"Stability Analysis of Three Species Ammensalism Model with Time Delay"

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***Abstract*:**
The present model is devoted to an analytical study of a three species syn-ecological model which the 1st species (N_1) ammensal on the 2nd species (N_2) and 2nd species (N_2) ammensal on the 3rd species (N_3). Here 1st species and 2nd species are neutral to each other. A time delay is established between 1st species and 2nd species and 2nd species and 3rd species. All attainable equilibrium points of the model are known and native stability for each case is mentioned and also the global stability of co-existing state is discussed by constructing appropriate Lyapunov operate. Further, precise solutions of perturbed equations are derived. The steadiness analysis is supported by numerical simulation victimization MatLab.

Keywords: Ammensalism, Time Delay, Equilibrium points, Global Stability, Lyapunov function, MATLAB.

I. INTRODUCTION

Ammensalism is a relationship in which a result of one life from adversary affects the other living being. It is explicitly a populace collaboration in which one creature is hurt, while the other is neither adversely nor decidedly influenced. The case for ammensalism, air contamination brought about via vehicles, power producing stations or metal smelters frequently causes extreme harm of plants in the influenced territory, while people get no immediate profit by this relationship. Tall trees that structure the woodland shelter prevent light from achieving littler plants howl. It is a fact that time delay in biological systems is a reality and it can have complex impact on the dynamics of the system namely loss of stability, induced oscillations and periodic solutions. It is a known fact that in any prey-predator system, the consumed prey does not contribute to the instant growth of the predator population, but with a time lag. This is reflected in the works of Cushing [4], Kuang [16], Gopalsammy [17] and some other authors have discussed models by incorporating delay terms. As far back as research in the order of a hypothetical environment was started by Lotka[1] and Volterra [2]. Later on, many mathematicians and ecologists contributed to the growth of this area as reported in the treaties of Meyer [3], Cushing [4] and Kapur [5, 6]. Lakshmi Narayan et al. [8, 9, 10] investigated prey-predator ecological models with a partial cover for the prey and alternative food for predator and Time Delay. Ravindra Reddy.B et al. [11] studied A Model of Two Mutually Interacting Species with Limited Resources and a Time Delay. Paparao. A. V. et al. [12, 13, 14] studied three species ecological models with time delay. Kondala Rao. K. et. Al [15] discussed a three species dynamical system of ammensal relationship of humans on plants and birds with time delay.

Ammensalism is a biological connection between the species where first species (N1) influence on the second species (N2) and second species (N2) influence on the third species (N3) without themselves being influenced in any capacity. Here first species (N1) and third species (N3) are impartial to one another. The model is represented by a system of three ordinary differential equations. All possible equilibrium points are identified and the stability of co-existing state is discussed using Routh-Hurwitz criteria. Further solutions of quasi-linearized equations and the results are simulated by Numerical examples using Mat Lab.

II. BASIC EQUATIONS.

The model equations for a system of three interacting species are given by the following set of non-linear first order simultaneous differential equations.

$$\begin{aligned} \frac{dN_1}{dt} &= f_1(N_1,N_2,N_3) = a_1N_1 - \alpha_{11}N_1^2 \\ \frac{dN_2}{dt} &= f_2(N_1,N_2,N_3) = a_2N_2 - \alpha_{22}N_2^2 - \alpha_{21}N_2\int_{-\infty}^{T} k_1(t-s)N_1(s)ds \\ \frac{dN_3}{dt} &= f_3(N_1,N_2,N_3) = a_3N_3 - \alpha_{33}N_3^2 - \alpha_{32}N_3\int_{-\infty}^{T} k_2(t-s)N_2(s)ds. \end{aligned} \quad (2.1)$$

Here $k_1(t-s)\,\&\,k_2(t-s)$ is giving weight factors to the influences at time t of N_1 and N_2 of time s.

That is $k_1(t-s)\,\&\,k_2(t-s)$ are rate of change of N_1 and N_2 after a time interval (t-s)

Let $t-s=z \Rightarrow s=t-z$ (2.2)

$k_1(z)\geq 0\,\&\,k_2(z)\geq 0$ and we normalize it, so that

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$$\int_0^\infty k_1(t-s)dz = 1 \,\&\, \int_0^\infty k_2(t-s)dz = 1 \quad (2.3)$$

Therefore

$$\begin{aligned}\frac{dN_1}{dt} &= f_1(N_1,N_2,N_3) = a_1N_1 - \alpha_{11}N_1^2\\ \frac{dN_2}{dt} &= f_2(N_1,N_2,N_3) = a_2N_2 - \alpha_{22}N_2^2 - \alpha_{21}N_2\int_0^\infty k_1(z)N_1(t-z)dz\\ \frac{dN_3}{dt} &= f_3(N_1,N_2,N_3) = a_3N_3 - \alpha_{33}N_3^2 - \alpha_{32}N_3\int_0^\infty k_2(z)N_2(t-z)dz\end{aligned} \quad (2.4)$$

with the following notation

N_i (t) : Population of the first, second and third species at a time " t", i=1, 2, 3.

a_i : Natural growth rate of first, second and third species, i=1, 2, 3.

α_{ii} : Rate of decrease of first, second and third species due to internal competitions, i= 1, 2, 3.

α_{21} : Rate of decrease of the second species due to attacks of first species.

α_{32} : Rate of decrease of the third species due to attacks of second species.

Further, the variables N_1, N_2, N_3 are non-negative and the model parameters $a_i, \alpha_{ii}, i = 1,2,3, \alpha_{21}, \alpha_{32}$ and α_{13} are assumed to be non-negative constants. $k_1(z) \ge 0 \,\&\, k_2(z) \ge 0$ and we normalize it, so that $\int_0^\infty k_1(z)dz = 1 \,\&\, \int_0^\infty k_2(z)dz = 1$.

