Quotient Square Sum Cordial Labeling

T.M. Selvarajan, Swapna Raveendran

Abstract—Let G = (V, E) be a simple graph and φ : V → {1, 2, ..., |V|} be a bijection, for each edge uv assigned the label 1 if \[\frac{[\phi(u)]^2 + [\phi(v)]^2}{\phi(u) + \phi(v)}\] is odd and 0 if \[\frac{[\phi(u)]^2 + [\phi(v)]^2}{\phi(u) + \phi(v)}\] is even. φ is called quotient square sum cordial labeling if |eφ(0) − eφ(1)| ≤ 1, where eφ(0) and eφ(1) denote the number of edges labeled with 0 and 1 respectively. A graph which admits a quotient square sum cordial labeling is called quotient square sum cordial graph. In this paper path Pn, cycle Cn, star K1,n, friendship graph F_n, bistar B_{n,n}, C_n∪P_2, K_{2,n} and K_{n,n}; UP_n are shown to be quotient square sum cordial graph.

Keywords—Quotient Square Sum Cordial Labeling, Friendship Graph, Wheel Graph and Double Fan Graph.

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In this paper, the authors defined quotient square sum cordial labeling and prove the existence or nonexistence of quotient square sum cordial labeling for some families of graphs.

I. INTRODUCTION

The field of Graph Theory plays an important role in various areas of pure and applied sciences. A labeling of a graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [3].

Graph labeling of a graph G is an assignment of integers either to the vertices or edges or both subject to certain conditions. Graph labeling is a very powerful tool that eventually makes things in different fields very easy to be handled in mathematical way. Nowadays graph labeling has much attention from different brilliant researches in graph theory which has rigorous applications in many disciplines, e.g., communication networks, coding theory, optimal circuits layouts, astronomy, radar and graph decomposition problems. Recently the concept of prime graceful labeling was introduced by T.M. Selvarajan and R. Subramoniam in the year 2018. They studied the prime graceful labeling of various graphs in [5]. In [6] M. Sudha and A. Chandra Babu studied the relation between different types of Graceful Graphs. The symbol [x] stands for largest integer less than or equal to x.

DISCUSSION & RESULTS

Definition 1.1

Let G be a (p,q) graph. Let φ : V(G) → {1, 2, ..., p} be a one- one map. For each edge uv assign the label \(\frac{[\phi(u)]^2 + [\phi(v)]^2}{\phi(u) + \phi(v)}\) or \(\frac{\phi(v)}{\phi(u)}\) according as φ(u) ≥ φ(v) or φ(v) ≤ φ(u). f is called a quotient cordial labeling of G, if |eφ(0) − eφ(1)| ≤ 1, where eφ(0) and eφ(1) respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with a quotient cordial labeling is called quotient cordial graph.

Definition 1.2

Let G = (V, E) be a simple graph and φ : V → {1, 2, ..., |V|} be a bijection, for each edge uv assigned the label 1 if \[\frac{[\phi(u)]^2 + [\phi(v)]^2}{\phi(u) + \phi(v)}\] is odd and 0 if \[\frac{[\phi(u)]^2 + [\phi(v)]^2}{\phi(u) + \phi(v)}\] is even. φ is called quotient square sum cordial labeling if |eφ(0) − eφ(1)| ≤ 1, where eφ(0) and eφ(1) denote the number of edges labeled with 0 and labeled with 1 respectively. A graph which admits a quotient square sum cordial labeling is called quotient square sum cordial graph.

Example 1.3

For e(23) (edge 23), \(\frac{(4+3)}{(2+3)}\) = 2.6

Example 1.4

eφ(0) = 3 and eφ(1) = 3 ⇒ |eφ(0) − eφ(1)| ≤ 1, therefore K_{2,3} admits quotient square sum cordial labeling.

Theorem 1.5

For n ≥ 3, the path P_n admits quotient square sum cordial labeling.
Define $\{v_1, v_2, \ldots, v_n\} \rightarrow \{1, 2, \ldots, n\} \text{ by } \varphi(v_k) = k.$
\[ \left[ \varphi(v_2)^2 + [\varphi(v_{k+1})]^2 \right] = k^2 + (k+1)^2 = k^2 + \frac{1}{2} - \frac{1}{2k+2} \times k 
\frac{1}{2} + \frac{1}{2k+2} < 1 \text{ for all } k > 0, \left[ \varphi(v_2)^2 + [\varphi(v_{k+1})]^2 \right] = k \]

$e(v_kv_{k+1}) = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$

If $n$ is even, $e(v_1v_2) = 1, e(v_1v_3) = 0, e(v_1v_4) = 1, \ldots, e(v_{n-1}v_n) = 1.$
If $n$ is odd, $e(v_1v_2) = 1, e(v_1v_3) = 0, e(v_1v_4) = 1, \ldots, e(v_{n-1}v_n) = 0.$
Then $e(\varphi(0)) = (\frac{n-1}{2})$ and $e(\varphi(1)) = (\frac{n-1}{2}) \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 0.$

**Theorem 1.6**

The star graph $K_{1,n}$ admits quotient square sum cordial labeling.

