24)$$

we know that

$$|w| \geq |z| \left(\frac{\beta\gamma|z|}{(1+|a|} - 1\right)$$

So $|w|^2 \geq |z|^2 \left(\frac{\beta\gamma|z|}{(1+|a|} - 1\right)^2 > |z|^2 > \beta\gamma|z|^2$.

This implies

$$|z_1| \geq \alpha\, |w^2| - (1 + |a|)|z|$$

$$\geq \alpha\beta\gamma\, |z|^2 - (1 + |a|)|z|$$

$$\geq |z|\, (\alpha\beta\gamma\, |z| - (1 + |a|))$$

$$|z_1| \geq |z| \left(\frac{\alpha\beta\gamma|z|}{(1+|a|} - 1\right) \quad (25)$$

Since $|z| \geq |c| > 2(1+|a|)/\alpha$, $|z| \geq |c| > 2(1+|a|)/\beta$, and $|z| \geq |c| > 2(1+|a|)/\gamma$, implies $|z| \geq |c| > 2(1+|a|)/\alpha\beta\gamma$, so that there exist $\lambda > 0$ such that $\alpha\beta\gamma\, |z| / (1 + |a|) > 2 + \lambda$, $\alpha\beta\gamma\, |z| / (1 + |a|) - 1 > 1 + \lambda$. Consequently $|z_1| > (1 + \lambda)|z|$. If we apply this argument until n-times, we find $|z_n| > (1 + \lambda)^n|z|$. So if $n \to \infty$, the orbit of z tendsto infinity or orbit of z is escape set. □

For cubic complex polynomials $f_c(z) = z^3 + az + c$, we will choose $T_c z = z^3 + c$ and $z' = az$ where a and c complex numbers.

Theorem 2. Let $|z| > |c| > 2(1 + |a|)/\alpha)^{1/2}$, $|z| > |c| > (2(1 + |a|)/\beta)^{1/2}$ $|z| > |c| > 2(1 + |a|)/\gamma)^{1/2}$ exist, where $0 < \alpha, \beta, \gamma < 1$ and a, c are complex numbers. Then there exist $\lambda > 0$, such that $|z_1| > (1 + \lambda)|z|$ and $|z_n| > (1 + \lambda)^n|z|$, for $n > 1$.

Proof. Define from (1)

$$z_{n+1} = (1-\alpha)z_n' + \alpha T w_n;$$

$$w_n = (1-\beta)z_n' + \beta T u_n;$$

$$u_n = (1-\gamma)z_n' + \gamma T z_n;\quad n = 0,1,2,\ldots \quad (26)$$

$f_c(z) = z^3 + az + c$, we will choose $T_c z = z^3 + c$ and $z' = az$

(iv) For $z_0 = z;\ w_0 = w;\ u_0 = u$

$$u = (1-\gamma)z_n' + \gamma T z_n$$

$$= (1 - \gamma)\, az + \gamma(z^3 + c)$$

$$= (1 - \gamma)\, az + (1-(1-\gamma)\, (z^3 + c)$$

$$|u| = |\, (1 - \gamma)\, az + (1-(1-\gamma)\, (z^3 + c)\, | \quad (27)$$

$$\geq (1 - (1 - \gamma)\, |(z^3 + c)| - |(1-\gamma)\, az\,| \quad (28)$$

$$\geq \gamma\, |z^3| - \gamma|c| - |az| + |\gamma az|$$

$$\geq \gamma\, |z^3| - \gamma|z| - |az| + |\gamma az|, \text{ because } |z| \geq c \quad (29)$$

$$\geq \gamma\, |z^3| - \gamma|z| - |az|, \text{ because } |a| \geq 0 \quad (30)$$

$$\geq \gamma\, |z^3| - |z| - |az|, \text{ because } \gamma < 1 \quad (31)$$

$$\geq \gamma\, |z^3| - |az| - |z|$$

$$= \gamma\, |z^3| - (1 + |a|)\, |z|$$

$$= |z|\, (\gamma\, |z^2| - (1 + |a|)$$

$$|u| \geq |z|\, (\gamma\, |z^2| - (1 + |a|) \quad (32)$$

$$|u| \geq |z| \left(\frac{\gamma|z^2|}{(1+|a|} - 1\right) \quad (33)$$

(v) For $z_0 = z;\ w_0 = w;\ u_0 = u$

$$w_n = (1-\beta)z_n' + \beta T u_n$$

$$w = (1 - \beta)\, az + \beta(u^3 + c) \quad (34)$$

$$= az - \beta az + uw^3 + \beta c$$

$$|w| = |az - \beta az + \beta u^3 + \beta c \quad (35)$$

$$\geq \beta\, |u^3| - \beta|c| - |az| + |\beta az|$$

$$\geq \beta\, |u^3| - \beta|z| - |az| + |\beta az| \ \ldots.\ |z| \geq c \quad (36)$$