III. EQUILIBRIUM POINTS:

For the system under investigation, eight equilibrium points are identified. They are given below.
The equilibrium points are identified by solving $\frac{dN_i}{dt} = 0, i = 1,2,3.$

All Equilibrium points are classified as bellow

(E1). Fully Extinct State: $\bar{N}_1 = 0; \bar{N}_2 = 0; \bar{N}_3 = 0$ (3.1)

(E2). First and Second Species Extinct State:

$$\bar{N}_1 = 0; \bar{N}_2 = 0; \bar{N}_3 = \frac{a_3}{\alpha_{33}} \quad (3.2)$$

(E3). First and Third Species Extinct State:

$$\bar{N}_1 = 0; \bar{N}_2 = \frac{a_2}{\alpha_{22}}; \bar{N}_3 = 0 \quad (3.3)$$

(E4). Second and Third Species Extinct State:

$$\bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = 0; \bar{N}_3 = 0 \quad (3.4)$$

(E5). Only First Species Extinct State:

$$\bar{N}_1 = 0; \bar{N}_2 = \frac{a_2}{\alpha_{22}}; \bar{N}_3 = \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{22}\alpha_{33}} \quad (3.5)$$

This state would exist only when $a_3\alpha_{22} - a_2\alpha_{32} > 0$. (3.6)

(E6). Only Second Species Extinct State:

$$\bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = 0; \bar{N}_3 = \frac{a_3}{\alpha_{33}} \quad (3.7)$$

(E7). Only Third Species Extinct State:

$$\bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = \frac{a_2\alpha_{11} - \alpha_{21}a_1}{\alpha_{11}\alpha_{22}}; \bar{N}_3 = 0 \quad (3.8)$$

This state would exist only when $a_2\alpha_{11} - a_1\alpha_{21} > 0$. (3.9)

(E8). Co-Existent State:

$$\bar{N}_1 = \frac{a_1}{\alpha_{11}}; \bar{N}_2 = \frac{a_2\alpha_{11} - \alpha_{21}a_1}{\alpha_{11}\alpha_{22}}; \bar{N}_3 = \frac{a_3\alpha_{11}\alpha_{22} - a_2\alpha_{11}\alpha_{32} + a_1\alpha_{21}\alpha_{32}}{\alpha_{11}\alpha_{22}\alpha_{33}} \quad (3.10)$$

This state would exist only when $a_2\alpha_{11} > a_1\alpha_{21}$ and $a_3\alpha_{11}\alpha_{22} + a_1\alpha_{21}\alpha_{32} > a_2\alpha_{11}\alpha_{32}$. (3.11)

IV. STABILITY OF THE SYSTEM AT EQUILIBRIUM POINTS:

To examine the stability of the equilibrium state $(\bar{N}_1, \bar{N}_2, \bar{N}_3)$ we consider a small perturbation (u_1, u_2, u_3), such that

$$N_1 = \bar{N}_1 + u_1, N_2 = \bar{N}_2 + u_2 \,\&\, N_3 = \bar{N}_3 + u_3. \quad (4.1)$$

After linearization, we get $\frac{dU}{dt} = AU$ (4.2)

Where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\bar{N}_1 & 0 & 0 \\ 0 & a_2 - 2\alpha_{22}\bar{N}_2 - \alpha_{21}\bar{N}_2 & 0 \\ -\alpha_{31}\bar{N}_3 & 0 & a_3 - 2\alpha_{33}\bar{N}_3 - \alpha_{32}\bar{N}_2 \end{bmatrix} \quad (4.3)$$

And $U = [u_1, u_2 u_3]^T$ (4.4)

The characteristic equation for the system is $|A - \lambda I| = 0$ (4.5)

The equilibrium state is stable if all the roots of the characteristic equation (4.5) are negative real parts.

4.1. Stability of the Fully Extinct state (E1):

Linearized equations are

$$\frac{du_1}{dt} = a_1u_1, \frac{du_2}{dt} = a_2u_2, \frac{du_3}{dt} = a_3u_3. \quad (4.1.1)$$

The characteristic equation for the fully washed out state is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_1 - \lambda & 0 & 0 \\ 0 & a_2 - \lambda & 0 \\ 0 & 0 & a_3 - \lambda \end{vmatrix} = 0 \begin{array}{l} \Rightarrow (\lambda - a_1)(\lambda - a_2)(\lambda - a) = 0 \\ \Rightarrow \lambda = a_1, a_2, a_3 \end{array} \quad (4.1.2)$$

The characteristic equation corresponding to the fully washed out state is $(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) = 0$, i.e., the Eigen values of this characteristic equation ar $\lambda_1 = a_1, \lambda_2 = a_2 \,\&\, \lambda_3 = a_3$. Here clearly $\lambda_1, \lambda_2 \,\&\, \lambda_3$ are positive. Hence the equilibrium state is unstable.

The solution of the perturbed equations is

$$u_1 = u_{10}e^{a_1t}, u_2 = u_{20}e^{a_2t}, u_3 = u_{30}e^{a_3t}. \quad (4.1.3)$$

4.2. Stability of First and Second Species Extinct state(E2):

Linearized equations for the existence of third species are

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$$\frac{du_1}{dt}=a_1u_1,\frac{du_2}{dt}=a_2u_2,\frac{du_3}{dt}=-\alpha_{32}\bar{N}_3\int_0^\infty k_2(z)u_2(t-z)dz-a_3u_3. \tag{4.2.1}$$

The characteristic equation for the equilibrium state is

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}\lambda-a_1 & 0 & 0\\ 0 & \lambda-a_2 & 0\\ 0 & \alpha_{32}\bar{N}_3k_2^*(\lambda) & \lambda+a_3\end{vmatrix}=0\begin{array}{l}\Rightarrow(\lambda-a_1)(\lambda-a_2)(\lambda+a_3)=0\\ \Rightarrow\lambda=a_1,a_2,-a_3\end{array} \tag{4.2.2}$$

The characteristic equation corresponding to the first and second species extinct state is $(\lambda-a_1)(\lambda-a_2)(\lambda+a_3)=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=a_1,\lambda_2=a_2\ \&\ \lambda_3=-a_3$. Here clearly λ_1,λ_2 are positive. Hence, the equilibrium state is unstable.

