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Stability Analysis of an Epidemic Model with Infected Immigrants and Optimal Vaccination

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Abstract*: *In this paper an SIR (Susceptible-infectious-recovered) epidemic model consisting of saturated incidence rate with vaccination to the susceptible individual in presence of infected immigrants is studied. Stabilities of disease free and endemic equilibrium are also analyzed. The impact of the infected immigrants in the spread of the illness in a populace is examined. A mathematical model has been used to investigate the inflow of the infected immigrants in a population who rapidly transmit the disease. By using appropriate vaccine level to the susceptible population, disease can be reduced. The main purpose of this work is minimizing the invectives and maximizes the recovered individuals. To attain this, apply optimal vaccination strategies by utilizing the pontryagin's maximum principle (PMP). Speculative results are demonstrated through the numerical simulations.

KEY WORDS: *Saturated incidence, Basic reproduction number, stability, vaccination, PMP.*

I. INTRODUCTION

The consequence of contagious disease to the world is one of the significant conflicts in the present time. Migration is one of the agents to spread the infectious diseases. There is a closer relationship between epidemic diseases and the movement of the individuals. Immigration from less developed countries to developed countries of the world for better economic chances is rising in number and could afford to the infectious diseases.

The main reason to study the mathematical modeling is to understand the transmission mechanism and requirement of more effective control strategies [1].The usage of incredible numerical models would be one of the better approaches to discover the elements of the infection by dissecting the framework. Mathematical epidemiology declares in a brief way the precise actions to help in eliminating diseases and the basis of such studies was given by D.clancy [2]. The basic epidemic model as proposed by W.O. kermack and A.G.Mckendrick has been dealt with various types of incidence rates as given by Ruan and Wang [3], li and Wang [4]. Capasso and Serio [5] established a saturated incidence where the interactions between susceptible and infective may immerse at high infective dimensions because of swarming of infective people.

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Optimal control theory is another part of mathematics that is used widely to control the spread of irresistible infections. It is a powerful tool to make verdicts concerning complex biological situation [6]. It is employ to control the spread of irresistible infections for which either vaccination or treatment is available. A few authors examined about the vaccination impact to control the spread of disease and some of them talk about the treatment models [7, 13, and 14]. The happening of infectious complexity makes mortality of many individuals as well as spending huge money to the health and disease control. The spread of several communicable diseases can be warding off by vaccination to the susceptible. M.K.Zaleta [8] examined a simple SIS model with immunization displayed in backward bifurcation. C.P.Farrington [9] obtained the vaccine efficacy against the spreading of diseases in huge population by analysing the effect of immunization with the perspective of transmission of the infection. Various modelling studies have been tried to find the performance of vaccine to stop the proliferation of the infections. Zaman et al [10] studied the vaccination as control in an SIR model. The main parts of mathematical models that employ the optimal control theory depend on the PMP [15], which is an important condition for finding the optimal solution. T.Malik[11] analyzed that the optimal control theory is an important tool to curtail the spread of the infectious diseases by using the various control strategies. Various optimal controlled strategies and optimality systems are analyzed with numerical techniques [16, 17]

T.k Kar et.al [12] investigated an SIR epidemic model with saturated incidence rate involving optimal control of vaccination. They evaluated the existence and steadiness of disease free and endemic equilibrium points. The effect of the vaccine is decreasing the vaccine induced reproduction number is discussed. We extended their work by including the constant rate of immigrants. In this paper, a mathematical model of the SIR dynamics is used to investigate the inflow of the infected immigrants in a population who rapidly transmit the disease. By using appropriate vaccine level to the susceptible population, the incidence of a disease can be eliminated or reduced in the presence of infected immigrants. Here vaccination is used as control strategy. Optimal control problem is formulated and solved by the PMP.

II. MATHEMATICAL MODEL

Consider the differential equations which describes the SIR model representing a susceptible individuals provided with vaccination in presence of infected immigrants and incidence rate is saturated

$$\frac{dS}{dt}=(1-p)A-\frac{\beta SI}{1+\alpha I}-dS-uS$$
$$\frac{dI}{dt}=\frac{\beta SI}{1+\alpha I}+pA-dI-\delta I-\gamma I$$
$$\frac{dR}{dt}=\gamma I-dR+uS \qquad (1)$$

A. *Variables and Parameters interpretation*:

$(1-p)A$	The susceptible recruitment corresponding to births and immigration
pA	The rate of infected recruitment
β	Disease transmission rate
α	Saturated parameter
d	Natural death rate
u	Vaccination rate
γ	Recovery rate
δ	Disease induced death rate

The population $N(t)=S(t)+I(t)+R(t)$

III. LOCAL STABILITY ANALYSIS OF THE SYSTEM

The system (1) has two equilibrium points. Let $E_0\left(S^*,0,0\right)$ be disease free equilibrium (DFE) point and $E_1(S^*,I^*,R^*)$ be the endemic equilibrium (EE) point [18].

DFE is locally asymptotically stable at E_0 if $R_0<1$.

