International Journal of Recent Technology and Engineering (IJRTE)
ISSN: 2277-3878 (Online), Volume-8 Issue-2, July 2019

Building the Stiffness Matrix and Load Vector of the Taper Elements with the I-Section That Considering the Effect of Shear Force and Stiffness of the Connection

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***Abstract*: This paper shows how to build stiffness matrix and load vector by energy method for taper, I section elements includes the effects of shear and semi-rigid connection, use for structural analysis problem. Mathematical model of the section is exponential. Thereby, programming the structural analysis program for taper elements and verify results stiffness matrix, load vector as well as assessing the effects of shear and semi-rigid connection to force and displacement of this element.**

***Keywords*: Taper elements, Stiffness matrix, Semi-rigid connection .**

I. INTRODUCTION

In the structure of constructions, bar components (with the height of the section that changing follow to the lengths) are used more widely in construction, sections meet the architectural requirements, more suiTab. for bearing capacity in the structures, and the making of changing sections is not difficult for manufacturers.

For steel frame structure, the type of beveled cross-section is used quite a lot, usually the beam and column components with I cross-section have the wing plate size and the thickness of the belly plate is constant, the section height (the height of vell plate) changes. The rate of change in the height of the section depends on the distribution of internal force in the section according to the length, the conditions of connection at the two ends of the component..

Documents on structural mechanics and numerical methods have presented how to construct the stiffness matrix and load vector for a constant section bar element, with different connection conditions at either end or applied With different load cases, the results have built a sample element Tab.. Recently, a number of domestic and foreign documents mentioned the establishment of variable section element stiffness matrices, but mainly rectangular sections with linear changing heights or based on the shape function of the regular section bar element and the analytical results are approximate, SAP 2000 commercial software also mentioned the problem of structural analysis with beveled structure, the structure is consistent with the section linear changes, or changes in parabola form with cross sectional characteristics are quadratic and tertiary exponential functions. However, in order to have the matrix and load vector of the taper element and I cross-section, we also need further studies, to achieve the results of the structural structural problem with elegant more accurate beveled face, and to approach more general problems of steel frame structural analysis that SAP 2000 software has not mentioned. In addition, when analyzing the stress and deformation state of bending structures such as beams, frames, plates ... the effect of shear force and sliding deformation is often overlooked.. However, for bar elements such as beams with a thin bell plate, the impact of the shear force needs to be considered. Especially for the I-shaped taper element, with semi-rigid connection at both ends, the effect of shear force should be considered clearly.. In addition, structural steel structures such as beams and columns are linked together to form a bearing structure. Many research results show that the connection has certain elasticity, also known as soft connection or semi-rigid connections. The problem of analyzing the steel frame structure with consideration of the linkage hardness has been interested recently by many domestic and foreign authors, however, for the taper element, taking into account the influence of the connection stiffness, there are almost no in-depth studies

II. THEORY

A. *Model of taper element, I cross-section*

1) *. Geometric characteristics of taper*

Due to beam and column with taper components, the cross section I is widely used as a bearing element in steel frame, the height of the section often changes. To determine the geometrical characteristics of sections at each position in length, including area, inertia moment, torsion moment, these cross sectional features are mathematically performed. The relational form of quantities of geometric dimensions in the section is quite complicated. In fact, to reduce numerical integration after each calculation and still give accurate results, the representation of the geometric characteristics of the section through the law of the exponential exponential function, or by exponential function with radix is the ratio of taper, also mentioned by some documents.

Revised Manuscript Received on 30 July 2019.
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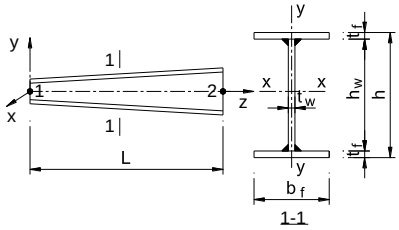

Fig. 1. Beam element with taper section

Accordingly, with the taper element with the I section, in the coordinate system Oyz, there are cross section characteristics defined.:

Area section bar, $A(z)$ Moment of inertia, $I(z)$:

$$A(z) = A_1(1+rz/L)^m \quad (1);$$

$$I(z) = I_1(1+rz/L)^n \quad (2)$$

Where: $A(z)$ – section area at position "z"; $I(z)$ – moment of inertia at position "z"; A_i, I_i, h_i – respectively the area of cross section, moment of inertia, height of beam section at position *z = 0*. Ratio of taper as:

$$z = L\ (i = 1, 2);\ r = h_2/h_1 - 1 \quad (3)$$

m, n – The shape coefficient, depending on the section and size of the element, corresponds to the area and moment of inertia of the section, determined as follows: Applying boundary conditions *z = L* and *z = 0*, we have: at *z = L*, có $A(z) = A_2$ and $I_x(z) = I_2$; *at z = 0*, we have $A(z) = A_1$ and $I_x(z) = I_1$;

$$m = \ln(A_2/A_1)/\ln(h_2/h_1) \text{ and } n = \ln(I_2/I_1)/\ln(h_2/h_1) \quad (4)$$

Notice that at the two ends of the element, the section-specific values are correct, but in the middle position there is a certain error. Below we examine some sections.

2) Shape coefficient of some sections

Considering section I, hollow box and solid box shape, for a constant width and thickness, change the section height, the value of the shape coefficients "n" and "m" are shown in Tab. 1..

Tab. 1. Value of shape coefficients

No	Section	Coefficients	
		n	m
1	rectangular	2,1 ÷ 2,6	Changing
2	hollow box	2,1 ÷ 2,6	Changing
3	solid box shape	3,0	1,0

3) Error when determining section characteristics through exponential function

As mentioned, the exponential exponential performance has the convenience of calculation, but there is a certain error compared to the exact formula. Accordingly, the survey of this error for some I-beam cross section is popular in practice, the survey position is the mid-beam point, the survey section characteristics include area, inertia moment in the bending plane, the coefficient of taper coefficient changes as follows 0; 1.0; 2.0; 3.0; 4.0. Survey results are shown in Tab. 2.

The following symbol: I-b_f×(h_1÷h_2)×t_w×t_f (mm) is used to denote the geometric parameters of the cross sectional bar I. Where: b_f – the width of the wing, h_1 – the height at the ends "1", h_2 – the height at the ends "2", t_w – the width of bell plate, t_f – the width of wing.

Tab. 2. The difference of the characteristic value of the section (%)

r	250×(350÷950)×6×8	250×(350÷950)×10×12	150×(350÷950)×10×12

	A(z)	$I_x(z)$	A(z)	$I_x(z)$	A(z)	$I_x(z)$
0.0	0.00	0.00	0.00	0.00	0.00	0.00
1.0	1.44	0.76	1.40	1.00	1.46	0.73
2.0	3.53	2.24	3.20	2.77	3.52	2.20
3.0	5.39	3.87	4.64	4.54	5.32	3.83
4.0	6.93	5.45	5.74	6.13	6.79	5.40

Tab. 2 shows when the raper ratio *r ≤3* the biggest error of *A*, I_x, is about 5%. The content of this article only takes into account the taper element with r ≤ 3.