The solution of the perturbed equations is

$$u_1=u_{10}e^{a_1t},u_2=u_{20}e^{a_2t},$$
$$u_3=-\frac{\alpha_{32}\bar{N}_3u_{20}k_2^*(a_2)}{a_2+a_3}e^{a_2t}+\left(u_{30}+\frac{\alpha_{32}\bar{N}_3u_{20}k_2^*(a_2)}{a_2+a_3}\right)e^{-a_3t}. \tag{4.2.3}$$

4.3. Stability of First and Third Species Extinct State (E$_3$):

Linearized equations for the existence of second species are

$$\frac{du_1}{dt}=a_1u_1,\frac{du_2}{dt}=-\alpha_{21}\bar{N}_2\int_0^\infty k_1(z)u_1(t-z)dz-a_2u_2,\frac{du_3}{dt}=(a_3-\alpha_{32}\bar{N}_2)u_3. \tag{4.3.1}$$

The characteristic equation corresponding first and third species extinct state is

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}\lambda-a_1 & 0 & 0\\ \alpha_{21}\bar{N}_2k_1^*(\lambda) & \lambda+a_2 & 0\\ 0 & 0 & \lambda-a_3\end{vmatrix}=0\begin{array}{l}\Rightarrow(\lambda-a_1)(\lambda+a_2)(\lambda-a_3)=0\\ \Rightarrow\lambda=a_1,-a_2,a_3\end{array} \tag{4.3.2}$$

The characteristic equation corresponding to the first and third species extinct state is $(\lambda-a_1)(\lambda+a_2)(\lambda-a_3)=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=a_1,\lambda_2=-a_2\ \&\ \lambda_3=a_3$. Here clearly $\lambda_1\ \&\ \lambda_3$ are positive. Hence the equilibrium state is unstable.

The solution of the perturbed equations is

$$u_1=u_{10}e^{a_1t},$$
$$u_2=-\frac{\alpha_{21}\bar{N}_2u_{10}k_1^*(a_1)}{a_1+a_2}e^{a_1t}+\left(u_{20}+\frac{\alpha_{21}\bar{N}_2k_1^*(a_1)u_{10}}{a_1+a_2}\right)e^{-a_2t},u_3=u_{30}e^{(a_3-\alpha_{32}\bar{N}_2)t} \tag{4.3.3}$$

4.4. Stability of Second and Third Species Extinct State (E$_4$):

Linearized equations for the existence of first species are

$$\frac{du_1}{dt}=-a_1u_1,\frac{du_2}{dt}=(a_2-\alpha_{21}\bar{N}_1)u_2,\frac{du_3}{dt}=a_3u_3. \tag{4.4.1}$$

The characteristic equation corresponding second and third species extinct state is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}\lambda+a_1 & 0 & 0\\ 0 & \lambda-(a_2-\alpha_{21}\bar{N}_1) & 0\\ 0 & 0 & \lambda-(a_3-\alpha_{31}\bar{N}_1)\end{vmatrix}=0\begin{array}{l}\Rightarrow(\lambda+a_1)(\lambda-(a_2-\alpha_{21}\bar{N}_1))(\lambda-a_3)=0\\ \Rightarrow\lambda=-a_1,(a_2-\alpha_{21}\bar{N}_1),a_3\end{array} \tag{4.4.2}$$

The characteristic equation corresponding to the second and third species extinct state is $(\lambda+a_1)(\lambda-(a_2-\alpha_{21}\bar{N}_1))(\lambda-a_3)=0$, i.e., the Eigen values of the above equations are $\lambda_1=-a_1$, $\lambda_2=a_2-\alpha_{21}\bar{N}_1$, $\lambda_3=a_3$. Here clearly λ_3 is positive. Hence, the equilibrium state is unstable using the Routh-Hurwitz criterion.

The solution of the perturbed equations is

$$u_1=u_{10}e^{-a_1t},u_2=u_{20}e^{(a_2-\alpha_{21}\bar{N}_1)t},u_3=u_{30}e^{a_3t}. \tag{4.4.3}$$

4.5. Stability of Only First Species Extinct State (E$_5$):

Linearized equations for the existence of second and third species are

$$\frac{du_1}{dt}=a_1u_1,\frac{du_2}{dt}=-\alpha_{21}\bar{N}_2\int_0^\infty k(z)u_1(t-z)dz-a_2u_2, \tag{4.5.1}$$
$$\frac{du_3}{dt}=-\alpha_{31}\bar{N}_3\int_0^\infty k(z)u_1(t-z)dz-a_3u_3.$$

The characteristic equation corresponding only first species extinct state is

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}\lambda-a_1 & 0 & 0\\ \alpha_{21}\bar{N}_2k_1^*(\lambda) & \lambda+a_2 & 0\\ \alpha_{31}\bar{N}_3k_2^*(\lambda) & 0 & \lambda-s_1\end{vmatrix}=0\begin{array}{l}\Rightarrow(\lambda-a_1)(\lambda+a_2)(\lambda-s_1)=0\\ \Rightarrow\lambda=a_1,-a_2,s_1.\end{array} \tag{4.5.2}$$

The characteristic equation corresponding to the only first species extinct state is $(\lambda-a_1)(\lambda+a_2)(\lambda-s_1)=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=a_1,\lambda_2=-a_2\ \&\ \lambda_3=s_1$. Here clearly λ_1 is positive. Hence the equilibrium state is unstable.

