Research of Periodic Orbits and Chaos Produced in 1-D Liner Piecewise-Smooth Maps with Single Discontinuity, Positive and Negative Slopes

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Abstract*— Last four decades have seen a major development in the theory of piece-wise smooth discontinuous maps for the analysis of typical bifurcation phenomena for systems that can be modelled as such. The major focus of this paper is the analysis of 1-D linear piecewise smooth maps with a discontinuity, one positive and another negative slope. Interestingly, this type of map analysis has been carried out by various authors and the results have been reported in literature. For example, the existence of period adding cascade in particular parameter regions specified by the parameters 'a' and 'b' was proven. The new range of parameters this work presents are a ϵ (0,1), b ϵ (-1, 0) and a ϵ (0,1), b ϵ (- •, -1). Elementary algebraic and geometric tools have been used to analyze the periodicities in the 1-D linear piecewise smooth discontinuous map with respect to parameters a, b, µ and l. Various examples have been illustrated along with the plotted bifurcation curves. The analysis of the behaviour of the system with varied parameter ranges indicates that non-trivial cases are present for negative values of one or more parameters. A sample basin of attraction plot is illustrated as well. Further, an analytic proof for the existence of LnR orbits for the region a ϵ (0,1), b ϵ (-1, 0) was successfully produced, which is unpublished till date. The research concentrates on the theoretical results.

I. INTRODUCTION

Mathematics is a very precise and concise language with well-defined assumptions and rules for manipulations. Models, on the other hand describe our beliefs of how approximately or exactly the world functions. Mathematical modelling is a combination of both, where we try to translate those *beliefs* into the real world, by employing algorithms, theorems, axioms and proofs. Mathematical modelling is used for a diverse range of phenomena, among which dynamical systems is one. If we give a thought, philosophically, we realize that the real question to ask about a model is whether the behaviour it exhibits is because of its simplifications or if it captures the behaviour despite the simplifications.

The study of the complex behaviour of dynamics and systems via mathematical modelling is very often used in science and technology. Originally, systems were classified as smooth dynamical systems for which the related mathematical theory has been well developed. However, in the present scenario, it has been observed that many systems exist whose internal processes are classified into non-smooth or only piece-wise smooth. Some of the examples of a piece-wise smooth curve include: heartbeats, bouncing, slipping, switching, Poincar´e maps, rocking blocks, friction, Chua circuit etc.

One of the elegant concept of a piece-wise smooth system is the 'Newton's cradle'. Five small silver balls are attached to strings that hang on two parallel bars, which are then attached to a silver base. If we release a ball from one end, then only one ball from the other end swings out and falls back making the other ones briefly reunite. This demonstrates laws of conservation of momentum, conservation of energy and friction, all leading to a piece-wise smooth system. This, and many other observations have been a source of motivation for researchers to study piecewise dynamical systems in depth.

Since years, inherently piece-wise smooth systems have been a source of great research giving a boost in the understanding, explanations and analysis of the rich and complex dynamics that they put forward. Examples of such systems belong not only to engineering but also in businesses, economics, biology, psychology etc. Parameter studies provide knowledge of how a particular fixed point and the periodic solutions change when a parameter is varied. Path following techniques, when used give us a branch of fixed points that the solution follows. The number and type of fixed points and periodic simulations can change at a certain parameter value. This qualitative change in the structural behaviour of the system is called *bifurcation* (introduced by Poincar´e). Although tools have been well developed to analyze the various bifurcation scenarios, it becomes increasingly clear that some of the peculiar bifurcation phenomena observed in the piece-wise smooth system cannot be explained by the dynamics of smooth systems. These dissimilarities have led to the development of yet another special bifurcation theory known as the *discontinuity induced bifurcation theory*. The border collision bifurcation, a sub-type of the discontinuity theory, is confronted in the linear piece wise smooth continuous and discontinuous maps.

A. Short Chronology of Piece-wise Smooth Systems

- *1959*

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– First known published results. – Leonov gave a recurrence relation to find analytical expressions for the family of bifurcations occurring in a 1-D linear piecewise-smooth map with one point of discontinuity[1].

- *1970*

– Feigin separately analysed the bifurcations in piecewise-smooth maps and termed them as sewing bifurcation or C-bifurcations[2].

- *1990*

– Deane published the analysis of bifurcations and chaos in power electronic circuits[3].