B. The making of stiffness matrix and load vector for taper element, I-section, two rigid connection ends

1) Method of making matrix

Using the making method of the stiffness matrix presented in Section 4.4 of the document [4], we have the stiffness matrix of the element built according to the following formula:

$$[k] = \begin{bmatrix} [d]^{-1} & [d]^{-1}[\Phi]^T \\ [\Phi][d]^{-1} & [\Phi][d]^{-1}[\Phi]^T \end{bmatrix} \quad (5)$$

Where: $[d] = [k_{ff}]^{-1}$ – matrix between pinned displacement vector and reaction force. $[\Phi]$ – matrix between the internal forces of the pinned.

2) Making stiffness matrix:

a) Making stiffness matrix with tension - compression bar element

The taper element has an I section (belly plate thickness and wing size are constant), length L and cross section characteristics at any position z: $I(z)$ và $A(z)$

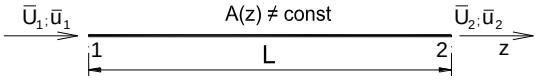

Fig. 2. Tension - compression bar element

The displacement at point 2 under the effect of force U2 is:

$$u_2 = \int_0^L \varepsilon dx = \int_0^L \frac{\sigma dx}{E} = \int_0^L \frac{U_2 dx}{EA(z)} = \frac{U_2 L}{EA(z)} \quad (6)$$

Where: E – Elastic modulus of bar material; $A(z)$ – bar section area at position z.

$$[d] = L/[EA(z)] \quad (7)$$

From balance equation we have: U_1 = -U_2 thus $[\Phi] = -1$. So, stiffness matrix of tension - compression bar element is defined as follows:

$$\bar{k}_{kn} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{14} \\ \bar{k}_{41} & \bar{k}_{44} \end{bmatrix} = \frac{EA(z)}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (8)$$

b) Making stiffness matrix for flat bending bar

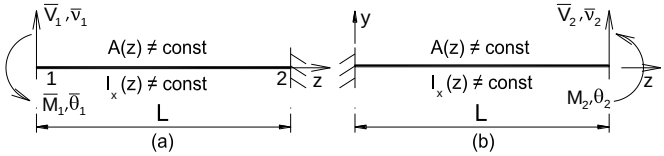

Fig. 3a.b. Internal forces and displacements

Make a connection matrix between displacement vector and reaction force of pinned. To determine $\bar{v}_1$ and $\bar{\theta}_1$, We fixed the first 2 of the bar as shown in Fig. 3b.

In the plane problem, with the moments, shear force, the potential energy expression of elastic deformation for the general force-bearing linear bar has length L as follows:

$$\bar{C}=\int_0^L \frac{M(z)^2 dz}{2EI(z)}+\int_0^L \frac{\gamma(z)V(z)^2 dz}{2GA(z)} \quad (9)$$

Where: $\bar{C}$ – elastic deformation potential energy; $M(z)$ – Expression of bending moment at the section with the z-coordinate; $V(z)$ – expression of shear force at the section with the z-coordinate; $EI(z)$ – Anti-bending stiffness of section (Anti-bending stiffness plane "yz") at section with z-coordinates ; G – Elastic slip modulus; $GA(z)$ – shear strength of section at section with z-coordinates;

$\gamma(z)$ – the coefficient depends on the y-shape of the cross-section at the section with z-coordinates, reflecting the uneven distribution of the stress; For I section or hollow rectangular $\gamma(z)=A(z)/A_t(z)$;

$A_t(z)$ – vertical section area at position z.

Coefficient $\gamma(z)$ can be expressed through the exponential rule with radix is the ratio of taper according to the following formula:

$$\gamma(z)=\gamma_1(1+rz/L)^p \quad (10)$$

γ_1, γ_2 – in turn, the coefficient depends on the sectional shape according to the y at z = 0 and at z = L;

$$p=\ln(\gamma_2/\gamma_1)/\ln(h_2/h_1) \quad (11)$$

According to the Fig. 3b, moment and shear force at the z position of the bar element are determined from the moment balance equation for pinned 1 and the shear balance equation.:

$$M(z)=\bar{V}_2(L-z)+\bar{M}_2;\ V(z)=-\bar{V}_2 \quad (12a.b)$$

Replace (12a.b) into (9) we have:

$$\bar{C}=\int_0^L \frac{\left[\bar{V}_2(L-z)+\bar{M}_2\right]^2 dz}{2EI(z)}+\int_0^L \frac{\gamma(z)\bar{V}_2^2 dz}{2GA(z)} \quad (12)$$

Applying Castigliano's theorem presented in the document [3], we have:

$$\bar{v}_2=\frac{\partial\bar{C}}{\partial\bar{V}_2}=\int_0^L \frac{\left[\bar{V}_2(L-z)+\bar{M}_2\right](L-z)dz}{EI(z)}+\int_0^L \frac{\gamma(z)\bar{V}_2 dz}{GA(z)}; \quad (13a)$$

$$\bar{\theta}_2=\frac{\partial\bar{C}}{\partial\bar{M}_2}=\int_0^L \frac{\left[\bar{V}_2(L-z)+\bar{M}_2\right]dz}{EI(z)}; \quad (13b)$$

Rewrite (13a) and (13b):

$$\bar{M}_2\int_0^L \frac{(L-z)dz}{EI(z)}+\bar{V}_2\left(\int_0^L \frac{(L-z)^2 dz}{EI(z)}+\int_0^L \frac{\gamma(z)dz}{GA(z)}\right)=\bar{v}_2 \quad (13c)$$

$$\bar{M}_2\int_0^L \frac{dz}{EI(z)}+\bar{V}_2\int_0^L \frac{(L-z)dz}{EI_x(z)}=\bar{\theta}_2 \quad (13d)$$

Where: $\beta=E/G$; $\bar{M}_1, \bar{M}_2$ –moment at the ends 1 and 2; $\bar{V}_1$, $\bar{V}_2$ – shear force at the ends 1 and 2; $\bar{\theta}_1, \bar{\theta}_2$ – rotation angle at the ends 1 and 2; $\bar{v}_1$, $\bar{v}_2$ – displacement in the direction perpendicular to the bar axis at the pinned 1 and 2.

Expression (13c.d) write in matrix form:

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{bmatrix}\begin{Bmatrix} \bar{V}_2 \\ \bar{M}_2 \end{Bmatrix}=\begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} \quad (20)$$

$$\bar{a}_{11}=\int_0^L \frac{(L-z)^2 dz}{EI(z)}+\int_0^L \frac{\beta\gamma(z)dz}{EA(z)};\ \bar{a}_{12}=\bar{a}_{21}=\int_0^L \frac{(L-z)dz}{EI(z)};\ \bar{a}_{22}=\int_0^L \frac{dz}{EI(z)}$$

Set basic integrals as follows:

- $T_1=\int_0^L \frac{dz}{I_x(z)}=\frac{L}{I_1 r}\frac{(1+r)^{1-n}-1}{1-n}$;

  $T_2=\int_0^L \frac{zdz}{I_x(z)}=\frac{L^2}{I_1 r^2}\frac{(1+r)^{1-n}\left[r(1-n)-1\right]+1}{(1-n)(2-n)}$

- $T_3=\int_0^L \frac{z^2 dz}{I_x(z)}=\frac{L^3}{I_1 r^3}\frac{\left\{\left[2+2(n-1)r+(n-1)(n-2)r^2\right](1+r)^{1-n}-2\right\}}{(1-n)(2-n)(3-n)}$;