The solution of the perturbed equations is

$$u_1=u_{10}e^{a_1t},u_2=-m_1e^{a_1t}+(u_{20}+m_1)e^{-a_2t},u_3=\left(u_{30}-\frac{m_2}{s_1-a_1}-\frac{m_3}{s_1+a_2}\right)e^{s_1t}-\frac{m_2}{s_1-a_1}e^{a_1t}+\frac{m_3}{s_1+a_2}e^{-a_2t}.$$

Where

$$m_1=\frac{\alpha_{21}\bar{N}_2u_{10}k_1^*(a_1)}{a_1+a_2},m_2=\frac{\alpha_{31}\bar{N}_3u_{10}k_2^*(a_2)}{s-a_1}m_1,m_3=\frac{\alpha_{22}\bar{N}_2k_2^*(a_2)}{s+a_2}(u_{20}+m_1)\ \&\ s_1=a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2 \tag{4.5.3}$$

4.6. Stability of Only Second Species Extinct State (E$_6$):

Linearized equations for the existence of first and third species are

$$\frac{du_1}{dt}=-a_1u_1,\frac{du_2}{dt}=(a_2-\alpha_{21}\bar{N}_1)u_2,\frac{du_3}{dt}=-a_3u_3-\alpha_{32}\bar{N}_3\int_0^\infty k_2(z)u_2(t-z)dz. \tag{4.6.1}$$

The characteristic equation corresponding only second species extinct state is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}\lambda+a_1 & 0 & 0\\ 0 & \lambda-(a_2-\alpha_{21}\bar{N}_1) & 0\\ 0 & \alpha_{32}\bar{N}_2k_2^*(\lambda) & \lambda+a_3\end{vmatrix}=0\begin{array}{l}\Rightarrow(\lambda+a_1)(\lambda-(a_2-\alpha_{21}\bar{N}_1))(\lambda+a_3)=0\\ \Rightarrow\lambda=-a_1,(a_2-\alpha_{21}\bar{N}_1),-a_3.\end{array} \tag{4.6.2}$$

The characteristic equation corresponding to the second species extinct state is $(\lambda+a_1)(\lambda-(a_2-\alpha_{21}\bar{N}_1))(\lambda+a_3)=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=-a_1$, $\lambda_2=a_2-\alpha_{21}\bar{N}_1$, $\lambda_3=-a_3$. Here clearly $\lambda_1\ \&\ \lambda_3$ are negative. When $a_2<\alpha_{21}\bar{N}_1$, λ_2 is negative. Hence the equilibrium state is conditionally asymptotically stable.

The solution of the perturbed equations is

$$u_1=u_{10}e^{-a_1t},u_2=u_{20}e^{(a_2-\alpha_{21}\bar{N}_1)t},$$
$$u_3=-\frac{\alpha_{32}u_{20}\bar{N}_3k_2^*(z)}{s_2+a_3}e^{s_2t}+\left(u_{30}+\frac{\alpha_{32}u_{20}\bar{N}_3k_2^*(z)}{s_2+a_3}\right)e^{-a_3t}. \tag{4.6.3}$$

Where $s_2=a_2-\alpha_{21}\bar{N}_1$

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4.7. Stability of Only Third Species Extinct State (E_7):

Linearized equations for the existence of first and second species are

$$\frac{du_1}{dt}=-a_1u_1,\ \frac{du_2}{dt}=(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1)u_2-\alpha_{21}\bar{N}_2\int_0^\infty k_1(z)u_1(t-z)dz,\qquad(4.7.1)$$

$$\frac{du_3}{dt}=(a_3-\alpha_{32}\bar{N}_2)u_3$$

The characteristic equation corresponding only third species washed out state is

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}\lambda+a_1 & 0 & 0\\ \alpha_{21}\bar{N}_2k_1^*(\lambda) & \lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1) & 0\\ 0 & 0 & \lambda-(a_3-\alpha_{32}\bar{N}_2)\end{vmatrix}=0\qquad(4.7.2)$$

$$\Rightarrow(\lambda+a_1)(\lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1))(\lambda-(a_3-\alpha_{32}\bar{N}_2))=0$$

$$\Rightarrow\lambda=-a_1,(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1),(a_3-\alpha_{32}\bar{N}_2).$$

The characteristic equation corresponding to the only third species extinct state is $(\lambda+a_1)(\lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1))(\lambda-(a_3-\alpha_{32}\bar{N}_2))=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=-a_1,\lambda_2=(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1),\lambda_3=(a_3-\alpha_{32}\bar{N}_2)$. Here clearly λ_1 is negative. When $a_2<2\alpha_{22}\bar{N}_2+\alpha_{21}\bar{N}_1\ \&\ a_3<\alpha_{32}\bar{N}_2$, $\lambda_2\ \&\ \lambda_3$ are negative. Hence, the equilibrium state is conditionally asymptotically stable.

The solution of the perturbed equations are

$$u_1=u_{10}e^{-a_1t},u_2=(u_{20}-m_4)e^{s_3t}+m_4e^{-a_1t},u_3=u_{30}e^{(a_3-\alpha_{32}\bar{N}_2)t}.\qquad(4.7.3)$$

Where $m_4=\dfrac{\alpha_{21}\bar{N}_2u_{10}k_1^*(z)}{(s_3+a_1)}\ \&\ s_3=a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1$ (4.7.4)

4.8. Stability of Co-existing State(E_8):

Linearized equations for the co-existing state are

$$\frac{du_1}{dt}=-a_1u_1,\ \frac{du_2}{dt}=(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1)u_2-\alpha_{21}\bar{N}_2\int_0^\infty k_1(z)u_1(t-z)dz,\qquad(4.8.1)$$

$$\frac{du_3}{dt}=(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2)u_3-\alpha_{32}\bar{N}_3\int_0^\infty k_2(z)u_2(t-z)dz.$$

The characteristic equation corresponding existence of all three species state is

$$|A-\lambda I|=0\Rightarrow\begin{vmatrix}\lambda+a_1 & 0 & 0\\ \alpha_{21}\bar{N}_2k^*(\lambda) & \lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1) & 0\\ 0 & \alpha_{32}\bar{N}_2k_2^*(\lambda) & \lambda-(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2)\end{vmatrix}=0\qquad(4.8.2)$$

$$\Rightarrow(\lambda+a_1)(\lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1))(\lambda-(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2))=0$$

$$\Rightarrow\lambda=-a_1,(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1),(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2).$$

The characteristic equation corresponding to the only co-existence state is $(\lambda+a_1)(\lambda-(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1))(\lambda-(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2))=0$, i.e., the Eigen values of this characteristic equation are $\lambda_1=-a_1,\lambda_2=(a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1),\lambda_3=(a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2)$.

