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Accretion of A New Bitopology

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Abstract: ***In this paper the authors have introduced a new function on a bitopological space which provides us with a tool to develop a new bitopology. Various characteristics of the derived bitopological spacehav been studied. two new separation axioms have also been introduced over the bitopological spaces. It is interesting to see that the derived bitopology accepts these new separation axioms in a very natural manner.***

Index Terms: ***R_1 closure, R_2 closure, Pairwise Ω closure, pairwise T_Ω separation axiom, pairwise T_{Ω^*} separation axiom,***

I. INTRODUCTION

The phenomenon of bitopological space, since its lineage from quasi metric spaces, has opened an outspread area at its broadside, which was original and fundamental enough to captivate interest of many contemporary topologists and many of them such as Reilly [7], Singhal and Singhal [10], and Fletcher et. al. [12] roped in to introduce a variety of covering properties and separation axioms on bitopological spaces. An interesting aspect of separation axioms was given by Kelley [8], when he introduced the concepts of pairwise Hausdorff, pairwise regular and pairwise normal space. Standard literatures on topology show importance of separation axioms and their uses [2] and these fundamental axioms paved way for further new axioms [11]. Along with these concepts, the basic notion of Kuratowski operator [4] already existed as the introduction of the notion of local function on A X as A*. Recently Hawary introduced ∈ closed sets [15] and ζ open sets[14]. Due to presence of two topologies in a bitopological space (X, $\Im_1$, $\Im_2$), it is always possible to consider the closure of a ($\Im_j$) open set with respect to the topology $\Im_i$, where i, j ∈ {1, 2}, j ≠ i. This opened way for various closed sets such as g_{ij} closed sets [1], generalized locally $\Im$*g closed sets [9], Ω-open sets [3] and weakly b-open fuctions [5]. Many topologies could be created with the help of these concepts [13]. This prompted us to introduce a new type of closure property on a bitopological space which proves to be a trailblazer and gives rise to a new bitopological space. We name this derived bitopological space as pairwise Ω bitopological space and study its behaviour in the presence of existing sepa-ration axioms and also introduce some new sepration axioms

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which behave in a peculiar manner with the parent bitopological space whereas they show a natural inherence in derived bitopological space.

For the sake of self completeness, in section 2, we introduce the necessary definitions. In section 3, we introduce the new closure property and show that it derives a new bitopological space. In section 4, we introduce two new separation axioms and study their behaviour in both the bitopologies.

II. PRELIMINARIES

Let we denote a bitopological space as (X, $\Im_1$, $\Im_2$) where X is a nonempty set equipped with two arbitrary topologies $\Im_1$ and $\Im_2$. The closure and interior of a subset of X are being considered in their general sense. To make the article self contained, we recall the following well known definitions:

Definition 2.1: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_0 if for every pair of distinct elements of X there exists an open set in any of the topologies containing only one of the points.

Definition 2.2: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise T_1 if for every pair of distinct elements of X say x and y there exists an open set U in $\Im_i$ containing x but not y and an open set say V in $\Im_j$ containing y but not x, for i, j ∈ {1,2} i ≠ j.

It is obvious that if a bitopological space is T_1 then every singleton is closed both in $\Im_1$ and $\Im_2$.

Definition 2.3: A subset A of X in a bitopological space (X, $\Im_1$, $\Im_2$) is called pairwise - open if A is both $\Im_1$-open and $\Im_2$-open.

Definition 2.4: A bitopological space (X, $\Im 1$, $\Im 2$) is pairwise $T_{1/2}$ if for every pair of distinct elements of X say x and y there exists an open or closed set say U in $\Im_i$ containing x but not y and an open or closed set say V in $\Im_j$ containing y but not x, i ,j ∈ {1,2} i ≠ j.

Pairwise $\mathbf{T_1}$ ⇒ Pairwise $\mathbf{T_{1/2}}$ ⇒ Pairwise $\mathbf{T_0}$

Definition 2.5: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise Hausdorff if for every pair of distinct elements of X say x and y there exists an open set U in $\Im_i$ containing x and an open set say V in $\Im_j$ containing y such that U ∩ V = ∅, for i, j ∈ {1,2} i ≠ j.