**Proof**

Let $v_1$ be the central vertex of degree $n$.

Let $v_2, \ldots, v_n, v_{n+1}$ be the $n$ pendant vertices attached to it.

Define $\varphi: \{v_1, v_2, \ldots, v_n, v_{n+1}\} \rightarrow \{1, 2, \ldots, n, n+1\} \text{ by } \varphi(v_k) = k.$

Define $\varphi: \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\} \rightarrow \{1, 2, \ldots, n, n+1, n+2\} \text{ by } \varphi(v_1) = 1, \varphi(v_2) = 2 \text{ and } \varphi(v_k) = k \text{ for } 3 \leq k \leq n+2.$

By Theorem 1.6, $\left[ \varphi(v_2)^2 + [\varphi(v_{k+1})]^2 \right] = k - 1 \text{ for } k \geq 2.$

$e(v_1v_k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$

For $k > 6, e(v_2v_k) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd} \end{cases}$

**Case 1** If $n$ is even $e(v_1v_k) = 1, \text{ for } k = 4, 6, \ldots, n+2,$ $e(v_1v_k) = 0, \text{ for } k = 3, 5, \ldots, (n-1) \text{ and } e(v_2v_k) = 0, \text{ for } k = 8, 10, \ldots, n+2.$

For $k > 6, e(v_1v_k) = 1, \text{ for } k = 7, 9, \ldots, (n+1).$

For $k > 6, e(v_1v_k) = 0, \text{ for } k = 3, 5, \ldots, (n-1) \text{ and } e(v_2v_k) = 0.$

$e(\varphi(0)) = 0 \text{ and } e(\varphi(1)) = 0 \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 0.$

**Theorem 1.8**

The fan graph $F_n = P_n + K_1$ is a quotient square sum cordial graph.

**Proof**

The fan graph $F_n = K_1 + P_n$ contains $n + 1$ vertices and $(2n - 1)$ edges. Label the vertex having degree $n$ of $F_n$ with 1 and the remaining vertices by $2, 3, 4, \ldots, n.$

**Case 1** If $n$ is even, using Theorem 1.5 and Theorem 1.6, $e(\varphi(0)) = n$ and $e(\varphi(1)) = (n - 1) \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 1.$

**Case 2** If $n$ is odd, using Theorem 1.5 and Theorem 1.6, $e(\varphi(0)) = (\frac{n-1}{2})$ and $e(\varphi(1)) = (\frac{n-1}{2}) \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 1.$ **Theorem 1.9.** The Double fan graph $F_{2,n}$ is a quotient square sum cordial graph for $n \neq 3.$

**Proof**

The Double fan graph $F_{2,n}$ contains $n+2$ vertices and $(3n-1)$ edges. Label the vertices having degree $n$ of double fan graph $F_{2,n}$ with 1 and 2, the remaining vertices by $3, 4, \ldots, n, (n+1), (n+2)$

**Case 1** If $n$ is even, using Theorem 1.5 and Theorem 1.7, $e(\varphi(0)) = (\frac{3n-1}{2})$ and $e(\varphi(1)) = (\frac{3n-1}{2}) - 1 \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 1.$

**Case 2** If $n$ is odd, using Theorem 1.5 and Theorem 1.7, $e(\varphi(0)) = (\frac{3n-1}{2})$ and $e(\varphi(1)) = (\frac{3n-1}{2}) - 1 \Rightarrow |e(\varphi(0)) - e(\varphi(1))| = 1.$

**Theorem 1.10**

For $n \geq 3,$ the cycle $C_n$ is a quotient square sum cordial graph.