- *1992*

– Nusse coined the term border collision bifurcation to describe the atypical bifurcation phenomena in the piecewise-smooth maps[4].

- *1999*

– di Bernardo showed a set of classification techniques for different types of bifurcations with respect to variation in map parameters for 1-D, 2-D and n-D linear piecewise-smooth continuous maps.

- *2003*

– First classification of border collision bifurcation in the 1-D linear piecewise-smooth discontinuous map with respect to variation in map parameters.

- *2006*

Dynamics of a Piecewise Linear Map with a Gap was published by Hogan.[5]

- *2008*

– di Bernardo published a textbook of the techniques to analyze various types of bifurcations in piecewise-smooth dynamical systems.

- *2006*

– Avrutin in his publication, analysed the periodic orbits in the 1-D linear piecewise-smooth discontinuous map. It was shown that the periodic orbits appear and disappear in a very specific manner as certain parameters of the map are varied. This specific arrangement of periodic orbits is termed as period adding cascade.[6].

- *2010*

– The term BCB curves was coined for the first time.

– Avrutin and Bischi analysed the periodic or-bits in a wide variety of linear and non-linear piecewise-smooth maps using border collision bifurcation curves (BCB curves)[7].

– In Gardinis publication, it is shown that BCB curve technique can be used to explain the results published by Leonov in 1959.[8]

- *2012*

– Rajpathak, Pillai and Bandyopadhyay published the analysis of stable periodic orbits in the one dimensional linear piece-wise smooth discontinuous map in American Institute of Physics Chaos
Journal[9].

- *2015*

– Rajpathak, Pillai and Bandyopadhyay analysed unstable periodic orbits and chaotic orbits in the one-dimensional linear piecewise-smooth discontinuous map[10].

In the subsequent sections we will briefly review the piecewise-smooth system theory and its applications in various fields e.g. engineering sciences, live sciences and finance.

B. The 1-D linear piecewise-smooth map

The real number line is divide into two sets of real number patterns, one the left side, denoted by L ϵ (-∞, 0) and the other on the right side, denoted R ϵ (0, ∞). We included $x_{n=}$ 0 in the left half of the number line because the equations that are used are affine and these affine pieces are strictly defined according to this partition of the real number line. The equations that will be formed are as follows The 1-*D* linear piecewise-smooth map is written as[11]:

$$x_{n+1} = \begin{cases} ax_n + \mu \quad \ldots\ldots\ldots.. for\ x_n \leq L \\ bx_n + \mu + l \ldots\ldots\ldots.. for\ x_n > R \end{cases}$$

When μ and l are varied, one gets three possible scenarios viz. either two fixed points (one on either side of discontinuity), or one fixed point on one of the two sides or no fixed points at all.

For calculation of the various values of μ for a particular given pattern, σ such that the periodic orbit exists. This range of values of μ is defined as P_σ. This range of existence for the periodic orbits plays a crucial role for the analysis and understanding of periodic orbits. An algorithm for the calculation of the range of μ *i.e.* P_σ is explained below:

Consider some pattern σ. a, b and l remain fixed. Assume that the length of the pattern is n, i.e. it contains n different points before the orbit is repeated. The definition of orbit gives us the knowledge of that any pattern consists of n distinct periodic points in the orbit namely, x_0, x_1,... x_{n-1}, x_n and $x_{n=}\,x_0$.

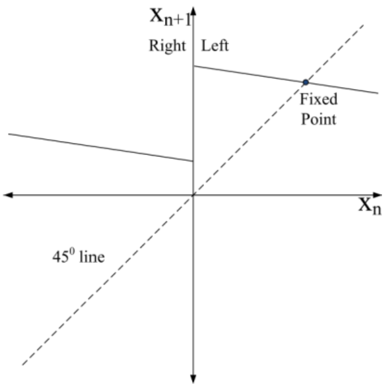

FIG. 1. The parameter as per the range of slopes of maps *a* and *b*

Let us assume, for now without losing the generality that there are p points in Land q points in R. From the above equation, the value of x_{i+1} will depend on whether the point is in the left half i.e take up the form of $axi+\mu$, or the point is in the right half of the plane and takes up the form of $bxi+\mu+l$.

C. Definitions of Basic Terms used in the field of piecewise smooth systems

Definition 1.