Here λ_1 is clearly negative. When $a_2<2\alpha_{22}\bar{N}_2+\alpha_{21}\bar{N}_1,a_3<2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2$, then $\lambda_2\ \&\ \lambda_3$ are negative. Hence, the equilibrium state is conditionally asymptotically stable if $a_2<2\alpha_{22}\bar{N}_2+\alpha_{21}\bar{N}_1\ \&\ a_3<2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2$.

The solution of the perturbed equations are

$$u_1=u_{10}e^{-a_1t},u_2=(u_{20}-m_4)e^{s_1t}+m_4e^{-a_1t},u_3=(u_{30}-m_5-m_6)e^{-s_1t}+m_5e^{s_3t}+m_6e^{-a_1t}.\qquad(4.8.3)$$

Where

$$m_4=\frac{\alpha_{21}\bar{N}_2u_{10}k_1^*(z)}{(s_3+a_1)},m_5=\frac{\alpha_{32}\bar{N}_3k_2^*(z)(u_{20}-m_4)}{(s_1-s_3)},s_3=a_2-2\alpha_{22}\bar{N}_2-\alpha_{21}\bar{N}_1\qquad(4.8.4)$$

$$\&m_6=\frac{\alpha_{32}\bar{N}_3k_2^*(z)}{(s_1+a_1)},s_1=a_3-2\alpha_{33}\bar{N}_3-\alpha_{32}\bar{N}_2$$

V. GLOBALLY STABILITY:

Theorem: The system is globally stable at the co-existing state at $E_8(\overline{N}_1,\overline{N}_2,\overline{N}_3)$

Proof: Let the Lyapunov's function be

$$V(N_1,N_2,N_3)=\begin{Bmatrix}\log(N_1)+\log(N_2)+\log(N_3)\\ -\alpha_{21}\int_0^\infty k_1(z)\int_{t-z}^t N_1(u)dudz-\alpha_{32}\int_0^\infty k_2(z)\int_{t-z}^t N_1(u)dudz\end{Bmatrix}\qquad(5.1)$$

The time derivate of V along the solutions of equations (5.1) is

$$\frac{dV}{dt}=\begin{Bmatrix}\frac{N_1^1}{N_1}+\frac{N_2^1}{N_2}+\frac{N_3^1}{N_3}-\alpha_{21}\int_0^\infty k_1(z)[N_1(t)-N_1(t-z)]dz\\ -\alpha_{32}\int_0^\infty k_2(z)[N_2(t)-N_2(t-z)]dz\end{Bmatrix}\qquad(5.2)$$

Using the system of equations (5.1) and the relations

$$\int_0^\infty k_1(z)dz=1\ \&\int_0^\infty k_2(z)dz=1\qquad(5.3)$$

Then equation (5.2) is

$$\frac{dV}{dt}=\begin{Bmatrix}\frac{1}{N_1}\left[a_1N_1-\alpha_{11}N_1^2\right]+\frac{1}{N_2}\left[a_2N_2-\alpha_{22}N_2^2-\alpha_{21}N_2\int_0^\infty k_1(z)N_1(t-z)dz\right]\\ +\frac{1}{N_3}\left[a_3N_3-\alpha_{33}N_3^2-\alpha_{32}N_3\int_0^\infty k_2(z)N_2(t-z)dz\right]-\alpha_{21}\int_0^\infty k_1(z)N_1(t)dz\\ +\alpha_{21}\int_0^\infty k_1(z)N_1(t-z)dz-\alpha_{32}\int_0^\infty k_2(z)N_2(t)dz+\alpha_{32}\int_0^\infty k_2(z)N_2(t-z)dz\end{Bmatrix}\qquad(5.4)$$

$$\frac{dV}{dt}=\begin{Bmatrix}a_1-\alpha_{11}N_1+a_2-\alpha_{22}N_2-\alpha_{21}\int_0^\infty k_1(z)N_1(t-z)dz\\ +a_3-\alpha_{33}N_3-\alpha_{32}\int_0^\infty k_2(z)N_2(t-z)dz-\alpha_{21}\int_0^\infty k_1(z)N_1(t)dz\\ +\alpha_{21}\int_0^\infty k_1(z)N_1(t-z)dz-\alpha_{32}\int_0^\infty k_2(z)N_2(t)dz+\alpha_{32}\int_0^\infty k_2(z)N_2(t-z)dz\end{Bmatrix}\qquad(5.5)$$

$$\frac{dV}{dt}=\begin{Bmatrix}a_1-\alpha_{11}N_1+a_2-\alpha_{22}N_2z+a_3-\alpha_{33}N_3\\ -\alpha_{21}N_1(t)\int_0^\infty k_1(z)dz-\alpha_{32}N_2(t)\int_0^\infty k_2(z)dz\end{Bmatrix}\qquad(5.6)$$

Since $\int_0^\infty k_1(z)dz=1\ \&\int_0^\infty k_2(z)dz=1$ then

$$\frac{dV}{dt}=(a_1-\alpha_{11}N_1)+(a_2-\alpha_{22}N_2+\alpha_{21}N_1)+(a_3-\alpha_{33}N_3-\alpha_{32}N_2)\qquad(5.7)$$

By proper choice of $a_1,a_2\ \&\ a_3$

$$\left(a_1=\alpha_{11}\overline{N_1}\right),\left(a_2=\alpha_{22}\overline{N_2}+\alpha_{21}\overline{N_1}\right)\&\left(a_3=\alpha_{33}\overline{N_3}+\alpha_{32}\overline{N_2}\right)\qquad(5.5)$$

Then Substitute (5.5) in (5.4), we get

$$\frac{dV}{dt}=-\alpha_{11}\left(N_1-\overline{N_1}\right)-\alpha_{22}\left(N_2-\overline{N_2}\right)-\alpha_{21}\left(N_1-\overline{N_1}\right)-\alpha_{33}\left(N_3-\overline{N_3}\right)-\alpha_{32}\left(N_2-\overline{N_2}\right)\qquad(5.6)$$

$$\Rightarrow\frac{dV}{dt}=-(\alpha_{11}+\alpha_{21})\left(N_1-\overline{N_1}\right)-(\alpha_{22}+\alpha_{32})\left(N_2-\overline{N_2}\right)-\alpha_{33}\left(N_3-\overline{N_3}\right)\qquad(5.7)$$