- $T_4=\int_0^L \frac{z^3 dz}{I_x(z)}=$

$$=\frac{L^4}{I_1 r^4}\frac{1}{\binom{n}{-1}\binom{n}{-2}\binom{n}{-3}\binom{n}{-4}}\left|(r+1)^{-n}\left\{r\begin{Bmatrix}-6n-3(n-1)nr\\ -n\left[2+(n-3)n\right]r^2\\ -(n-3)(n-2)(n-1)r^3\end{Bmatrix}+6\left[-1+(r+1)^n\right]\right\}\right|;$$

- $T_5=\int_0^L \frac{z^4 dz}{I_x(z)}=$

$$=\frac{L^5(1+r)^{-n}}{\binom{n}{-5}\binom{n}{-4}\binom{n}{-3}\binom{n}{-2}\binom{n}{-1}r^5}\left(r\begin{pmatrix}-24n-12(-1+n)nr\\ -4n\left(2+(-3+n)n\right)r^2\\ -(-3+n)(-2+n)(-1+n)nr^3\\ -(-4+n)(-3+n)(-2+n)(-1+n)r^4\end{pmatrix}+24\left(-1+(1+r)^n\right)\right)$$

- $T_6=\int_0^L \frac{\gamma(z)}{A(z)}dz=\frac{\gamma_1 L}{A_1 r}\frac{-1+(1+r)^{(1-m+p)}}{1-m+p}$;

- $T_7=\int_0^L \frac{\gamma(z)}{A(z)}zdz=\frac{\gamma_1 L^2}{A_1 r^2}\frac{1+(1+r)^{(1-m+p)}\left[-1+(1-m+p)r\right]}{(-2+m-p)(-1+m-p)}$;

- $T_8=\int_0^L \frac{\gamma(z)}{A(z)}z^2 dz=\frac{\gamma_1 L^3}{A_1 r^3}\frac{\binom{1}{+r}^{-m}\left|2(1+r)^m-\binom{1}{+r}^{(1+p)}\left\{2+\binom{-1}{+m-p}r\ \ \left[2+\binom{-2}{+m-p}r\right]\right\}\right|}{(-3+m-p)(-2+m-p)(-1+m-p)}$

Then the coefficients $\bar{a}_{11}, \bar{a}_{12}, \bar{a}_{21}, \bar{a}_{22}$ are rewritten as follows:

$$\bar{a}_{11}=\left(L^2T_1-2LT_2+T_3+\beta T_6\right)/E;\ \bar{a}_{12}=\bar{a}_{21}=(LT_1-T_2)/E;\ \bar{a}_{22}=T_1/E; \quad (21)$$

The connection matrix between displacement and external forces $[d]=\begin{bmatrix}\bar{a}_{11} & \bar{a}_{12}\\ \bar{a}_{21} & \bar{a}_{22}\end{bmatrix}$;

Making a matrix between the reaction force components at the pinned

With Firgure 3a, Considering the balancing of moments and shear force around pinned 1, we get the following equation between the reaction force at pinned 1 and pinned 2:

$$\bar{V}_1=-\bar{V}_2;\ \bar{M}_1=-\bar{V}_2L-\bar{M}_2 \quad (22)$$

Expression (22) write in matrix form:

$$\begin{Bmatrix}\bar{V}_1\\ \bar{M}_1\end{Bmatrix}=\begin{bmatrix}-1 & 0\\ -L & -1\end{bmatrix}\begin{Bmatrix}\bar{V}_2\\ \bar{M}_2\end{Bmatrix} \quad (23)$$

Therefore, the matrix $[\Phi]$ as follows:

$$[\Phi]=\begin{bmatrix}-1 & 0\\ -L & -1\end{bmatrix} \quad (24)$$

Making the stiffness matrix Applying formula (5) stiffness matrix is defined as follows:

$$\left[\bar{k}_u'\right]=\begin{bmatrix}[d]^{-1} & [d]^{-1}[\Phi]^T\\ [\Phi][d]^{-1} & [\Phi][d]^{-1}[\Phi]^T\end{bmatrix} \quad (25)$$

with: $[d]^{-1}=\begin{bmatrix}\bar{k}_{55} & \bar{k}_{56}\\ \bar{k}_{65} & \bar{k}_{66}\end{bmatrix}$; $[d]^{-1}[\Phi]^T=\begin{bmatrix}\bar{k}_{52} & \bar{k}_{53}\\ \bar{k}_{62} & \bar{k}_{63}\end{bmatrix}$;

$[\Phi][d]^{-1}=\begin{bmatrix}\bar{k}_{25} & \bar{k}_{26}\\ \bar{k}_{35} & \bar{k}_{36}\end{bmatrix}$; $[\Phi][d]^{-1}[\Phi]^T=\begin{bmatrix}\bar{k}_{22} & \bar{k}_{23}\\ \bar{k}_{32} & \bar{k}_{33}\end{bmatrix}$;

We define the elements of the stiffness matrix $\bar{k}_{ij}$. Then we rewrite the stiffness matrix in the following order:

$$\left[\bar{k}_u\right]=\begin{bmatrix}\bar{k}_{22} & \bar{k}_{23} & \bar{k}_{25} & \bar{k}_{26}\\ & \bar{k}_{33} & \bar{k}_{35} & \bar{k}_{36}\\ & & \bar{k}_{55} & \bar{k}_{56}\\ dx & & & \bar{k}_{66}\end{bmatrix} \quad (26)$$

where:

$$\bar{k}_{22}=\frac{12EI_1}{L^3}\frac{L^3T_1}{12I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};\ \bar{k}_{23}=\frac{6EI_1}{L^2}\frac{L^2T_2}{6I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};$$

$$\bar{k}_{33}=\frac{4EI_1}{L}\frac{L(T_3+\beta T_6)}{4I_1\left[-T_2^2+T_1\times(T_3+\beta T_6)\right]}$$

$$\bar{k}_{25}=-\frac{12EI_1}{L^3}\frac{L^3T_1}{12I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};$$

$$\bar{k}_{26}=\frac{6EI_1}{L^2}\frac{L^2(LT_1-T_2)}{6I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]}\quad \bar{k}_{35}=-\frac{6EI_1}{L^2}\frac{L^2T_2}{6I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};$$

$$\bar{k}_{36}=\frac{2EI_1}{L}\frac{L(-LT_2+T_3+\beta T_6)}{2I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]}\quad \bar{k}_{55}=\frac{12EI_1}{L^3}\frac{L^3T_1}{12I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};$$

$$\bar{k}_{56}=-\frac{6EI_1}{L^2}\frac{L^2(-LT_1+T_2)}{6I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]}\quad \bar{k}_{66}=\frac{4EI_1}{L}\frac{L\left(L^2T_1-2LT_2+T_3+\beta T_6\right)}{4I_1\left[-T_2^2+T_1(T_3+\beta T_6)\right]};$$

(27)

The composition of the stiffness matrix for bending resistant bar elements are values $\bar{k}_{ij}$ (i,j = 2,3,5,6) obtained above.