**Proof**

Let $C_n = \{v_1, v_2, \ldots, v_n, v_1\}$ be the cycle, where $v_i$ and $v_{i+1}$ are adjacent, $v_n$ and $v_1$ are adjacent for $k = 1, 2, 3, \ldots, n.$

**Case(i):** $n$ is odd

Define $\varphi: \{v_1, v_2, \ldots, v_n\} \rightarrow \{1, 2, \ldots, n\} \text{ by } \varphi(v_k) = k \text{ for } k = 1, 2, 3, \ldots, n.$

\[ \left[ \varphi(v_2)^2 + [\varphi(v_{k+1})]^2 \right] = k^2 + (k+1)^2 = k + \frac{1}{2} - \frac{1}{2k+2} \times k \]

\[ \frac{1}{2} + \frac{1}{2k+2} < 1 \text{ for all } k > 0 \]

For $k > 6, e(v_2v_k) = 0 \text{ if } k \text{ is even} \text{ and } 1 \text{ if } k \text{ is odd}.$
\[
\frac{[\varphi(v_0)^2 + \varphi(v_{k+1})^2]}{[\varphi(v_k) + \varphi(v_{k+1})]^2} = k
\]

\[
e(v_k v_{k+1}) = \begin{cases} 
1 & \text{if } k \text{ is odd} \\
0 & \text{if } k \text{ is even}
\end{cases}
\]

Case (ii): \( n \) is even

Define \( \varphi : \{v_1, v_2, \ldots, v_n\} \rightarrow \{1, 2, \ldots, n\} \) by

\[
\varphi(v_k) = \begin{cases} 
k & \text{if } 1 \leq k \leq n - 4 \\
k - 1 & \text{if } n = k - 3 \\
(\frac{n-2}{2}) + \frac{1}{2} & \text{if } 2 \leq k \leq n
\end{cases}
\]

\[
e(v_1 v_2) = 1, e(v_2 v_3) = 0, e(v_3 v_4) = 1, \ldots, e(v_{n-1} v_n) = 0 \text{ and } e(v_n v_1) = 0.
\]

\[
e_\varphi(0) = \frac{n-1}{2} \text{ and } e_\varphi(1) = \frac{2n+1}{2} \Rightarrow |e_\varphi(0) - e_\varphi(1)| = 1
\]

Case (i): \( n \) is odd

Define \( \varphi : \{v_1, v_2, \ldots, v_n\} \rightarrow \{1, 2, \ldots, n\} \) by

\[
\varphi(v_k) = \begin{cases} 
k & \text{if } 1 \leq k \leq n - 3 \\
k - 1 & \text{if } n - 2 \leq k \leq n
\end{cases}
\]

\[
e(v_1 v_2) = 1, e(v_2 v_3) = 0, e(v_3 v_4) = 1, \ldots, e(v_{n-1} v_n) = 0 \text{ and } e(v_n v_1) = 0.
\]

\[
e_\varphi(0) = \frac{n-1}{2} \text{ and } e_\varphi(1) = \frac{2n+1}{2} \Rightarrow |e_\varphi(0) - e_\varphi(1)| = 1
\]

Theorem 1.11

For \( n > 3 \), the wheel graph \( W_n = K_1 + C_n \) admits quotient square sum cordial labeling. 

Proof

The wheel graph \( W_n = K_1 + C_n \) contains \( n + 1 \) vertices and \( 2n \) edges. Label the vertex having degree \( n \) of \( W_n \) with 1 and the remaining vertices by 2,3,4,...,n. Label the vertices of degree \( n - 1 \) and \( n - 2 \) with \( \frac{n-2}{2} + \frac{1}{2} \) and \( \frac{n-2}{2} \) respectively.

Case 1

If \( n \) is even, using Theorem 1.6 and Theorem 1.11, \( e_\varphi(0) = n \)

and \( e_\varphi(1) = n \Rightarrow |e_\varphi(0) - e_\varphi(1)| = 0. \)

Case 2

If \( n \) is odd, using Theorem 1.6 and Theorem 1.11, \( e_\varphi(0) = n \)

and \( e_\varphi(1) = n \Rightarrow |e_\varphi(0) - e_\varphi(1)| = 0. \)

Theorem 1.13

Bistar graph \( B_{n,n} \) admits quotient square sum cordial labeling.

Proof

Bistar graph \( B_{n,n} \) contains \( (2n+2) \) vertices and \( (2n+1) \) edges. Let \( V(G) = 2n + 2 \) and \( E(G) = 2n + 1 \). Bistar graph \( B_{n,n} \) has exactly two vertices of degree \( n \), let them be \( u_0 \) and \( v_0 \).

Define \( \varphi : \{v_1, v_2, \ldots, v_n, u_0, u_1, u_2, \ldots, u_n\} \rightarrow \{1, 2, \ldots, 2n+2\} \) by \( \varphi(v_0) = 1, \varphi(u_0) = 2, \varphi(v_i) = k + 2 \text{ for all } k > 2, \varphi(u_k) = k + 1 \text{ for } k > 2, \varphi(v_{n+1}) = 2n+2 \text{ for } k > 3 \).