$$\geq \beta\, |u^3| - |z| - |az|$$

$$|w| \geq \beta\, |u^3| - (1 + |a|)\, |z| \quad (37)$$

We know that

$$|u| \geq |z| \left(\frac{\gamma|z^2|}{(1+|a|} - 1\right)$$

so $|u|^3 \geq |z|^3 \left(\frac{\gamma|z|}{(1+|a|} - 1\right)^3 > |z|^3 > \gamma\, |z|^3$

this implies

$$|w| \geq \beta\, |u^3| - (1 + |a|)\, |z|$$

$$\geq \beta\gamma|z|^3 - (1 + |a|)|z| \quad (38)$$

$$= |z|(\,\beta\gamma|z^2| - (1 + |a|)) \quad (39)$$

$$|w| \geq |z| \left(\frac{\beta\gamma|z^2|}{(1+|a|} - 1\right) \quad (40)$$

(vi) For $z_0 = z;\ w_0 = w;$

$$z_n = (1-\alpha)z_{n-1}' + \alpha T w_{n-1}$$

$$|z_n| = |(1-\alpha)z_{n-1}' + \alpha T w_{n-1}| \quad (41)$$

$$|z_1| = |(1-\alpha)z' + \alpha T w| \quad (42)$$

$$= |(1-\alpha)az + \alpha(w^3 + c)| \quad (43)$$

$$= |az - \alpha az + \alpha\, w^3 + \alpha c|$$

$$\geq (1-(1-\alpha))\,|3^2 + c| - |(1-\alpha)az| \quad (44)$$

$$\geq \alpha\,|w^3| - \alpha|z| - |az| + |\alpha az| \quad (45)$$

$$\geq \alpha\,|w^3| - \alpha|z| - |az|$$

Hence, $|z_1| \geq \alpha\,|w^3| - (1 + |a|)|z|$.

We know that

$$|w| \geq |z|\,\left(\frac{\beta\gamma|z^2|}{(1+|a|)} - 1\right), \text{ therefore}$$

$$|w|^3 \geq |z|^3\,\left(\frac{\beta\gamma|z^2|}{(1+|a|)} - 1\right)^3 > |z|^3 > \beta\gamma|z|^3.$$

This implies

$$|z_1| \geq \alpha\,|w^3| - (1 + |a|)|z|.$$

$$\geq |z|\,(\alpha\beta\gamma\,|z^2| - (1 + |a|))$$

$$|z_1| \geq |z|\,\left(\frac{\alpha\beta\gamma|z^2|}{(1+|a|)} - 1\right).$$

Since $|z| \geq |c| > (2(1+|a|)/\alpha)^{1/2}$, $|z| \geq |c| > (2(1+|a|)/\beta)^{1/2}$, and $|z| \geq |c| > (2(1+|a|)/\gamma)^{1/2}$, this implies $|z| \geq |c| > (2(1+|a|)/\alpha\beta\gamma)^{1/2}$, so that there exist $\lambda > 0$ such that $\alpha\beta\gamma\,|z^2|\,/\,(1 + |a|) > 2 + \lambda$, and $\alpha\beta\gamma\,|z^2|\,/\,(1 + |a|) - 1 > 1 + \lambda$. Consequently, $|z_1| > (1 + \lambda)|z|$. If we apply this argument until n-times, we find $|z_n| > (1 + \lambda)^n|z|$. So if $n \to \infty$, the orbit of z tendsto infinity or orbit of z is escape set.

A. Graphical Representation of Cubic Julia Set

- **The fractal generated from equation $z \to \cos(z^*) + c$possesses symmetry along the real axis.**

Fig. 2 shows symmetric orbit along the real axis in Julia Set for α=1.333, $\beta=\gamma$=1.0005, a=0.5+0.255i, c=1.000555$i$convergent to the point 0.427 + 0.3141i after the 73rd iteration.

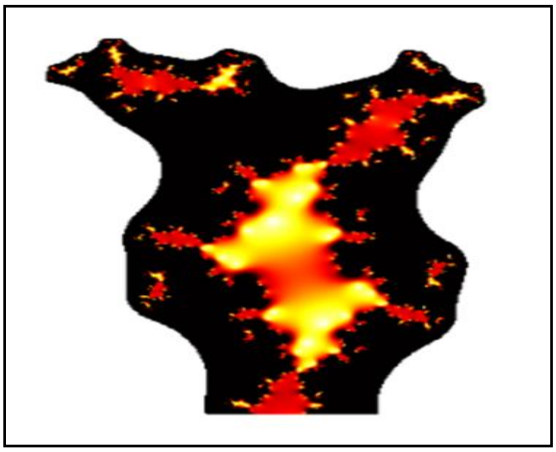

Fig. 2: Cubic Julia Set for α=1.333, $\beta=\gamma$=1.0005, a=0.5+0.255i, and$c$=1.000555i.

But if the initial point is taken in the escape set represented by α=1.333,$\beta=\gamma$=1.0005, and a=1.000555i (see Fig. 3), along the imaginer axis, the orbit is not convergent. The Julia Set for this function in the complex plane behaves chaotic.