Fixed Point: A point p is a *FIXED POINT* of the map if $f(p)$ *i.e.* p (Fixed points are found by solving the equation $f(x)=x$).

Let f be a map on R and let p be a real number such that $f(p) = p$. i.e. p is a fixed point. Now consider definitions 5 and 6.

Definition 2.

Stable Fixed Point: If the starting point is in vicinity after iterations the solution come at this point and remains there. Then it is a stable fixed point.

Definition 3.

Unstable Fixed Point If the starting point is at the fixed point it remains there. All other starting points will go to infinity after iterations. It is an unstable point.

Definition 4.

Periodic Orbit: Let f be a map from R to R. We call p a periodic point of order k if $f^k(p)= p$, where k is the smallest such positive integer. An orbit of k distinct points is called a periodic orbit of period-k.[12]

Definition 5.

Period of an Orbit: Let f be a map on R. We call p a periodic point of period k if $f^k(p)= p$, and if k is the smallest such positive integer. The orbit with initial point p (which consists of k points) is called a periodic orbit of period k.

Definition 6.

Eventually Periodic Points: A point x is *eventually periodic* with period p for the map f if for some positive integer N, $f^{(n+p)}= f^{(n)}(x)$, for all $n \geq N$, and if p is the smallest such positive integer.

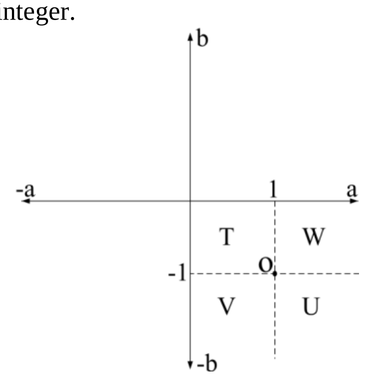

FIG. 2. The *a*,*b* plane divided into four parts

Definition 7.

Stable and Unstable periodic orbits: Let f be a map from R to R. A periodic orbit $\{p_1, p_2, .. .p_k\}$ is called a *stable periodic orbit* if $|f'(p_1)f'(p)...f'(p_k)| > 1$ and an *unstable periodic orbit* if $|f'(p_1)f'(p)...f'(p_k)| < 1$.

Definition 8.

Pattern: Every point through which map evolve is a real number (R). Now if the real number is divided into two groups like close left half L:= $(-\infty, 0)$ and the open right half R:=$(0, \infty)$. The point x_n 0 is included in left side. The points in an Orbit also can be shown by the sequence of L*s* and R*s* by showing the set (L or R) the corresponding point belongs to. Therefore, the sequence of L*s* and R*s* is called as the *Pattern*. Examples of pattern is LLLRR or L^3R^2 in generalized way it is L^nR^m.

A pattern is described by different names by different authors for example: *periodic cycle*[7], *symbolic sequence*[13] etc.

Definition 9.

Chaotic Orbits Let f be a map from R to R and let $\{p_1, p_2, p_3...\}$ be a bounded orbit (i.e. the region in which the points lie is bounded). The orbit is said to be *chaotic* if it is NOT asymptotically periodic AND its Lyapunov exponent $h(p_1)$ is greater than zero. Hence, a chaotic orbit is a bounded, non-periodic orbit that displays sensitive dependence.

Definition 10.

Bifurcation Qualitative change in the structural behaviour of the system is called as bifurcation.

D. Parameter regions

Previously, analytical results proving the existence of various orbits have been produced for certain regions in the parameter plane having axes as a and b. The entire first quadrant has been exhausted (i.e. $a > 0$ and $b > 0$). So, we attempt analysis for the quadrant four (i.e. $a > 0$ and $b < 0$). The quadrant IV of a, b plane is divided in to four parts as: (see Fig. 2)

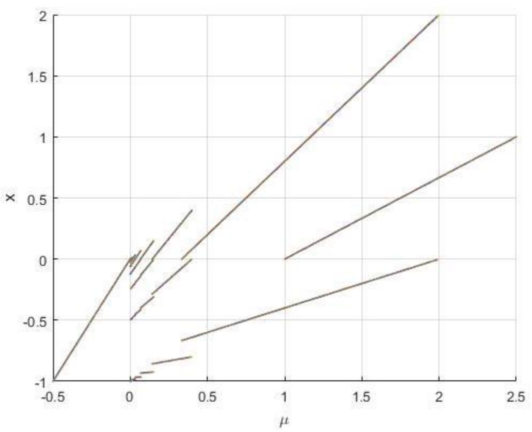

FIG. 3. $a \epsilon (0,1)$ $i.e.\ 0.5$ and $\epsilon (-1,0)$ $i.e. -0.5$, l<0 i.e. -1 and -0.5<μ<2.5 Period increment L^nR