Definition 2.6: A bitopological space (X, $\Im_1$, $\Im_2$) is pairwise regular if for every element of X say x and every open set U in $\Im_i$ containing x there exists an $\Im_i$ open neighbourhood of x say V_x whose $\Im_j$ closure is contained in U, for i, j ∈ {1,2} i ≠ j.

Definition 2.7: A cover $\mathcal{U}$ ={Uα | α ∈ A }is said to be pairwise open cover of X if $U_a \in \Im 1 \cup \Im 2$ and $\mathcal{U} \cap \Im_i$ contains a nonempty set i ∈ {1,2}.

Definition 2.8: A cover $\mathcal{U}$={Uα| α ∈ A }is said to be locally countable pairwise open cover of X if $\mathcal{U}$ is a pairwise open cover of X and each element of X is contained in countably many elements of $\mathcal{U}$.

Definition 2.9: A bitopological space X is said to be pairwise compact if every pairwise open cover of X has a finite subcover.

Definition 2.10: A bitopological space $(X, \Im_1, \Im_2)$ is pairwise antilocally countable if it is antilocally countable with respect to both the topologies.

III. PAIRWISE Ω BITOPOLOGICAL SPACES : DEFINITIONS AND PROPERTIES

Definition 3.1: Let $(X, \Im_1, \Im_2)$ be a bitopological space. Define a function $\Omega_{(i,j)} : P(X) \to P(X)$ as $\Omega_{(i,j)}(A) = A \cup \{y \in X \mid \forall U \in \Im_j \mid y \in U, \Im_i cl\text{-}\{U\} \cap A \neq \text{Countable}\}$. Then $\forall$ i, j∈ (1,2) i ≠j, the $\Omega_{(i,j)}$ operator holds following assertions:

(i) $A \subset \Omega_{(i,j)}(A)$,

(ii) $B \subset A \Rightarrow \Omega_{(i,j)}(B) \subset \Omega_{(i,j)}(A)$

(iii) $\Omega_{(i,j)}(A \cup B) = \Omega_{(i,j)}(A) \cup \Omega_{(i,j)}(B)$

(iv) $\Omega_{(i,j)}(A \cap B) \subset \Omega_{(i,j)}(A) \cap \Omega_{(i,j)}(B)$

Proof:

(i)By definition.

(ii) Let $x \notin \Omega_{(i,j)}(A)$ then there exists a $\Im_i$ open neighbourhood U_x of x such that $\Im_j cl\text{-}\{U_x\} \cap A =$ Countable . Then $\Im_j cl\text{-}\{U\} \cap B =$ Countable $\Rightarrow x \notin \Omega_{(i,j)}(B)$.

(iii) By (ii) $\Omega_{(i,j)}(A) \cup \Omega_{(i,j)}(B) \subset \Omega_{(i,j)}(A \cup B)$. Now let $x \notin \Omega_{(i,j)}(A) \cup \Omega_{(i,j)}(B)$. Then there exist $\Im_i$ open neighbourhoods U_x and V_x of x such that $\Im_j cl\text{-}\{U_x\} \cap A =$ Countable and $\Im_j cl\text{-}\{V_x\} \cap B =$ Countable. Then $W_x = U_x \cap V_x$ is $\Im_i$ open neighbourhood of x such that $\Im_j cl\text{-}\{V_x\} \cap (A \cup B) =$ Countable indicating that $x \notin \Omega_{(i,j)}(A \cup B)$.

(iv) Easy to prove.

Hence we can see that the $\Omega_{(i,j)}$ operator satisfies all the conditions of Kuratowski's closure operator and therefore we can define $\Omega_{(i,j)}$ operator as closure property. Thus by $\Omega_{(i,j)}$ operator, closed sets can be defined as :

Definition 3.2 : In a bitopological space $(X, \Im_i, \Im_j)$, $A \subset X$ is called $\Omega_{(i,j)}$ closed if $\Omega_{(i,j)}(A) = A$.