The stiffness matrix of the tension-compression-bending element in the plane is the "total" of the tension-compression stiffness matrix with the bending stiffness matrix. "Total" means to sort in order: $\bar{K}_e=\text{"}\bar{K}_{kn}\text{"}+\text{"}\bar{K}_u\text{"}$

3) Load vector

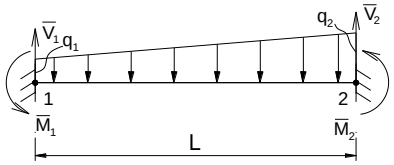

Fig. 4. Bending element bar, with rigid connection, trapezoidal load

Determine displacement and rotation angle at pinned 1 ($\bar{v}_1$; $\bar{\theta}_1$)

The moment balancing equation for pinned 2 and the shear force balancing equation of the bar element is:

$$\bar{M}_1+\bar{M}_2-\bar{V}_1L+(2q_1+q_2)L^2/6=0;\ \bar{V}_2+\bar{V}_1=0{,}5(q_1+q_2)L \quad (27a.b)$$

Moment and shear force at the z-position of the bar are determined from the moment balancing equation for the left part and the shear force balance equation, we get:

$$\bar{M}(z)=\bar{V}_1z-\bar{M}_1-q_1z^2/2+(q_1-q_2)z^3/(6L) \quad (27c)$$

$$\bar{V}(z)=\bar{V}_1-\left[q_1+q_1-(q_1-q_2)z/L\right]z/2 \quad (27d)$$

Replace (27c.d) into (9) we have:

$$\bar{C}=\int_0^L\frac{\left(\begin{matrix}\bar{V}_1z-\bar{M}_1-q_1z^2/2\\+(q_1-q_2)z^3/6L\end{matrix}\right)^2dz}{2EI_x(z)}+\int_0^L\frac{\gamma(z)\left(\begin{matrix}\bar{V}_1-q_1z\\+(q_1-q_2)z^2/2L\end{matrix}\right)^2dz}{2GA(z)} \quad (28)$$

Applying Castigliano's theorem, we have:

$$\bar{v}_1=\frac{\partial\bar{C}}{\partial\bar{V}_1}=\int_0^L\frac{\left(\begin{matrix}\bar{V}_1z-\bar{M}_1-\dfrac{q_1z^2}{2}\\+\dfrac{(q_1-q_2)z^3}{6L}\end{matrix}\right)zdz}{EI_x(z)}+\int_0^L\frac{\gamma(z)\left[\begin{matrix}\bar{V}_1-q_1z\\+(q_1-q_2)\dfrac{z^2}{2L}\end{matrix}\right]dz}{GA(z)} \quad 29a)$$

$$\bar{\theta}_1=\frac{\partial\bar{C}}{\partial\bar{M}_1}=\int_0^L\frac{\left(-\bar{V}_1z+\bar{M}_1+\dfrac{q_1z^2}{2}-\dfrac{(q_1-q_2)z^3}{6L}\right)dz}{EI_x(z)} \quad (29b)$$

Compact expression; change basic integrals and apply boundary conditions at pinned 1: linear displacement perpendicular to the bar axis and rotation at the 1st pinned equal to 0 ($\bar{v}_1=0$, $\bar{\theta}_1=0$) obtain the equation as follows:

$$M_1(-T_2)+V_1(T_3+\beta T_6)-\frac{q_1}{2}T_4-q_1\beta T_7+(T_5+3\beta T_8)(q_1-q_2)/(6L)=0;$$

$$M_1T_1-V_1T_2+q_1T_3/2-T_4(q_1-q_2)/(6L)=0; \quad (30a.b)$$

Solve the system of equations (30a.b) with unknown values $\bar{M}_1$ and $\bar{V}_1$ that we have:

$$\bar{V}_1=\frac{\left\{3Lq_1\left[T_2T_3-T_1(T_4+2\beta T_7)\right]-(q_1-q_2)\left[T_2T_4-T_1(T_5+3\beta T_8)\right]\right\}}{6L\left[T_2^2-T_1(T_3+\beta T_6)\right]}$$

$$\bar{M}_1=\frac{\left\{3Lq_1\begin{bmatrix}T_3^2+\beta T_3T_6\\-T_2(T_4+2\beta T_7)\end{bmatrix}-(q_1-q_2)\begin{bmatrix}T_3T_4+\beta T_4T_6\\-T_2\begin{pmatrix}T_5+\\3\beta T_8\end{pmatrix}\end{bmatrix}\right\}}{6L\left[T_2^2-T_1(T_3+\beta T_6)\right]} \quad (31a.b)$$

Replace (31a.b) into (27a.b) that we obtain $\bar{M}_2$ and $\bar{V}_2$:

$$\bar{V}_2=\frac{\left\{\begin{matrix}3L^2(q_1+q_2)\left[T_2^2-T_1(T_3+\beta T_6)\right]+3Lq_1\left[-T_2T_3+T_1(T_4+2\beta T_7)\right]\\+(q_1-q_2)\left[T_2T_4-T_1(T_5+3\beta T_8)\right]\end{matrix}\right\}}{6L\left[T_2^2-T_1(T_3+\beta T_6)\right]}$$

$$\bar{M}_2=\frac{\left|\begin{matrix}-L^3(2q_1+q_2)\left[T_2^2-T_1\begin{pmatrix}T_3\\+\beta T_6\end{pmatrix}\right]+3L^2q_1\left[T_2T_3-T_1\begin{pmatrix}T_4\\+2\beta T_7\end{pmatrix}\right]\\+(q_1-q_2)\left[T_3T_4+\beta T_4T_6-T_2(T_5+3\beta T_8)\right]\\+L\left\{\begin{matrix}q_2\left[T_2T_4-T_1(T_5+3\beta T_8)\right]\\+q_1\left[-3T_3^2-3\beta T_3T_6+2T_2(T_4+3\beta T_7)+T_1(T_5+3\beta T_8)\right]\end{matrix}\right\}\end{matrix}\right|}{6L\left[T_2^2-T_1(T_3+\beta T_6)\right]}$$

(32a.b)

From the above result, the load vector of the bending bar is obtained with uniformly distributed load:

$$\{\bar{F}\}=\{\bar{V}_1\ \ \bar{M}_1\ \ \bar{V}_2\ \ \bar{M}_2\}^T \quad (33)$$

C. Making the stiffness matrix and load vector for I-shaped taper bar element with semi-rigid connection at both ends

1) Making stiffness matrix of bending bar

The potential energy part of the pinned takes into account the stiffness:

$$C^*=0{,}5k_1\left(\bar{M}_1^*/k_1\right)^2+0{,}5k_1\left(\bar{M}_2^*/k_2\right)^2 \quad (34)$$

Where: k_1 ; k_2 ; $\bar{M}_1^*$; $\bar{M}_2^*$ respectively the stiffness and bending moment of pinned 1 and 2.