Region T: $a \in (0, 1)$, $b \in (0, -1)$

Region U : $a \in (0, \infty)$, $b \in (-\infty, -1)$

Region V : $a \in (0, 1)$, $b \in (-\infty, -1)$

Region W : $a \in (1, \infty)$, $b \in (0, -1)$

The 1-*D* piece-wise smooth maps exhibit different behaviour as per the range of parameters. We consider Region T and Region V for analysis in this paper.

Bifurcation diagrams produced using *iterative numerical methods* indicate that high periodic orbits characterized by L^nR and chaotic orbits exist for this region. Hence, an analytical proof has been attempted for this.

II. BIFURCATION DIAGRAMS

Consider the bifurcation diagrams given below. Note that here, we always assume that $l < 0$, since only trivial results exist for $l>0$.

Figure 3 shows that period increment of the form L^nR when $a \epsilon (0,1)$ i.e. 0.5, $b \epsilon (-1, 0)$ i.e. -0.5, $l < 0$ i.e. -1 and $\mu \epsilon (0, 0.5)$. An analytical proof has been successfully attempted for this later. It provides a starting point for the next proof. As the parameter μ approaches zero, very high period orbits are obtained. When $\mu \epsilon (0.4, 1)$ there is a period-2 orbit and when $\mu \epsilon (1,2)$ there is co-existence of period-2 and a fixed point.

Consider the Figure 4 which provides an example orbit for the above region. It has been obtained by simply iterating points as per the definition of the map and the given values of parameters.

The basin of attraction is also plotted using numerical simulation for parameter values a ϵ (0,1) i.e. 0.5, and b ϵ (-1,0) i.e. -0.5, l < 0 i.e. -1 and μ=2 is shown in figure 5.

Figure 5. The starting points marked by the green region are attracted to period-1 orbits whereas points marked by the red

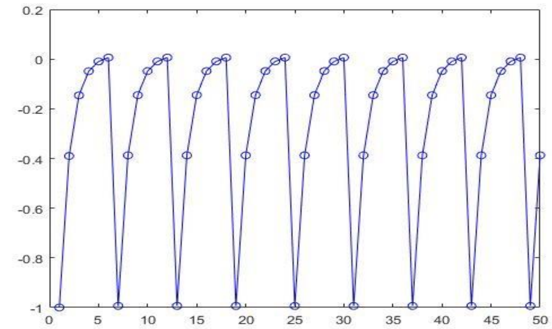

FIG. 4. Orbit L^5R for a = 0.5 and b = -0.5

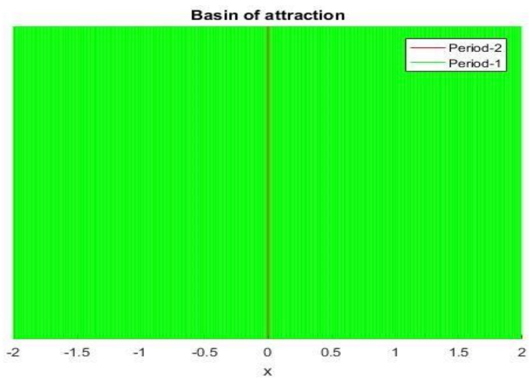

FIG. 5. $a \epsilon (0,1) i.e. 0.5$ and $\epsilon (-1,0)$ $i.e. -0.5$, $l <$ 0 i.e. -1 and $\mu = 2$

region attracted to period-2 orbits. Period-2 points lies at 0, 2 and a fixed point at 0.6667.

The two Figures 6, 7 given below show bifurcation diagrams for the range $a \epsilon (0,1)$ and $b \epsilon (-\infty, -1)$. For $0 < \mu < l$, both high periodic orbits (characterized by L^nR and chaotic orbits exist whereas for $\mu > -l$ only chaotic orbits are present.