Using the inequality $ab\le\dfrac{a^2+b^2}{2}$

$$\frac{dV}{dt}\le-\frac{1}{2}\left\{(\alpha_{11}+\alpha_{21})^2+\left(N_1-\overline{N_1}\right)^2\right\}-\frac{1}{2}\left\{(\alpha_{22}+\alpha_{32})^2+\left(N_2-\overline{N_2}\right)^2\right\}-\frac{1}{2}\left\{\alpha_{33}^2+\left(N_3-\overline{N_3}\right)^2\right\}\qquad(5.8)$$

$$\Rightarrow\frac{dV}{dt}\le-\frac{1}{2}\left\{(\alpha_{11}+\alpha_{21})^2+\left(N_1-\overline{N_1}\right)^2+(\alpha_{22}+\alpha_{32})^2+\left(N_2-\overline{N_2}\right)^2+\alpha_{33}^2+\left(N_3-\overline{N_3}\right)^2\right\}\qquad(5.9)$$

$$\Rightarrow\frac{dV}{dt}<0\qquad(5.10)$$

Hence the system is globally stable at the positive equilibrium point $E_8(\overline{N}_1,\overline{N}_2,\overline{N}_3)$.

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VI. NUMERICAL SIMULATION:

The systems of equations (2.1) are simulated using Matlab using ode45.

Example 1: Let a_1=0.02, a_2=0.05, a_3=0.08, α_{11}=0.04, α_{22}=0.4, α_{33}=0.4, α_{21}=0.09, α_{32}=0.09, N_{10}=5, N_{20}=10 and N_{30}=15.

The system dynamics with the above parameter values without the delay argument is shown the graphs 6.1.1(A) and 6.1.1(B).

Fig 6.1.1(A) shows the time series analysis of the model (2.1) is a stable system with fixed equilibrium point converging to E (1.9, 0.08, 0.32). The phase portrait of the system is shown in Fig 6.1.1(B)

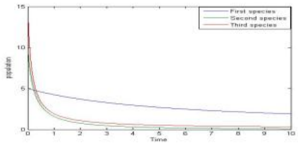
Fig 6.1.1(A)

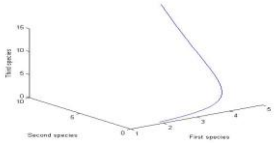
Fig 6.1.1(B)

Let us assume the delay kernels the above model $k(\lambda)=e^{-\alpha\lambda},\alpha>0$ for different values of 'α' the system dynamics is studied and placed in the following table

Fig (A) : shows the Time series analysis of the model (2.2)

Fig (B) : shows the phase portrait of the model (2.2)

Table 1:

S.No	Parameters values α&β and Converging equilibrium point E	Nature of system
1	α=0.005, E (1.9, 0.00, 0.33)	The second species is extinct. The system is stable and converging to equilibrium point E (1.9, 0.00, 0.33).
2	α=0.05, E (1.9, 0.00, 0.34)	The second species is extinct. The system is stable and converging to equilibrium point E (1.9, 0.00, 0.34).
3	α=0.5, E (1.9, 0.12, 0.33)	The system is stable and converging to a fixed equilibrium point E (1.9, 0.12, 0.33)
4	α=1.5, E (1.9, 0.14, 0.33)	The system is stable and converging to a fixed equilibrium point E (1.9, 0.14, 0.33)
5	α=5 , E (1.9, 0.25, 0.35)	The system is stable and converging to a fixed equilibrium point E (1.9, 0.25, 0.35)
6	α=15, E (1.9, 0.29, 0.36)	The system is stable and converging to a fixed equilibrium point E (1.9, 0.29, 0.36)
7	α=100, E (1.9, 0.31, 0.36)	The system is stable and converging to a fixed equilibrium point E (1.9, 0.31, 0.36)

1. α=0.005, E (1.9, 0.00, 0.33)

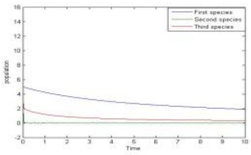
Fig 6.1.2(A)

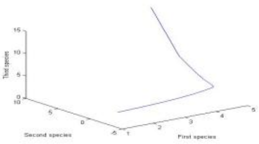
Fig 6.1.2(B)

2. α=0.05, E (1.9, 0.00, 0.34)

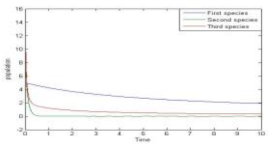
Fig 6.1.3(A)

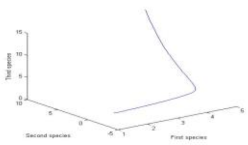
Fig 6.1.3(B)

3. α=0. 5, E (1.9, 0.12, 0.33)

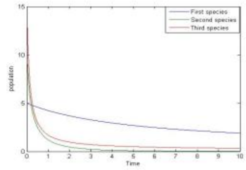
Fig 6.1.4(A)

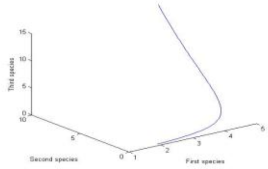
Fig 6.1.4(B)

4. α=1.5, E (1.9, 0.14, 0.33)

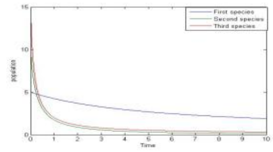
Fig 6.1.5(A)

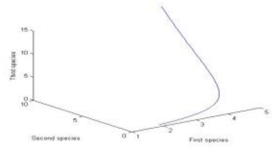
Fig 6.1.5(B)

5. α=5, E (1.9, 0.25, 0.35)

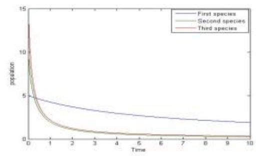
Fig 6.1.6(A)