Thus $\Omega_{(i,j)}$ operator can be applied to define a topology on the bitopological space $(X, \Im_1, \Im_2)$ $\forall$ i, j∈ (1,2) i ≠j. We denote this topology by $\Im_{\Omega(i,j)}$ and the topological space equipped with this topology is denoted as $(X, \Im_{\Omega(i,j)})$. The members of $\Im_{\Omega(i,j)}$ are defined as :

Definition 3.3 : If $U \in P(X)$ then $U \in \Im_{\Omega(i,j)}$ or U is called $\Omega_{(i,j)}$ open if for every x∈ U there exists a $\Im_j$ open neighbourhood U_x of x such that $\Im_i cl(U_x) - U$ is countable.

Based on the definitions, the closure and interior of a subset A of X can be defined as

Definition 3.4 : Union of all $\Omega_{(i,j)}$ open sets in $A \subset X$ is defined as $\Omega_{(i,j)}$ interior of A.

Definition 3.5 : Intersection of all $\Omega_{(i,j)}$ closed sets containing $A \subset X$ is defined as $\Omega_{(i,j)}$ closure of A.

Further it can also be seen that by changing the choices of i and j one can define two new topologies on a bitopological space $(X, \Im_1, \Im_2)$. Thus for a bitopological space $(X, \Im_1, \Im_2)$ a new bitopological space $(X, \Re_1, \Re_2)$ can be defined where $\Re_1$ is the topology defined by $\Omega_{(1,2)}$ open sets and $\Re_2$ is the topology defined by $\Omega_{(2,1)}$ open sets. Thus

Theorem 3.1 : In a bitopological space $(X, \Im_1, \Im_2)$, let $\Re_i$ be the collection of all $\Omega_{(i,j)}$ open sets in X then,

(i) $\emptyset, X \in \Re_i$

(ii) $A, B \in \Re_i \Rightarrow A \cap B \in \Re_i$

(iii) $\{A_\alpha \mid \alpha \in \Delta\} \in \Re_i \Rightarrow \cup \{A_\alpha \mid \alpha \in \Delta\} \in \Re_i$

Proof : Easy

Following results are easy to prove :

Result 3.1: A is $\Re_i$ open iff A can be written as union of all $\Re_i$ open sets in A.

Result 3.2: A is $\Re_i$ closed iff A can be written as intersection of all $\Re_i$ closed sets in A.

Result 3.3: If A is $\Re_i$ closed then

(i) $A = \Re_i cl(A)$

(ii) $\Re_i cl(X-A) = X - \Re_i int(A)$

(iii) $\Re_i cl(A)$ is $\Re_i$ closed.

(iv) $x \in \Re_i cl(A)$ iff $\{U_x\} \cap A = \emptyset$ $\forall \Re_i$ open set U_x containing

The concepts of pairwise open and pairwise closed sets can also be defined on $(X, \Re_1, \Re_2)$ as:

Definition 3.6: In a bitopological $(X, \Im_1, \Im_2)$, $A \subset X$ is called pairwise Ω closed if $\Re_i cl(A) = A = \Re_j cl(A)$.

Hence A is pairwise Ω closed if $\forall$ $x \notin A$ there exist $U_x \in \Im_i$ and $V_x \in \Im_j$ such that $(\Im_i cl\text{-}\{U_x\} \cup \Im_j cl\text{-}\{V_x\}) \cap A$ is Countable.

Definition 3.7: In a bitopological $(X, \Im_1, \Im_2)$, $A \subset X$ is called pairwise Ω open if $\Re_i int(A) = A = \Re_j int(A)$.

Hence A is pairwise Ω open if $\forall$ x ∈ A there exist $U_x \in \Im_i$ and $V_x \in \Im_j$ such that $(\Im_i cl\text{-}\{U_x\} \cup \Im_j cl\text{-}\{V_x\})$-A is Countable.