Points marked by the green region are attracted to period-1 orbits whereas points marked by the red region are attracted to period-2 orbits. Period-2 points at 0, 2and a fixed point at 0.6667

The bifurcation diagram shown in the Figure 7 is plotted for $\mu \epsilon (2, 1.6)$. It can be clearly seen that only chaotic orbits are present for $\mu > 1$.

As per the bifurcation diagram shown in Figure 8 when $a \epsilon (0,1)$ i.e. 0.5, $b \epsilon (-\infty, -1)$ i.e. -2, $l > 0$ i.e. 1 and $\mu \epsilon (-0.2 , 0.6)$. there exists Period adding cascade, high periodic orbits and chaotic orbits.

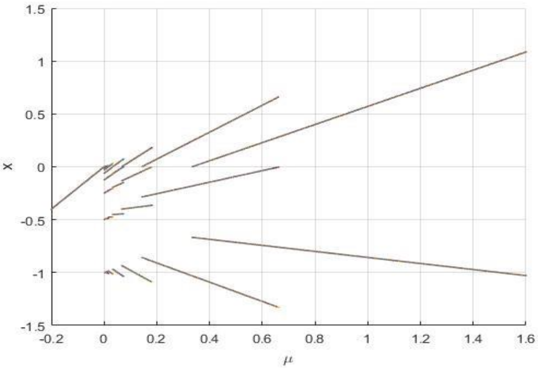

FIG. 6. $a \epsilon (0,1) i.e. 0.5$, and $\epsilon (-\infty,-1)$ $i.e. -1.5$, l<0 i.e. -1 and -0.2<μ<1.6

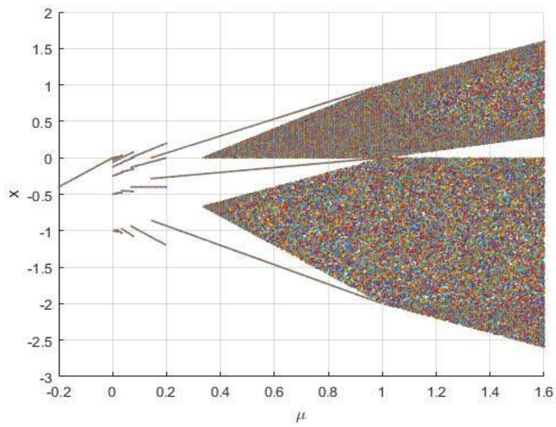

FIG. 7. $a \epsilon (0,1) i.e. 0.5$, and $\epsilon (-\infty,-1)$ $i.e. -2$, l<0 i.e. -1 and -0.2<μ<1.6

III. ANALYSIS & RESULTS

Consider the initial point x_0 and the map

$$x_{n+1} = \begin{cases} ax_n + \mu \quad \ldots\ldots\ldots.. for\ x_n \leq 0 \\ bx_n + \mu + l \ldots\ldots\ldots.. for\ x_n > 0 \end{cases}$$

The existence of the orbit L^nR has to be proved for high periodic orbits and chaotic orbits.

Both high periodic orbits and chaotic orbits are present a ϵ (0, 1) and b ϵ (-1, 0).

$$x_0 \leqslant 0$$

$$x_1 = ax_0 + \mu \leqslant 0$$

$$x_2 = ax_1 + \mu = a(ax_0 + \mu) + \mu = a^2x_0 + (1+a)\mu \leqslant 0$$

$$x_3 = ax_2 + \mu = a(a^2x_0 + (1+a)\mu) + \mu = a^3x_0 + (1+a+a^2)\mu$$

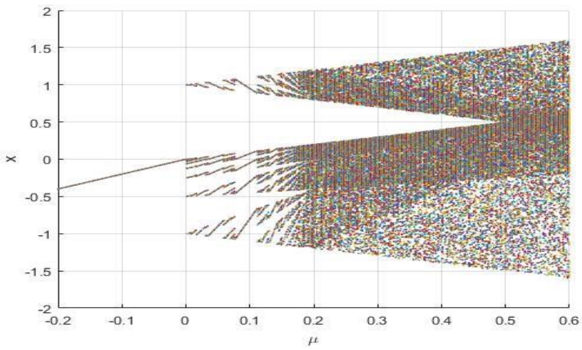

FIG. 8. $a\ \epsilon\ (0,1)\ i.e.\ 0.5$, and $\epsilon\ (-\infty,-1)\ i.e. -2$, , l>0 i.e. 1 and -0.2<μ<0.6