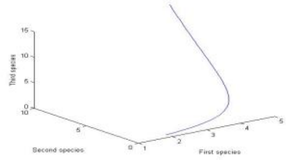
Fig 6.1.6(B)

6. α=15, E (1.9, 0.29, 0.35)

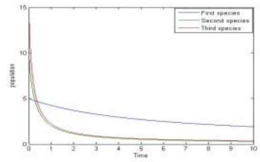
Fig 6.1.7(A)

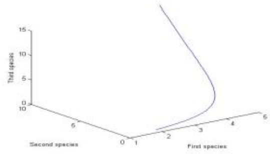
Fig 6.1.7(B)

7. α=100, E (1.9, 0.31, 0.36)

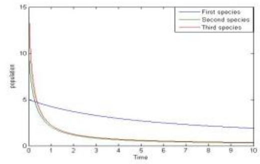
Fig 6.1.8(A)

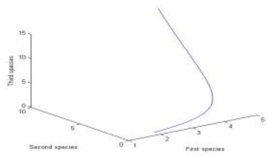
Fig 6.1.8(B)

As on delay kernel value, 'a' increases the growth rates of second and third species are slightly increasing. The system is stable and delay further stabilizes the system for the above mentioned parametric values.

Observations:

The parameters for the model are identified shows the stable system and the impact delay parameter 'α'

(i) As α ranges from 0.005 to 0.05 , the second is washed out, a slight increase in the third species population and first species population is stabilizes, hence the system is being stable.

(ii) As α value ranges from [1.5 , 100] , the second and third species populations are slightly increasing from zeros, Hence the delay is having a slow impact in the population growth rates of second and third species.

(iii) No variation in the first species population growth even though the delay is imposed

Example 2:

Let a_1=1, a_2=2, a_3=3, α_{11}=0.9, α_{22}=0.8, α_{33}=0.8, α_{21}=0.9, α_{32}=0.9, N_{10}=5, N_{20}=10 and N_{30}=15.

The system dynamics with the above parameter values without the delay argument is shown the graphs 6.2.1(A) and 6.2.1(B).

Fig 6.2.1(A) shows the time series analysis of the model (2.1) is a stable system with fixed equilibrium point converging to E (1.11, 1.23, 2.35). The phase portrait of the system is shown in Fig 6.2.1(B)

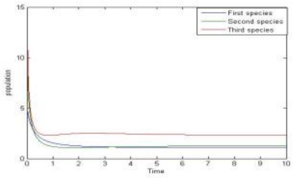

Fig 6.2.1 (A)

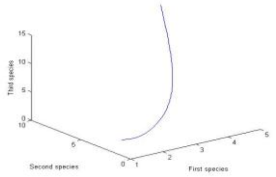

Fig 6.2.1 (B)

For different delay kernel 'α' the system dynamics is studied and placed in the following table

Table 2:

S.No	Parameters values α&β and Converging equilibrium point E	Nature of system
1	α=0.005, E (1.11, 0.00, 3.75)	The second species is extinct, There is a significant growth in third species . The system is stable and converging to equilibrium point E (1.11, 0.00, 3.75).
2	α=0.05, α=0.05, E (1.11, 0.00, 3.75)	The second species is extinct. The system is stable and converging to equilibrium point E (1.11, 0.00, 3.75).
3	α=0.5, E (1.11, 0.07, 3.58)	The system is stable and converging to a fixed equilibrium point E(1.11,0.07, 3.58)
4	α=1.5, E (1.11, 1.65, 2.5)	The system is stable and converging to a fixed equilibrium point E(1.11, 1.65, 2.5)
5	α=5 , E (1.11, 2.25, 3.24)	The system is stable and converging to a fixed equilibrium point E(1.11, 2.25, 3.24)
6	α=15, E (1.11, 2.42, 3.57)	The system is stable and converging to a fixed equilibrium point E (1.11, 2.42, 3.57)
7	α=100, E (1.11, 2.49, 3.72)	The system is stable and converging to a fixed equilibrium point E (1.11, 2.49, 3.72)

1. α=0.005, E (1.11, 0.00, 3.75)

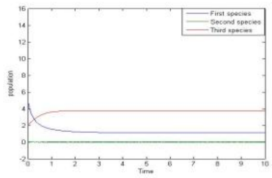

Fig 6.2.2(A)

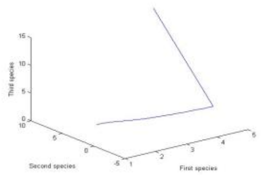

Fig 6.2.2(B)

2. α=0.05, E (1.11, 0.00, 3.75)

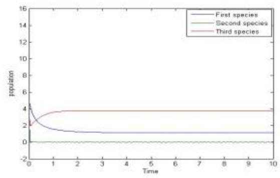

Fig 6.2.3(A)

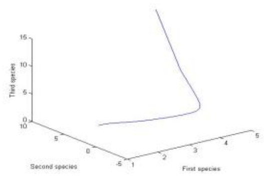

Fig 6.2.3(B)

3. α=0.5, E (1.11, 0.07, 3.58)

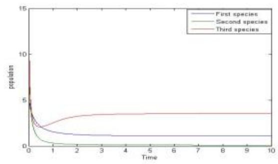

Fig 6.2.4(A)

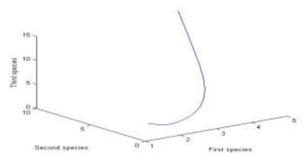

Fig 6.2.4(B)

4. α=1.5, E (1.11, 1.65, 2.5)

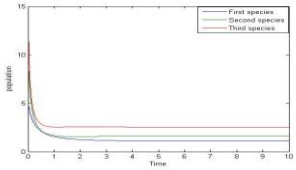

Fig 6.2.5(A)

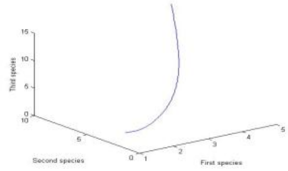

Fig 6.2.5(B)

5. α=5, E (1.11, 2.25, 3.24)