Example 3.1 : Let we consider two topologies on the set of real numbers **R** as follows :

$(\mathbf{R}, \Im_1) = \{\emptyset, \mathbf{R}, \{0\}, (-\infty, 0), (-\infty, 0]\}$

$(\mathbf{R}, \Im_2) = \{\emptyset, \mathbf{R}, \{0\}, (0, \infty), [0, \infty)\}$

It can be checked easily that $(\mathbf{R}, \Re_1)$ consists of all the sets of the type ∅, [0, ∞)-C, **R**-C whereas $(\mathbf{R}, \Re_2)$ consists of all the sets of the type ∅, (-∞, 0] - C, **R**-C where C is any countable subset of **R**. We can see that all these four topologies are independent to each other.

Example 3.2 : Let **R** be the set of real numbers and let $\Im_1$ and $\Im_2$ be the usual topology and right order topology respectively. Then, $\Re_1$ is the topology finer than $\Im_2$ as it consists of all the members of it along with all the elements of the type A-C and R-C where A is any member of $\Im_2$. Members of $\Re_2$ will be of the type ∅ (-∞, a) – C and **R** – C for each element a of **R**.

Example 3.3 : On the set of real numbers let we define two topologies $\Im_1$ and $\Im_2$ where $\Im_1$ is the usual topology and $\Im_2$ is defined as $(\mathbf{R}, \Im_2) = \{\emptyset\} \cup \{U \cup (a, \infty) \mid U \in \Im_1$ and $a \in \mathbf{R}\}$, then $(\mathbf{R}, \Re_1) = \{\emptyset, (a, \infty) - C, \mathbf{R} - C \mid a \in \mathbf{R}$ and C is any countable subset of R$\}$ and $(\mathbf{R}, \Re_2)$ is the usual topology.

Example 3.4 : If $(\mathbf{R}, \Im_1)$ is the usual topology and $(\mathbf{R}, \Im_2)$ is the cocountable topology then $(\mathbf{R}, \Re_1)$ is cocountable topology whereas $(\mathbf{R}, \Re_2)$ is the usual topology.

Theorem 3.2 : If a $\Im_j$ open set U has countably many boundaries in $\Im_i$ then, U is $\Re_i$ open also.

Proof : U is $\Re_i$ open if for every x∈ U there exists a $\Im_j$ open neighbourhood U_x of x such that $\Im_i cl(U_x) - U$ is countable. Take $U_x = U$ then $\Im_i cl(U_x) - U = \Im_i cl(U) - U$ is countable.

Corollary 3.1 : If a $\Im_j$ open set U is closed in $\Im_i$ then, U is $\Re_i$ open also.

Corollary 3.2 : If U is $\Im_j$ open set and $\Im_i$ closed then, U is both $\Re_i$ open and $\Re_j$ closed.

Hence a pairwise open and closed set in X is pairwise Ω open and Ω closed.

Theorem 3.3 : If a bitopological space $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ is pairwise antilocally countable then the bitopological space $(X, \mathfrak{R}_1, \mathfrak{R}_2)$ is pairwise anti locally countable.

Proof : It suffices to show that if $(X,\mathfrak{I}_1)$ is pairwise anilocally countable then so is $(X, \mathfrak{R}_2)$. Let $U\in \mathfrak{R}_2$ and let $x \in U$ then there exists a $\mathfrak{I}_1$neighbourhood U_x of x such that $\mathfrak{I}_2$ cl(U_x)- U is countable. But U is uncountable and so is $\mathfrak{I}_2$ cl(U_x) therefore U has to be uncountable.

Theorem 3.4 : Every countable set in a bitopological space $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ is pairwise Ω-closed.

Proof : Let A be a countable subset of X then $\mathfrak{R}_1$ (A) =A ∪ {y ∈ X | ∀ U ∈ $\mathfrak{I}_2$ | y ∈ U, $\mathfrak{I}_1$cl-{U}∩ A ≠ Countable} ⇒ $\mathfrak{R}_1$ (A) =A. Similarly $\mathfrak{R}_2$ (A) =A. Hence A is pairwise Ω-closed.