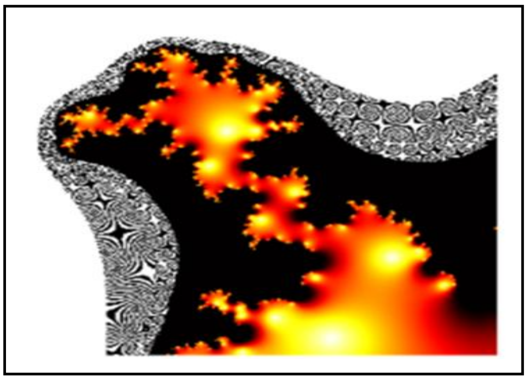

Fig. 3 : Cubic Julia Set for α=1.333, $\beta=\gamma$=1.0005, a=1.000555i, and c =0.5+0.255i.

- **The fractal generated from $z \to \sin(z^*) + c$possesses symmetry along the real axis.**

Fig. 4 and Fig. 5 show symmetric orbit along the real axis in Julia Set for α= 0.000, $\beta=\gamma$ =0.000, a = -0.5555557i, c = -0.5 – 0.255i and the Cubic Julia Set for α=0.000, $\beta=\gamma$=0.000, a=-0.5-0.255i, c=-0.555557i is convergent.

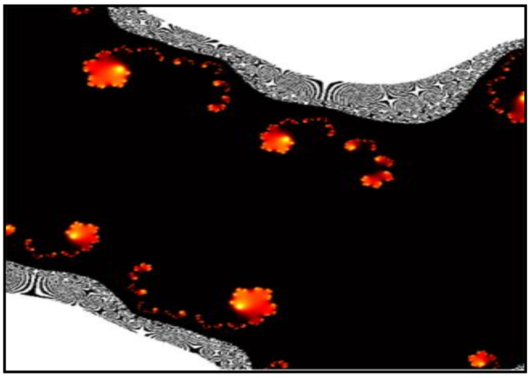

Fig.4. Cubic Julia Set for α=0.000, $\beta=\gamma$=0.000, a=-0.555557i, c=-0.5-0.255i.

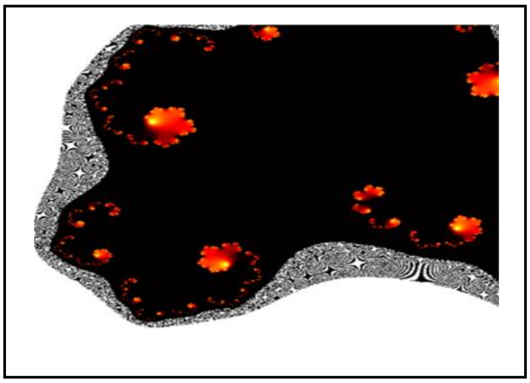

Fig. 5. Cubic Julia Set for α=0.000, $\beta=\gamma$=0.000, a=-0.5-0.255i, c=-0.555557i.

In case of $z \to \sin(z^*) + c$, it shows a bigger part that has a shape like a lobe and similar shape smaller parts. The bigger part consists of six wings with same rotating orientation. The orbit of this Julia Set shows a symmetry of both axes.

B. Fixed Points of CubicPolynomial

(i) Cosine Function

Table 1. Orbit of *f(z)*

Number of Iteration	f(z)	Number of Iteration	f(z)
1	0.5+0.255i	11	0.509+0.314i
2	0.489+0.250i	12	0.416+0.344i
3	0.406+0.379i	13	0.469+0.358
4	0.485+0.346i	...	....
5	0.438+0.335i	...	....
6	0.456+0.355i	73	0.427+0.314i
7	0.454+0.340i	74	0.427+0.314i
8	0.499+0.510i	75	0.427+0.314i
9	0.494+0.245i	76	0.427+0.314i
10	0.434+0.121i	77	0.427+0.314i

Here we observe that the value converges to a fixed point after 73 iterations.

(ii) Sine Function

Table 2. Orbit of *f(z)*

Number of Iteration	*f(z)*	Number of Iteration	*f(z)*
51	0.523+0.022i	61	0.9503+0.042i
52	0.991+0.014i	62	....
53	0.561+0.155i	63	....
54	0.981+0.016i	64	....
55	0.542+0.047i	65	....
56	0.988+0.006i	66	....
57	0.425+0.066i	67	0983+0.045i
58	0.843+0.055i	68	0983+0.045i
59	0.9918+0.001i	69	0983+0.045i
60	0.9503+0.044i	70	0983+0.045i

Here we skipped 50 iteration and observed that the value converges to fixed point after 67 iterationn.

IV. CONCLUSION

Julia set is collection limit of several points between escape set and prison set. In this paper we studied the cosine function which is one of the members of transcendental family. Orbit for these functions on the real axis tend to 0, but on the imaginary axis tends to infinity. The results thus obtained are innovative and studies about different behavior of two basic trigonometry.

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