Similarly,

$$x_k = a^k x_0 + (1 + a + a^2 + \cdots + a^{k-1}))\,\mu$$

and

$$x_n = a^n x_0 + (1 + a + a^2 + \cdots + a^{n-1})\mu$$

$$x_k = \frac{\mu b\{a^k S_a^{n-1} - a^n S_a^{k-1}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu b\{a^k \frac{1-a^n}{1-a} - a^n \frac{1-a^k}{1-a}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu b\{\frac{a^k - a^{n+k} - a^n + a^{n+k}}{1-a}\} + S_a^k \mu - a^k}{1 - a^n b}$$

Period adding cascade, high periodic orbits and chaotic orbits are present.

The steps to find range of μ *i.e.* P_σ for which $L^n R^m$ is possible:

$x_{n+1} = x_0 = b x_n + \mu + l$

$x_{n+1} = b(a^n x_0 + (1 + a + a^2 + \cdots + a^{n-1})) + \mu + l$

$x_0 = a^n b x_0 + (1 + b(1 + a + a^2 + \cdots + a^{n-1})) + \mu + l$

Put

$1 + a + a^2 + \cdots + a^{n-1} = S_a^{n-1}$

$l = -1$

The above does not lead to any loss of generality. Now consider the following set of equations. Each equation logically follows from the one above it in a single step.

$$x_0(1 - a^n b) = \mu(1 + b S_a^{n-1}) - 1$$

$$x_0 = \frac{\mu(1 + b S_a^{n-1}) - 1}{1 - a^n b}$$

$$x_k = a^k x_0 + (1 + a + a^2 + \ldots + a^{k-1})\mu$$

$$x_k = a^k x_0 + S_a^{k-1}\mu$$

$$x_k = a^k\{\frac{\mu(1 + b S_a^{n-1}) - 1}{1 - a^n b}\} + S_a^{k-1}\mu$$

$$x_k = \frac{a^k \mu + b a^k \mu S_a^{n-1} - \mu a^n b S_a^{k-1} - a^k + S_a^{k-1}\mu}{1 - a^n b}$$

$$x_k = \frac{\mu b\{\frac{a^k - a^n}{1-a}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu b\{a^k \frac{1 - a^{n-k}}{1-a}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu b\{a^k S_a^{n-k-1}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu b\{a^k S_a^{n-k-1}\} + S_a^k \mu - a^k}{1 - a^n b}$$

$$x_k = \frac{\mu\{b a^k S_a^{n-k-1} + S_a^k\} - a^k}{1 - a^n b} \qquad (1)$$

Note that this is the general term for all x_k for k = 0,1,2...n. Using this term, we can find μ. The question is which term gives us the tightest bound according to the P_σ algorithm. The lower bound will always be given by $x_n = \frac{a^n}{S_a^n}$. As for the lower bound, we have, by rearranging the above equation:

$$x_k \leq 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

Note that the denominator is always positive due to 'b' being negative. Hence, we have:

$$\mu\{b a^k S_a^{n\ \ k-1} + S_a^k\} \leqslant a^k \qquad (3)$$

Which bound it gives depends on the value of the expression in the bracket after μ. If the value is negative, it will give a lower bound less than zero, which will be discarded anyway. Hence, we will only look at the values of k for which we get positive values of the expression in the braces. These shall give us the upper bound. Now the task is to find which x_k gives the us the tightest upper bound, i.e. the lowest value. Putting k = n - 1, we get:

$$\mu \leqslant \frac{a^{n-1}}{a^{n-1} b + S_a^{n-1}} \qquad (4)$$

By graphical analysis, we can be sure that this is the tightest upper bound. Hence, an analytic proof has been produced for one of the two desired regions.

IV. CONCLUSION

The analysis of periodic orbits in the 1-*D* linear piecewise smooth discontinuous map with one positive and another negative slop have been worked upon. A mathematical proof was produced for one of the 'new' regions along with numerical and simulation results. Numerical simulation indicated the existence of $L^n R$ orbits for the region a ϵ (0,1) and b ϵ (-1,0). Hence, it was analysed and the existence of $L^n R$ was proven. Further, a numerical simulation was carried out for $a\ \epsilon\ (0,1)$ and $b\ \epsilon\ (-\infty,-1)$.

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