Theorem 3.5 : Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space. Let U be a subset of X containing x then U is called $\mathfrak{R}_i$ open if and only if for every x∈ U there exists a $\mathfrak{I}_j$ open neighbourhood U_x of x and a countable subset C of X such that $\mathfrak{I}_i$cl(U_x) –C ⊂ U.

Proof : Let U be $\Omega_{(i,j)}$ open then for every x∈ U there exists a $\mathfrak{I}_j$ open neighbourhood U_x of x such that $\mathfrak{I}_i$cl(U_x) –U= C ⇒ $\mathfrak{I}_i$cl(U_x) –C ⊂ U. Conversely, let for every x∈ U there exist a $\mathfrak{I}_j$ open neighbourhood U_x of x and a countable subset C of X such that $\mathfrak{I}_i$cl(U_x) –C ⊂ U ⇒ $\mathfrak{I}_i$cl(U_x) –U ⊂ C ⇒ $\mathfrak{I}_i$cl(U_x) –U has to be countable.

Theorem 3.6 : If a bitopological space $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ is pairwise antilocally countable and A is $\mathfrak{I}_i$ open then A ⊂ $\mathfrak{I}_j$cl(A) ⊂ $\mathfrak{R}_j$cl(A) ∀ {i, j}∈{1,2}, i ≠ j.

Proof : Let x ∈ $\mathfrak{I}_j$cl(A). Let U_x be $\mathfrak{R}_j$ neighbourhood of x ⇒ there exists a $\mathfrak{I}_i$ neighbourhood U of x and a countable subset V of X such that $\mathfrak{I}_j$cl(U) –V ⊂ U_x ⇒ $\mathfrak{I}_j$cl(U) ∩ A –V ⊂ U_x ∩ A. Now, both U and A are $\mathfrak{I}_i$ neighbourhoods of x therefore U ∩ A is non empty ⇒ U ∩ A is uncountable and so is U_x ∩ A. Hence x ∈ $\mathfrak{R}_j$cl(A).

It is obvious that if X is a countable set or a finite set then each of the derived topologies will be discrete topology. Therefore it is useful to consider X as an uncountable set in order to make the results meaningful.

IV. DEFINITIONS AND PROPERTIES OF Ω PRIME POINTS IN BITOPOLOGICAL SPACES

Definition 4.1: Let $(X, \mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space equipped with two topologies $\mathfrak{I}_1$ and $\mathfrak{I}_2$. Let A be a subset of X. We define (i, j) prime point of A as elements of $\Omega_{(i,\ j)}$(A) . Thus a is (i, j) prime point of A if ∀ U ∈ $\mathfrak{I}_j$ | a ∈ U, $\mathfrak{I}_i$cl-{U}∩ A ≠ Countable. If A has all its (i, j) prime points within A then A is said to be $\Omega_{(i,\ j)}$ closed.

Theorem 4.1: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space such that $\mathfrak{I}_j$ is second countable then, every uncountable subset of X contains at least one (i, j) prime point.

Proof : Let A be an uncountable subset of X such that A contains no (i, j) prime point then, for every a in A there exists U such that U ∈ $\mathfrak{I}_j$ | a ∈ U, $\mathfrak{I}_i$cl-{U}∩ A = Countable. Since $\mathfrak{I}_j$ is second countable and a ∈ U there exists a base element B_j such that a ∈ B_j ⊂ U⇒ $\mathfrak{I}_i$cl-{ B_j }∩ A = Countable. Then A= $\bigcup_{j\in I} B_j \cap A$ implying that A is countable union of countably many sets and therefore is countable which is a contradiction. Hence every uncountable subset of X contains at least one (i, j) prime point.

Corollary 4.1: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space such that $\mathfrak{I}_j$ is second countable then, every uncountable subset of X contains uncountably many one (i, j) prime points.