The total elastic energy potential of the taper section element is semi-rigid connection:

$$\bar{C}^* = \bar{C} + C^* = \int_0^L \frac{\bar{M}^*(z)^2 dz}{2EI(z)} + \int_0^L \frac{\gamma(z)\bar{V}^*(z)^2 dz}{2GA(z)} + 0{,}5\begin{bmatrix} k_1\left(\bar{M}_1^*/k_1\right)^2 \\ +k_2\left(\bar{M}_2^*/k_2\right)^2 \end{bmatrix} \quad (35)$$

According to the Fig. 3b, moment and shear force at the z position of the bar determined from the moment balance equation for the left part and the shear force balance equation:

$$\bar{M}^*(z) = \bar{V}_2^*(L-z) + \bar{M}_2^*;\ \bar{V}^*(z) = -\bar{V}_2^*; \quad (36a.b)$$

Replace (36a.b) into (35) that we have:

$$\bar{C}^* = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^*\right]^2 dz}{2EI(z)} + \int_0^L \frac{\gamma(z)\bar{V}_2^{*2} dz}{2GA(z)} + \bar{M}_1^{*2}/(2k_1) + \bar{M}_2^{*2}/(2k_2) \quad (37)$$

Balance equation for moment and shear force follows Fig. 3c is:

$$\bar{M}_1^* + \bar{M}_2^* + \bar{V}_2^* L = 0;\quad \bar{V}_1^* + \bar{V}_2^* = 0 \quad (38a.b)$$

Replace (38a.b) into (37) that we have:

$$\bar{C}^* = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^*\right]^2 dz}{2EI(z)} + \int_0^L \frac{\gamma(z)\bar{V}_2^{*2} dz}{2GA(z)} + \left(\bar{M}_2^* + \bar{V}_2^* L\right)^2/(2k_1) + \bar{M}_2^{*2}/(2k_2)$$

Applying Castigliano's theorem, we have:

$$\bar{v}_2^* = \frac{\partial \bar{C}^*}{\partial \bar{V}_2^*} = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^*\right](L-z)dz}{EI(z)} + \int_0^L \frac{\gamma(z)\bar{V}_2^* dz}{GA(z)} + \frac{\left(\bar{M}_2^* + \bar{V}_2^* L\right)}{k_1} L$$

$$\bar{\theta}_2^* = \frac{\partial \bar{C}}{\partial \bar{M}_2^*} = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^*\right]dz}{EI(z)} + \frac{\left(\bar{M}_2^* + \bar{V}_2^* L\right)}{k_1} + \frac{\bar{M}_2^*}{k_2} \quad (39a.b)$$

Set: $\alpha_1 = k_1/E;\ \alpha_2 = k_2/E;\ \beta = E/G$ we have:

$$\bar{M}_2^*\left(\int_0^L \frac{(L-z)dz}{EI(z)} + \frac{L}{E\alpha_1}\right) + \bar{V}_2^*\left(\int_0^L \frac{(L-z)^2 dz}{EI(z)} + \int_0^L \frac{\beta\gamma(z)dz}{EA(z)} + \frac{L^2}{E\alpha_1}\right) = \bar{v}_2^*$$

$$\bar{M}_2^*\left(\int_0^L \frac{dz}{EI(z)} + \frac{1}{E\alpha_1} + \frac{1}{E\alpha_2}\right) + \bar{V}_2^*\left(\int_0^L \frac{(L-z)dz}{EI(z)} + \frac{L}{E\alpha_1}\right) = \bar{\theta}_2^* \quad (40a.b)$$

Expression (40a.b) written in matrix form:

$$\begin{Bmatrix} \bar{v}_2^* \\ \bar{\theta}_2^* \end{Bmatrix} = \begin{bmatrix} \bar{a}_{11}^* & \bar{a}_{12}^* \\ \bar{a}_{21}^* & \bar{a}_{22}^* \end{bmatrix} \begin{Bmatrix} \bar{V}_2^* \\ \bar{M}_2^* \end{Bmatrix}$$

where :

$$\bar{a}_{11}^* = \int_0^L \frac{(L-z)^2 dz}{EI(z)} + \int_0^L \frac{\beta\gamma(z)dz}{EA(z)} + \frac{L^2}{E\alpha_1} \quad (41a)$$

$$\bar{a}_{12}^* = \bar{a}_{21}^* = \int_0^L \frac{(L-z)dz}{EI(z)} + \frac{L}{E\alpha_1}; \quad (41b)$$

$$\bar{a}_{22}^* = \int_0^L \frac{dz}{EI(z)} + \frac{1}{E\alpha_1} + \frac{1}{E\alpha_2} \quad (41c)$$

Replace the basic integral values from T_1 to T_8 into (41a.b.c) that we have:

$$\bar{a}_{11}^* = \left(L^2T_1 - 2LT_2 + T_3 + \beta T_6 + L^2/\alpha_1\right)/E;\ \bar{a}_{12}^* = \bar{a}_{21}^* = \left(LT_1 - T_2 + L/\alpha_1\right)/E$$

$$\bar{a}_{22}^* = \left(T_1 + 1/\alpha_1 + 1/\alpha_2\right)/E$$

Stiffness matrix: $$\left[\bar{K}_U^*\right] = \begin{bmatrix} \bar{k}_{22}^* & \bar{k}_{23}^* & \bar{k}_{25}^* & \bar{k}_{26}^* \\ & \bar{k}_{33}^* & \bar{k}_{35}^* & \bar{k}_{36}^* \\ & & \bar{k}_{55}^* & \bar{k}_{56}^* \\ dx & & & \bar{k}_{66}^* \end{bmatrix} \quad (42)$$

$$MS = L^2(1+\alpha_1T_1) - 2\alpha_1LT_2 - \alpha_1\alpha_2T_2^2 + (\alpha_1+\alpha_2)T_3 + \alpha_1\alpha_2T_1T_3 + \beta\begin{pmatrix}\alpha_1+\alpha_2 \\ +\alpha_1\alpha_2T_1\end{pmatrix}T_6$$

$$\bar{k}_{22}^* = \frac{E(\alpha_1+\alpha_2+\alpha_1\alpha_2T_1)}{MS};\ \bar{k}_{23}^* = \bar{k}_{32}^* = \frac{\alpha_1E(L+\alpha_2T_2)}{MS}$$

$$\bar{k}_{25}^* = \bar{k}_{52}^* = -\frac{E(\alpha_1+\alpha_2+\alpha_1\alpha_2T_1)}{MS};\ \bar{k}_{26}^* = \bar{k}_{62}^* = \frac{E\alpha_2(L+\alpha_1LT_1-\alpha_1T_2)}{MS}$$

$$\bar{k}_{33}^* = \frac{E\alpha_1\left[L^2+\alpha_2(T_3+\beta T_6)\right]}{MS};\ \bar{k}_{35}^* = \bar{k}_{53}^* = -\frac{\alpha_1E(L+\alpha_2T_2)}{MS}$$

$$\bar{k}_{36}^* = \bar{k}_{63}^* = \frac{\alpha_1\alpha_2E(LT_2-T_3-\beta T_6)}{MS};\ \bar{k}_{55}^* = \frac{E(\alpha_1+\alpha_2+\alpha_1\alpha_2T_1)}{MS}$$

$$\bar{k}_{56}^* = \bar{k}_{65}^* = -\frac{E\alpha_2(L+\alpha_1LT_1-\alpha_1T_2)}{MS}\quad \bar{k}_{66}^* = \frac{E\alpha_2\begin{pmatrix}L^2(1+\alpha_1T_1)-2\alpha_1LT_2 \\ +\alpha_1(T_3+\beta T_6)\end{pmatrix}}{MS};$$

2) *Making the load vector for taper element, semi-rigid connection*

a) *Trapezoidal load distribution*

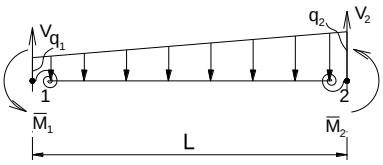

Fig. 5. The bar element has a semi-rigid connection, the trapezoidal load

Determine displacement and rotation angle at pinned 2 ($\bar{v}_2^*;\bar{\theta}_2^*$)

Moment balance equation with pinned 1 and shear force balance equation:

$$\bar{M}_1^* + \bar{M}_2^* + \bar{V}_2^*L - (q_1+2q_2)L^2/6 = 0;\ \bar{V}_2^* + \bar{V}_1^* = 0{,}5(q_1+q_2)L \quad (43)$$