If \( n \) is even, \( e(v_0 v_k) = 1 \text{ for } k = 1, 3, 5, \ldots, 2n \) and \( e(v_0 v_k) = 0 \text{ for } k = 2, 4, 6, \ldots, n \).

If \( n \) is odd, \( e(v_0 v_k) = 1 \text{ for } k = 1, 3, 5, \ldots, 2n \) and \( e(v_0 v_k) = 0 \text{ for } k = 2, 4, 6, \ldots, n \).

Theorem 1.14

The friendship graph \( F_n^{(3)} \) admits quotient square sum cordial labeling.

Proof

The friendship graph \( F_n^{(3)} \) is a planar undirected graph with \( 2n + 1 \) vertices and \( 3n \) edges constructed by joining \( n \) copies of the triangle graph \( T \) with a common vertex.
Let $v_1$ be the central vertex of degree $2n$.

Friendship graph contains $n$ triangles $T_1, T_2, \ldots, T_n$.

Define $f: V(G) \rightarrow \{1, 2, \ldots, 2n, 2n+1\}$ by $\varphi(v_1) = 1$, label the outer vertices of the triangles $T_1, T_3, T_5, \ldots$ by $\varphi(v_i) = 2k, 1 \leq k \leq n$ and label the outer vertices of the triangles $T_2, T_4, T_6, \ldots$ by $\varphi(v_i) = 2k+1, 1 \leq k \leq n$.

$$\frac{2k^2}{(k+1)} < 1 \text{ for } k \geq 2$$

$$\frac{[\varphi(v_1)]^2 + [\varphi(v_2)]^2}{[\varphi(v_1)]^2 + [\varphi(v_2)]^2} = (k - 1) + \frac{2}{(k+1)}$$

Case 1: If $n$ is even, $e_{\varphi}(0) = \frac{3n}{2}$ and $e_{\varphi}(1) = \frac{3n}{2} \Rightarrow |e_{\varphi}(0) - e_{\varphi}(1)| = 0$.

Case 2: If $n$ is odd, $e_{\varphi}(0) = \frac{3n-1}{2}$ and $e_{\varphi}(1) = \frac{3n+1}{2}$

$\Rightarrow |e_{\varphi}(0) - e_{\varphi}(1)| = 1$.

Theorem 1.15

The friendship graph $F_{3n}^{(3)}$ admits quotient square sum cordial labeling.

Proof

The friendship graph $F_{3n}^{(3)}$ is an undirected graph with $3n + 1$ vertices and $5n$ edges constructed by joining $n$ copies of the fan graph $F_3$ with a common vertex.

Let $v_1$ be the central vertex of degree $3n$.

Define $\varphi: V(G) \rightarrow \{1, 2, \ldots, 2n, 3n, 3n+1\}$ by $\varphi(v_1) = 1$, label the outer vertices $v_2, v_3, \ldots, v_{2n}, v_{3n+1}$ of the friendship graph $F_{3n}^{(3)}$ by $\varphi(v_k) = k, 2 \leq k \leq 3n + 1$

$$\frac{[\varphi(v_1)]^2 + [\varphi(v_2)]^2}{[\varphi(v_1)]^2 + [\varphi(v_2)]^2} = (k - 1) \text{ for } k \geq 2$$

$$\frac{[\varphi(v_1)]^2 + [\varphi(v_2)]^2}{[\varphi(v_1)]^2 + [\varphi(v_2)]^2} = k + \frac{1}{4k^2} - \frac{[\varphi(v_1)]^2 + [\varphi(v_2)]^2}{[\varphi(v_1)]^2 + [\varphi(v_2)]^2} = k$$

Case 1: If $n$ is even, $e_{\varphi}(0) = \frac{5n}{2}$ and $e_{\varphi}(1) = \frac{5n}{2} \Rightarrow |e_{\varphi}(0) - e_{\varphi}(1)| = 0$

Case 2: If $n$ is odd, $e_{\varphi}(0) = \frac{5n-1}{2}$ and $e_{\varphi}(1) = \frac{5n+1}{2}$

$\Rightarrow |e_{\varphi}(0) - e_{\varphi}(1)| = 1$.

II. CONCLUSION

The investigation of labeled graph is very important due to its various applications in many fields. It is very interesting to study the various types of graphs which admits quotient square sum cordial labeling. It is an open area of research to discuss some more similar results for various graphs.

REFERENCES