Proof : Let A be an uncountable subset of X containing countably many (i, j) prime points. Let C be the collection of (i, j) prime points then C is countable and therefore A-C is an uncountable set containing no (i, j) prime points contrary to the above theorem which asserts that every uncountable subset of X contains uncountably many (i, j) prime points.

Corollary 4.2: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space such that $\mathfrak{I}_j$ is second countable. Let A be an uncountable set and C be the collection of (i, j) prime points of A then the set A- C is countable.

Proof : If A-C is uncountable then there will exist an (i, j) prime point of A-C in itself and consequently of A in A but not in C contrary to the construction of A-C.

Corollary 4.3: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space. Let A be a countable set then A has no (i, j) prime points.

Theorem 4.2: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space such that $\mathfrak{I}_j$ is a Lindeloff space then, every uncountable subset of X contains at least one (i, j) prime point.

Proof : Let C be a subset of X such that C has no (i, j) prime point. To prove the theorem it will suffice to show that C is countable. For every a ∈ X there exists a nbd U_a of a in $\mathfrak{I}_j$ such that $\mathfrak{I}_i$cl-{U_a}∩ A is Countable. Then {U_a | a ∈ X }is $\mathfrak{I}_j$ open cover of X. Since $\mathfrak{I}_j$ is Lindeloff, it has a countable subcover {U_k | k∈ I}. Then C ⊂ $(\bigcup_{k\in I} U_k) \cap C \subset \bigcup_{k\in I}(\mathfrak{I}_j cl(U_k) \cap C) \subset C$ which shows that C is countable being union of countably many countable sets. This proves the theorem.

Corollary 4.4: Let $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ be a bitopological space such that $\mathfrak{I}_j$ is a compact space then, every uncountable subset of X contains at least one (i, j) prime point.

Theorem 4.3: In a bitopological space $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ the set of all (i, j) prime points of a subset of X is $\mathfrak{I}_j$ closed.

Proof : :Let A be a subset of X and let D be the collection of all (i, j) prime points. We show that X-D is $\mathfrak{I}_j$ open. Let a ∈ X-D then there exists U_a such that U_a ∈ $\mathfrak{I}_j$ | a ∈ U_a, $\mathfrak{I}_i$cl-{U_a}∩ A = Countable. We claim that {U_a}∩ D = ∅ for if x∈ {U_a}∩ D then x∈ D and U_a is $\mathfrak{I}_j$ neighbourhood of a and therefore $\mathfrak{I}_i$cl-{U_a}∩ A should be uncountable. This contradiction leads to the fact that the set of all (i, j) prime points of a subset of X is $\mathfrak{I}_j$ closed.

V. TWO NEW SEPARATION AXIOMS

Definition 5.1: A bitopological space $(X,\mathfrak{I}_1,\mathfrak{I}_2)$ is said to be pairwise T_{Ω}^{*} if for every element x of X there exists an open set U_i in $\mathfrak{I}_i$ and a countable subset C of X such that $\mathfrak{I}_j$cl(U) - (U_1∩ U_2) = C where U_i ∈ $\mathfrak{I}_i$ and x ∉ U_i ∀ {i, j} ∈{1,2}, i ≠ j.

Definition 5.2: A bitopological space (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) is pairwise T_{Ω} if for every element x of X there exists an open set U_i in $\mathfrak{I}_i$ such that $\mathfrak{I}_j$cl(U_i)∩($\mathfrak{I}_i$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) is countable.

Theorem 5.1: A pairwise T_{Ω}^{*} bitopological space (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) is pairwise T_{Ω}.

Proof: Let x ∈ X and let there exist an open set U_i in $\mathfrak{I}_i$ such that $\mathfrak{I}_j$cl(U_i) - (V_1∩ V_2) = C. Since x ∉ V_i ∀ {i, j} ∈{1,2}, i ≠ j ⇒ V_i ∩ $\mathfrak{I}_i$ cl-{x}=ϕ= V_j ∩ $\mathfrak{I}_j$ cl-{x}⇒ (V_i ∩ V_j)∩($\mathfrak{I}_i$ cl-{x}∪$\mathfrak{I}_j$ cl-{x})=ϕ ⇒($\mathfrak{I}_j$ cl(U) - C)∩($\mathfrak{I}_i$ cl-{x}∪$\mathfrak{I}_j$

cl-{x})=ϕ ⇒ $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) is countable.