Moment and shear force at the z position of the bar are determined from the moment balance equation for the left part and the shear force balance equation, we get:

$$\bar{M}^*(z) = \bar{V}_2^*(L-z) + \bar{M}_2^* - q_2(2L+z)(L-z)^2/6L - q_1(L-z)^3/6L; \quad (44.a)$$

$$\bar{V}^*(z) = \left[q_1 - (q_1-q_2)z/L + q_2\right](L-z)/2 - \bar{V}_2^*; \quad (44.b)$$

Replace (44a.b) into (35) we get:

$$\bar{C}^* = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^* - q_2(2L+z)(L-z)^2/6L - q_1(L-z)^3/6L\right]^2 dz}{2EI(z)}$$
$$+\int_0^L \frac{\gamma(z)\left[q_2\left(L^2-z^2\right)/2L + q_1(L-z)^2/2L - V_2\right]^2 dz}{2GA(z)}$$
$$+\left(L^2(q_1+2q_2)/6 - \bar{M}_2^* - \bar{V}_2^*L\right)^2/2k_1 + \bar{M}_2^{*2}/2k_2$$

Applying Castigliano's theorem, we have the formula (45a.b) as follows:

$$\bar{v}_2^* = \frac{\partial \bar{C}^*}{\partial \bar{V}_2^*} = \int_0^L \frac{\begin{bmatrix}\bar{V}_2^*(L-z) + \bar{M}_2^* \\ -q_2(2L+z)(L-z)^2/6L - q_1(L-z)^3/6L\end{bmatrix}(L-z)dz}{EI(z)}$$
$$+\int_0^L \frac{\gamma(z)\begin{bmatrix}\bar{V}_2^* - q_2\left(L^2-z^2\right)/2L \\ -q_1(L-z)^2/2L\end{bmatrix}dz}{GA(z)} + \frac{\left[-L^2(q_1+2q_2)/6 + \bar{M}_2^* + \bar{V}_2^*L\right]L}{k_1}$$

$$\bar{\theta}_2^* = \frac{\partial \bar{C}^*}{\partial M_2} = \int_0^L \frac{\left[\bar{V}_2^*(L-z) + \bar{M}_2^* - \frac{q_2}{6L}(2L+z)(L-z)^2 - \frac{q_1}{6L}(L-z)^3\right]dz}{EI(z)}$$
$$+\frac{\left[-\frac{L^2}{6}(q_1+2q_2) + \bar{M}_2^* + \bar{V}_2^*L\right]}{k_1} + \frac{\bar{M}_2^*}{k_2}$$

Determine displacement and rotation angle at pinned 1 ($\bar{v}_1^*;\bar{\theta}_1^*$)

The moment balance equation for node 2 and the shear force balance equation of the bending bar element is as above Fig. 5 as: $\bar{M}_1^* + \bar{M}_2^* - \bar{V}_1^* L + (2q_1 + q_2)L^2/6 = 0$; $\bar{V}_2^* + \bar{V}_1^* = 0{,}5(q_1 + q_2)L$ (46)

$$\bar{M}^*(z) = \bar{V}_1^* z - \bar{M}_1^* - q_1 z^2/2 + (q_1 - q_2)z^3/(6L) \quad (47a)$$

$$\bar{V}^*(z) = \bar{V}_1^* - \left(q_1 + q_1 - (q_1 - q_2)z/L\right)z/2 \quad (47b)$$

Replace (47a.b) into (37) we get:

$$\bar{C}^* = \int_0^L \frac{\left(\bar{V}_1^* z - \bar{M}_1^* - \frac{q_1 z^2}{2} + \frac{(q_1 - q_2)z^3}{6L}\right)^2 dz}{2EI(z)} + \int_0^L \frac{\gamma(z)\left[\bar{V}_1^* - q_1 z + (q_1 - q_2)\frac{z^2}{2L}\right]^2 dz}{2GA(z)} + \frac{\bar{M}_1^{*2}}{2k_1} + \frac{\left(-\bar{M}_1^* + \bar{V}_1^* L - (2q_1 + q_2)L^2/6\right)^2}{2k_2}; \quad (48)$$

Applying Castigliano's theorem, we have:

$$\bar{v}_1^* = \frac{\partial \bar{C}^*}{\partial \bar{V}_1^*} = \int_0^L \frac{\left(\bar{V}_1^* z - \bar{M}_1^* - q_1 z^2/2 + (q_1 - q_2)z^3/6L\right)z\,dz}{EI(z)} + \int_0^L \frac{\gamma(z)\left[\bar{V}_1^* - q_1 z + (q_1 - q_2)z^2/2L\right]dz}{GA(z)} + \frac{-\bar{M}_1^* + \bar{V}_1^* L - (2q_1 + q_2)L^2/6}{k_2}L;$$

$$\bar{\theta}_1^* = \frac{\partial \bar{C}^*}{\partial M_1} = \int_0^L \frac{\left(-\bar{V}_1^* z + \bar{M}_1^* + q_1 z^2/2 - (q_1 - q_2)z^3/6L\right)dz}{EI(z)} + \frac{\bar{M}_1^*}{k_1} + \frac{\left(\bar{M}_1^* - \bar{V}_1^* L + (2q_1 + q_2)L^2/6\right)}{k_2};$$

Shorten the above expression and replace the basic integrals and use boundary conditions: straight-forward displacement perpendicular to the bar axis at pinned 1 and pinned 2 equal to 0 ($\bar{v}_1^* = 0$, $\bar{v}_2^* = 0$) we get:

$$\bar{M}_1^*\left(-L/\alpha_2 - T_2\right) + \bar{V}_1^*\left(L^2/\alpha_2 + T_3 + \beta T_6\right) - (2q_1 + q_2)L^3/(6\alpha_2) - q_1 T_4/2 - q_1\beta T_7 + \left(T_5 + 3\beta T_8\right)(q_1 - q_2)/(6L) = 0;$$

$$\bar{M}_1^*\left(-L/\alpha_1 - LT_1 + T_2\right) + \bar{V}_1^*\left(LT_2 - T_3 - \beta T_6\right) - q_1\left(LT_3 - T_4\right)/2 + \left(LT_4 - T_5 - 3\beta T_8\right)(q_1 - q_2)/(6L) + q_1\beta T_7 = 0; \quad (50a.b)$$