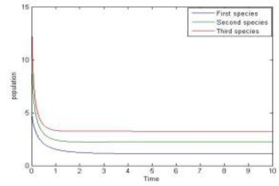

Fig 6.2.6(A)

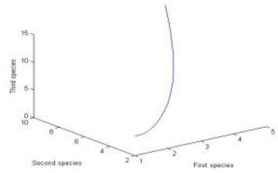

Fig 6.2.6B)

6. α =15, E (1.11, 2.42, 3.57)

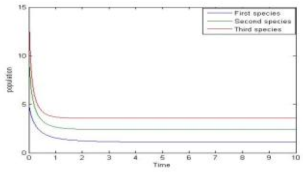

Fig 6.2.7(A)

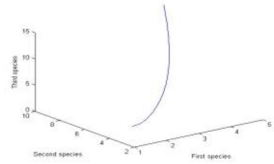

Fig 6.2.78(B)

7. α =100, E (1.11, 2.49, 3.72)

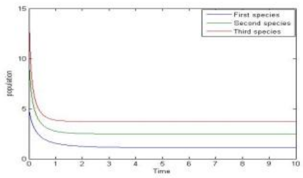

Fig 6.2.8(A)

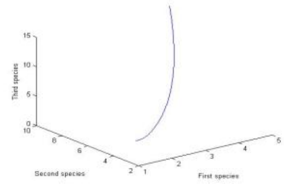

Fig 6.2.8(B)

Observations:

The parameters for the model are identified shows the stable system and the impact delay parameter 'α'

(i) As α ranges from 0.005 to 0.05 , the second is washed out, a slight increase in the third species population and first species population is stabilizes, hence the system is being stable.

(ii) As α value ranges from [0.5 , 1.5] , the second species population is increasing and third species populations is slightly decreasing, Hence the delay is having a slow impact in the population growth rates of second and third species.

(iii) As α value ranges from [1.5 , 100] , the second and third species population is increasing, Hence the delay is having a slow impact in the population growth rates of second and third species

(iv) No variation in the first species population growth even though the delay is imposed

VII. CONCLUSION:

In this present investigation, we studied three spices, ecological model, in which the first species ammensal on second species and second species ammensal on third species. Here first species and third species are neutral to each other. A time delay is introduced between first species and second species and second species and third species.

All possible equilibrium points were identified and the stability of co-existing state was discussed analytically. The analytical results were supported by numerical simulation. We observed that E_1, E_2, E_3, E_4, and E_5 are unstable. Further, we observed that E_6, E_7, and E_8 are conditionally asymptotically stable and we discussed stability of E_8 Co-existing state. The impact delay is studied by choosing suitable parameters in support of stability analysis using Mat Lab simulation. Two examples are chosen for analysis and detailed investigation with results is shown in table 1 & 2. The results are compared with different delay arguments of the model with the system without delay arguments. The delay argument further stabilizes the system. In two examples the system is asymptotically stable. Even we employee the delay kernel the system still exhibits the same nature. Hence the delay argument further stabilizes the system.

REFERENCES

1. Lotka A. J. 1925. Elements of Physical Biology, Williams and Wilking, Baltimore.
2. Voltera V, Leconseen La Theori Mathematique De La Leitte Pou Lavie, Gauthier-Villars, Paris, 1931.
3. Meyer. W. J 1925, Concepts of Mathematical Modeling, McGraw-Hill.
4. Cushing. J. N. 1977 Integro-Differential Equations and Delay Models in Population Dynamics. Lecture Notes in Bio-Mathematics. Vol. 20. Springer Verlag.
5. Kapur. J. N. 1985. Mathematical Modeling in Biology Affiliated East West.
6. Kapur. J. N. 1985. Mathematical Modeling, Wiley Easter.
8. Lakshmi Narayan K, 2005. A Mathematical Steady of a Prey-Predator Ecological Model with a Partial Cover and Alternate food for the Predator, Ph. D. Thesis. J. N. T. University, India.
9. A Prey - Predator Model with cover for Prey and a Time Delay for Interaction by Lakshmi Narayan K., Pattabhiramacharyulu N.Ch. Caribb.J. Math. Coput. Sci. ISSN 1017-6764, Vol no: 9,(2006) pp.56-63
10. A Prey - Predator Model with cover for Prey and an Alternative Food for the Predator and Time Delay by Lakshmi Narayan K., Pattabhiramacharyulu N.Ch., Int. J. of Scientific Computing, **ISSN: 0973-578X,** Vol.1, No.1, 2007 pp. 7-14
11. A Model of Two Mutually Interacting Species with Limited Resources and a Time Delay by Lakshmi Narayan.K, Ravindra Reddy.B, PattabhiRamacharyulu.N.Ch, Advances in Theoretical and Applied Mathematics ISSN 0973-4554, Volume 5, Number 2, (2010), pp. 121–132.
12. A prey, predator and a competitor to the predator model with time delay by Papa Rao A.V., Lakshmi Narayan K., International Journal of Research In Science & Engineering e-ISSN: 2394-8299, Special Issue –NCRAPAM March 2017, Pp 27-38.
13. Dynamics of Prey predator and competitor model with time delay by Papa Rao A.V., Lakshmi Narayan K., International Journal of Ecology& Development Year 2017; Int. J. Ecol. Dev.ISSN 0972-9984(Print); ISSN 0973-7308 (Online), Volume 32, Issue No. 1, Pp 75-86.
14. Stability Analysis of Three species Ecological Model with Time delay by Papa Rao A.V., Lakshmi Narayan K., Proceedings of international conference IMBIC at Kolkata during December 19th -21st 2013, Vol.2.
15. Dynamical System of Ammensal Relationship of Humans on Plants and Birds with Time Delay by Kondala Rao. K., Lakshmi Narayan. K., Bulletin of Calcutta Mathematical Society**,** Volume-109, No-6, December 2017, pp 485-500 with ISSN: 0008-0659.
16. Kuang, Y., 1993, "Delay differential equations with application in population dynamics," Academic Press, New York.
17. Gopalsamy, K., 1992, "Stability and oscillations in delay differential equations of population dynamics," Kluwer Academic, Netherland.

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