Theorem 5.2: A pairwise T_1 bitopological space (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) is pairwise T_{Ω}^{*}.

Proof : Since X is pairwise T_1, each singleton is closed in both the topologies. Let x∈ X. Let U_i be an open set in $\mathfrak{I}_i$ containing x. Take $V_1=V_2=$ X-{x}. Then, $\mathfrak{I}_j cl(U_i)$ - ($U_1 \cap U_2$) = {x} which is countable. Hence (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) is pairwise T_{Ω}^{*}.

Theorem 5.3: A pairwise T_{Ω} bitopological space (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) is pairwise T_0 if the countable set C-{x}, where C is the set defined in definition 2, is closed in both the topologies.

Proof : Let (X, $\mathfrak{I}_1$, $\mathfrak{I}_2$) be pairwise T_{Ω}^{*} then for every element x of X there exists an open set U_i in $\mathfrak{I}_i$ such that $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) =C ⇒ (U_i)∩($\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) =C ⇒ (U_i-C)∩($\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) = {x} ⇒ (V_i)∩($\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) = {x}. If V_i does not contain y then we are done. If V_i contains y then at least one of $\mathfrak{I}_\iota$ cl-{x} and $\mathfrak{I}_j$ -cl{x} does not contain y and consequently their complement will be an open set containing y but not x.

Theorem 5.4: In a bitopological space (X, $\mathfrak{I}_1, \mathfrak{I}_2$), the bitopological space (X, $\mathfrak{R}_1, \mathfrak{R}_2$) is always pairwise T_1.

Proof : Since a countable set is closed with respect to both $\mathfrak{R}_1$ and $\mathfrak{R}_2$ in particular for every pair of x,y ∈ X | x ≠ y, each of {x} and {y} are also pairwise closed and therefore X-{x} is the pairwise open set in (X, $\mathfrak{R}_1, \mathfrak{R}_2$) containing y but not x and X-{y} is the pairwise open set in (X, $\mathfrak{R}_1, \mathfrak{R}_2$) containing x but not y.

Corollary 5.1: In a bitopological space (X, $\mathfrak{I}_1, \mathfrak{I}_2$), the bitopological space (X, $\mathfrak{R}_1, \mathfrak{R}_2$) is pairwise T_{Ω}^{*}.

Corollary 5.2: In a bitopological space (X, $\mathfrak{I}_1, \mathfrak{I}_2$), the bitopological space (X, $\mathfrak{R}_1, \mathfrak{R}_2$) is pairwise T_{Ω}.

Corollary 5.3: In a bitopological space (X, $\mathfrak{I}_1, \mathfrak{I}_2$), the bitopological space (X, $\mathfrak{R}_1, \mathfrak{R}_2$) is pairwise T_0.

Theorem 5.5 : In a pairwise T_{Ω} bitopological space the derived set of each element in both the topologies is pairwise Ω closed.

Proof : Let x∈ X and let $\{x\}_i$' be the derived set of {x} in $\mathfrak{I}_i$. Since X is pairwise T_{Ω} it has a neighbourhood in $\mathfrak{I}_i$ say U_i such that $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) = C where C is some countable subset of X. Therefore $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_\iota$ cl-{x} ∪ $\mathfrak{I}_j$ -cl{x}) = C ⇒ $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_\iota$ cl-{x}) = Countable and $\mathfrak{I}_j cl(U_i) \cap (\mathfrak{I}_j$ cl-{x}) = Countable ⇒ $\mathfrak{I}_j cl(U_i) \cap$ ($\{x\}_i$') = Countable and $\mathfrak{I}_j cl(U_i) \cap (\{x\}_i$') = Countable ⇒ Each $\{x\}_i$' is pairwise Ω closed.

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