Solve the system of magnetic equations (50a.b) with the unknown factors and, apply to the load distributed problem, we get:

$$\bar{V}_1^* = \frac{\left\{\begin{array}{l} L^4(2q_1 + q_2)(1 + \alpha_1 T_1) \\ -\alpha_1 L^3(2q_1 + q_2)T_2 - 3\alpha_1 L^2 q_1 T_3 \\ +L\begin{bmatrix} -3\alpha_1\alpha_2 q_1 T_2 T_3 + 4\alpha_1 q_1 T_4 + 3\alpha_2 q_1 T_4 \\ -\alpha_1 q_2 T_4 + 3\alpha_1\alpha_2 q_1 T_1 T_4 + 6\beta q_1(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_7 \end{bmatrix} \\ +(q_1 - q_2)\begin{bmatrix} \alpha_1\alpha_2 T_2 T_4 - \alpha_1 T_5 - \alpha_2 T_5 \\ -\alpha_1\alpha_2 T_1 T_5 - 3\beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_8 \end{bmatrix} \end{array}\right\}}{6L\begin{bmatrix} L^2(1 + \alpha_1 T_1) - 2\alpha_1 LT_2 - \alpha_1\alpha_2 T_2^2 \\ +\alpha_1 T_3 + \alpha_2 T_3 + \alpha_1\alpha_2 T_1 T_3 + \beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_6 \end{bmatrix}} \quad (51a)$$

$$\bar{M}_1^* = \frac{\alpha_1\left\{\begin{array}{l} L^4(2q_1 + q_2)T_2 - L^3\begin{bmatrix} (5q_1 + q_2)T_3 \\ +\beta(2q_1 + q_2)T_6 \end{bmatrix} \\ +L^2(4q_1 T_4 - q_2 T_4 + 6\beta q_1 T_7) \\ +L\begin{bmatrix} -3\alpha_2 q_1\left[T_3^2 + \beta T_3 T_6 - T_2(T_4 + 2\beta T_7)\right] \\ -(q_1 - q_2)(T_5 + 3\beta T_8) \end{bmatrix} \\ +\alpha_2(q_1 - q_2)\left[T_3 T_4 + \beta T_4 T_6 - T_2(T_5 + 3\beta T_8)\right] \end{array}\right\}}{6L\begin{bmatrix} L^2(1 + \alpha_1 T_1) - 2\alpha_1 LT_2 - \alpha_1\alpha_2 T_2^2 \\ +\alpha_1 T_3 + \alpha_2 T_3 + \alpha_1\alpha_2 T_1 T_3 + \beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_6 \end{bmatrix}} \quad (51b)$$

Repalce (51a.b) into the balance equation, we have $\bar{M}_2^*$ and $\bar{V}_2^*$:

$$\bar{V}_2^* = \frac{\left\{\begin{array}{l} L^4(q_1 + 2q_2)(1 + \alpha_1 T_1) - \alpha_1 L^3(4q_1 + 5q_2)T_2 \\ +3L^2\left|\begin{array}{l} \alpha_2(q_1 + q_2)(T_3 + \beta T_6) \\ +\alpha_1\left\{\begin{array}{l} (2q_1 + q_2)T_3 + \beta(q_1 + q_2)T_6 \\ -\alpha_2(q_1 + q_2)\begin{bmatrix} T_2^2 \\ -T_1(T_3 + \beta T_6) \end{bmatrix} \end{array}\right\} \end{array}\right| \\ +L\begin{bmatrix} 3\alpha_1\alpha_2 q_1 T_2 T_3 - 4\alpha_1 q_1 T_4 - 3\alpha_2 q_1 T_4 \\ +\alpha_1 q_2 T_4 - 3\alpha_1\alpha_2 q_1 T_1 T_4 - 6\beta q_1(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_7 \end{bmatrix} \\ +(q_1 - q_2)\begin{bmatrix} -\alpha_1\alpha_2 T_2 T_4 + \alpha_1 T_5 \\ +\alpha_2 T_5 + \alpha_1\alpha_2 T_1 T_5 + 3\beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_8 \end{bmatrix} \end{array}\right\}}{6L\begin{bmatrix} L^2(1 + \alpha_1 T_1) - 2\alpha_1 LT_2 - \alpha_1\alpha_2 T_2^2 \\ +\alpha_1 T_3 + \alpha_2 T_3 + \alpha_1\alpha_2 T_1 T_3 + \beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_6 \end{bmatrix}} \quad (52a)$$

$$\bar{M}_2^* = \frac{-\alpha_2 L\left|\begin{array}{l} \begin{bmatrix} L^2(2q_1 + q_2)(T_3 + \beta T_6) \\ -3Lq_1(T_4 + 2\beta T_7) \\ +(q_1 - q_2)(T_5 + 3\beta T_8) \end{bmatrix} + \alpha_1\left\{\begin{array}{l} -L^3(2q_1 + q_2)\left[T_2^2 - T_1(T_3 + \beta T_6)\right] \\ +3L^2 q_1\left[T_2 T_3 - T_1(T_4 + 2\beta T_7)\right] \\ +(q_1 - q_2)\begin{bmatrix} T_3 T_4 + \beta T_4 T_6 \\ -T_2(T_5 + 3\beta T_8) \end{bmatrix} \\ +L\left\{\begin{array}{l} q_2\left[T_2 T_4 - T_1(T_5 + 3\beta T_8)\right] \\ +q_1\begin{bmatrix} -3T_3^2 - 3\beta T_3 T_6 \\ +2T_2(T_4 + 3\beta T_7) \\ +T_1(T_5 + 3\beta T_8) \end{bmatrix} \end{array}\right\} \end{array}\right\} \end{array}\right|}{6L\begin{bmatrix} L^2(1 + \alpha_1 T_1) - 2\alpha_1 LT_2 - \alpha_1\alpha_2 T_2^2 \\ +\alpha_1 T_3 + \alpha_2 T_3 + \alpha_1\alpha_2 T_1 T_3 + \beta(\alpha_1 + \alpha_2 + \alpha_1\alpha_2 T_1)T_6 \end{bmatrix}} \quad (52b)$$

Thus, we obtain the load vector for the taper element, section I, two semi-rigid connection ends subject to the trapezoidal distribution load as follows:

$$\left\{\bar{F}^*\right\} = \left\{\bar{V}_1^* \quad \bar{M}_1^* \quad \bar{V}_2^* \quad \bar{M}_2^*\right\}^T \quad (53)$$

III. PROGRAMMING CALCULATIONS AND NUMERICAL EXAMPLES

A. *Programming base*

On the built-in stiffness matrix and load vector matrix, the author set up a computer program based on 2014 MathLab specialized software named SAC (Static Analysis with Rigid connections) for analyzing the flat bar with I-section with height varies with length.

B. *Example 1: Determination of internal force - displacement of taper cross-section steel beams with uniformly distributed load, two rigid connection ends*

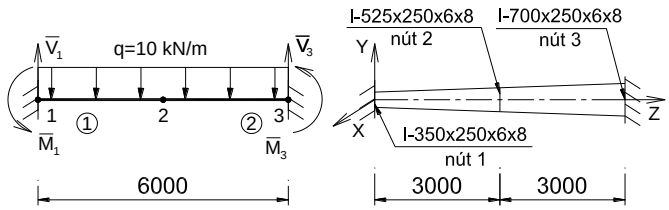

Fig. 6. Beam I-(350÷700)×250×6×8 bearing the distributed load q = 10kN/m

Known: I-beam cross section: I – (350÷700)×250×6×8 lenght L = 6 m, q = 10 kN/m, elastic modulus E = 21×10^4 MPa. Requirements: Determine the bending moment and shear force at nodes 1,2,3 and compare the calculation results of SAC program and SAP 2000 software (SAP),

in case of consideration and without considering the image effect of shear force.

1) The results are analyzed according to SAC and SAP and the difference is based on 2 software recorded (Tab. 7 - beam elements considering the impact of shear force, rigid connection); (Tab. 8 - beam elements without consideration of shear force, rigid connection)

Tab. 7. Results of internal forces, beam displacement

Elements	Internal force and díplacement	Pinned	Results		Difference (%)
			SAC	SAP	
1	$\bar{M}$ (kNm)	1	-21,37	-21,41	0,17
		2	14,29	14,28	0,07
	$\bar{V}$ (kN)	1	-26,89	-26,90	0,04
		2	3,11	3,10	0,32
2	$\bar{M}$ (kNm)	2	14,29	14,28	0,07
		3	-40,05	-40,03	0,05
	$\bar{V}$ (kN)	2	3,11	3,10	0,32
		3	33,11	33,10	0,03
1	C.vị(mm)	2	-0,721	-0,717	0,35
2	C.vị(mm)	2	-0,721	-0,717	0,35

Tab. 8. Results of internal forces, beam displacement

Elements	Internal force and díplacement	Pinned	Results		Difference (%)
			SAC	SAP	
1	$\bar{M}$ (kNm)	1	-21,47	-21,48	0,05
		2	14,32	14,31	0,07
	$\bar{V}$ (kN)	1	-26,93	-26,93	0,00
		2	3,07	3,07	0,00
2	$\bar{M}$ (kNm)	2	14,32	14,31	0,07
		3	-39,89	-39,91	0,03
	$\bar{V}$ (kN)	2	3,07	3,07	0,00
		3	33,07	33,07	0,00
1	C.vị(mm)	2	-0,526	-0,524	0,38
2	C.vị(mm)	2	-0,526	-0,524	0,38

C. The results are analyzed according to SAC and SAP and the difference according to the 2 software is recorded in (Tab. 9 - beam elements considering the impact of shear force,rigid connection); (Tab. 10 - beam elements without consideration of shear force, rigid connection)

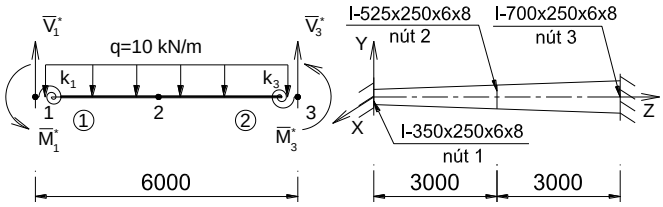

Fig. 7. Beam I-(350÷700)×250×6×8 bearing the distributed load q = 10kN/m

Tab. 9. Results of internal forces, beam displacement

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			SAC	SAP	
1	$\bar{M}^*$ (kNm)	1	-21,46	-21,50	0,19
		2	17,83	17,81	0,11
	$\bar{V}^*$ (kN)	1	-28,10	-28,11	0,03
		2	1,90	1,89	0,52
2	$\bar{M}^*$ (kNm)	2	17,83	17,81	0,11
		3	-32,87	-32,87	0,00
	$\bar{V}^*$ (kN)	2	1,90	1,89	0,52
		3	31,90	31,89	0,03
1	Displacement (mm)	2	-0,946	-0,942	0,00
2	Displacement (mm)	2	-0,946	-0,942	0,00

Tab. 10. Results of internal forces, beam displacement

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			SAC	SAP	
1	$\bar{M}^*$ (kNm)	1	-21,73	-21,75	0,09
		2	17,92	17,89	0,17
	$\bar{V}^*$ (kN)	1	-28,21	-28,21	0,00
		2	1,79	1,79	0,00
2	$\bar{M}^*$ (kNm)	2	17,92	17,89	0,17
		3	-32,44	-32,48	0,12
	$\bar{V}^*$ (kN)	2	1,79	1,79	0,00
		3	31,79	31,79	0,00
1	Displacement (mm)	2	-0,750	-0,752	0,31
2	Displacement (mm)	2	-0,750	-0,752	0,31

D. Comparing the calculation results of the case of beam elements consider (PA1) and not consider (PA2) the rigid connection

When consider the shear force effect

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			PA1	PA2	
1	$\bar{M}^*$ (kNm)	1	-21,46	-21,37	0,42
		2	17,83	14,29	24,77
	$\bar{V}^*$ (kN)	1	-28,10	-26,89	4,5
		2	1,90	3,11	38,9
2	$\bar{M}^*$ (kNm)	2	17,83	14,29	24,77
		3	-32,87	-40,05	17,93
	$\bar{V}^*$ (kN)	2	1,90	3,11	38,9
		3	31,90	33,11	3,65
1	Displacement (mm)	2	-0,946	-0,721	31,2
2	Displacement (mm)	2	-0,946	-0,721	31,2

When not consider the shear force effect

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			PA1	PA2	
1	$\bar{M}^*$ (kNm)	1	-21,73	-21,47	1,21
		2	17,92	14,32	25,14

	$\bar{V}^*$ (kN)	1	-28,21	-26,93	4,75
		2	1,79	3,07	41,69
2	$\bar{M}^*$ (kNm)	2	17,92	14,32	25,14
		3	-32,44	-39,89	18,68
	$\bar{V}^*$ (kN)	2	1,79	3,07	41,69
		3	31,79	33,07	3,87
1	Displacement (mm)	2	-0,750	-0,526	42,5
2	Displacement (mm)	2	-0,750	-0,526	42,5

E. Comparing the results of calculating the case of beam elements consider and not consider the effect of shear forces

When not consider semi-rigid connection.

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			PA1	PA2	
1	$\bar{M}^*$ (kNm)	1	-21,37	-21,47	0,46
		2	14,29	14,32	0,2
	$\bar{V}^*$ (kN)	1	-26,89	-26,93	0,14
		2	3,11	3,07	1,0
2	$\bar{M}^*$ (kNm)	2	14,29	14,32	0,2
		3	-40,05	-39,89	0,4
	$\bar{V}^*$ (kN)	2	3,11	3,07	1,0
		3	33,11	33,07	0,12
1	Displacement (mm)	2	-0,721	-0,526	37,07
2	Displacement (mm)	2	-0,721	-0,526	37,07

When conside semi-rigid connection.

Elements	Internal force and displacement	Pinned	Results		Difference (%)
			PA1	PA2	
1	$\bar{M}^*$ (kNm)	1	-21,46	-21,73	1,24
		2	17,83	17,92	0,5
	$\bar{V}^*$ (kN)	1	-28,10	-28,21	0,4
		2	1,90	1,79	6,14
2	$\bar{M}^*$ (kNm)	2	17,83	17,92	0,5
		3	-32,87	-32,44	1,33
	$\bar{V}^*$ (kN)	2	1,90	1,79	6,14
		3	31,90	31,79	0,34
1	Displacement (mm)	2	-0,946	-0,750	26,1
2	Displacement (mm)	2	-0,946	-0,750	26,1

F. Comments on calculation results

- When dividing the bar into 2 elements, the calculation results for internal force and displacement at the pinned according to the programs SAC and SAP2000 have the largest value difference of 0.52% and the smallest is 0.1%. Thereby, it shows that the stiffness matrix and load vector have been set to be accurate, and the SAC calculation program can be trusted..

When considering the effect of shear force, the internal force results are not much different, but the displacement result at pinned 2 has a difference of 43.5%.

When considering the stiffness of the connection with the assumed stiffness at the two ends of the connection is 2×10^5 kNm/rad, The result of the largest difference in internal force at pinned 2 is 41,7%; The difference in displacement results at pinned 2 is 42,5%..

IV. CONCLUSION

Through the content mentioned above, the author draws some conclusions and recommendations as follows:

Using the mathematical model is the general exponential function that represents the characteristic of the taper element section to achieve the permitted accuracy, and used in calculating the geometry characteristics for that section..

The result of stiffness matrices and load vectors establishes reliability, for structural analysis. Matrixes and load vectors give more accurate results than the results of SAP 2000 software without having to split multiple elements.

With the approach of taper element as above, we can use to analyze and solve all the problems for the taper section such as the problem of oscillation and stability..etc

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