The Jacobian matrix of (1) at $E_0\left(\frac{(1-p)A}{d+u},0,0\right)$ is given by

$$J=\begin{bmatrix} -d-u & \frac{-\beta(1-p)A}{d+u} & 0 \\ 0 & \frac{\beta(1-p)A}{d+u}-d-\delta-\gamma & 0 \\ u & \gamma & -d \end{bmatrix} \qquad (2)$$

The characteristic equation of the Jacobian is

$$(-d-u-\lambda)\left(\frac{\beta(1-p)A}{d+u}-d-\delta-\gamma-\lambda\right)(-d-\lambda)=0$$

The obtained Eigen values are

$$\lambda_1=-(d+u),\ \lambda_2=\frac{\beta(1-p)A}{d+u}-d-\delta-\gamma,\ \lambda_3=-d$$

λ_2 be negative for $\beta(1-p)A<(d+\delta+\gamma)(d+u)$

$$\frac{\beta(1-p)A}{(d+u)(d+\delta+\gamma)}<1$$

Let $$R_0=\frac{\beta(1-p)A}{(d+\delta+\gamma)(d+u)} \qquad (3)$$

If $R_0<1$ then $\lambda_2<0$

Since $\lambda_1<0$, $\lambda_3<0$ and $\lambda_2<0$ if $R_0<1$

Then DFE is locally asymptotically stable.

The solution of following equations is the EE

$$(1-p)A-\frac{\beta S^*I^*}{1+\alpha I^*}-dS^*-uS^*=0 \qquad (4)$$
$$\frac{\beta S^*I^*}{1+\alpha I^*}+pA-dI^*-\delta I^*-\gamma I^*=0 \qquad (5)$$
$$\gamma I^*-dR^*+uS^*=0 \qquad (6)$$

$$S^*=\frac{(1-p)A(1+\alpha I^*)}{\beta I^*+(d+u)(1+\alpha I^*)},\qquad R^*=\frac{uS^*+\gamma I^*}{d}$$

Substitute S^* in (5) then we get

$$B_1I^{*2}+B_2I^*-B_3=0 \qquad (7)$$

Where $B_1=(\beta+(d+u)\alpha)(d+\delta+\gamma)$,

$$B_2=(d+u)(d+\delta+\gamma)-\alpha pA(d+u)-\beta A$$

$$B_3=\frac{pA^2(1-p)\beta}{R_0(d+\delta+\gamma)}$$

If $R_0>1$ then only one positive solution of (7) exists by Descartes rule at $E_1(S^*,I^*,R^*)$

$$I^*=\frac{-B_2\pm\sqrt{B_2^{\,2}-4B_1B_3}}{2B_1}$$

If $R_0>1$ then $B_2^{\,2}-4B_1B_3>0$

III. GLOBAL STABYLITY ANALYSIS OF DISEASE FREE EQUILIBRIUM

Consider the Lyapunov function

$$L(v)=V_1(S-S^*)+V_2(I-I^*)+V_3(R-R^*) \qquad (8)$$

Where V_1,V_2,V_3 are positive constants to be determined.

differentiating (8) with respect to time t

We obtain

$$L'=V_1\left[(1-p)A-\frac{\beta SI}{1+\alpha I}-dS-uS\right]$$
$$+V_2\left[\frac{\beta SI}{1+\alpha I}+pA-dI-\delta I-\gamma I\right]+V_3(\gamma I-dR+uS)$$
$$L'=\frac{\beta SI}{1+\alpha I}(V_2-V_1)+pA(V_2-V_1)+\gamma I(V_3-V_2)$$
$$+uS(V_3-V_1)+AV_1-dsV_1-V_2\delta I-V_3dR-V_2dI$$

Choose positive constants $V_1=V_2=V_3=1$

$$L'=A-dS-dI-\delta I-dR$$
$$L'=A-dN-\delta I-dR$$

$$L' = -(dN - A) - \delta I - dR < 0$$

Thus the DFE of the system (1) is globally asymptotically stable if $R_0 < 1$

IV. OPTIMAL VACCINATION

An optimal control problem is constructed to make minimize the objective functional.

$$J(u) = \int_0^T \left[A_2 I(t) + A_1 S(t) + \frac{1}{2}\tau u^2(t) \right] dt \quad (9)$$

Subjected to the system (1) with the initial conditions

$$I(0) \geq 0, R(0) \geq 0, S(0) \geq 0, \tau > 0$$

Where A_2, A_1 are small positive constants to stay a steadiness in the size of $I(t)$ and $S(t)$ respectively. The reactions of vaccination are reflected by the square of control variable. τ is a weight parameter which is associated with control $u(t)$.

For existence, a control system (1) can be recast as

$$\phi_t = F\phi + B\phi \quad (10)$$

Where

$$\phi = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}, B = \begin{bmatrix} -d-u & 0 & 0 \\ 0 & -(d+\delta+\gamma) & 0 \\ u & \gamma & -d \end{bmatrix}$$

$$F(\phi) = \begin{bmatrix} (1-p)A - \dfrac{\beta SI}{1+\alpha I} \\ \dfrac{\beta SI}{1+\alpha I} + pA \\ 0 \end{bmatrix}$$

ϕ_t Indicate the derivative of ϕ with time t. Equation (10) is surely a nonlinear equation with a boundary coefficient.

We set

$$K(\phi) = F\phi + B\phi \quad (11)$$

Now

$$F(\phi_1) - F(\phi_2) = \begin{bmatrix} (1-p)A - \dfrac{\beta S_1 I_1}{1+\alpha I_1} \\ \dfrac{\beta S_1 I_1}{1+\alpha I_1} + pA \\ 0 \end{bmatrix} - \begin{bmatrix} (1-p)A - \dfrac{\beta S_2 I_2}{1+\alpha I_2} \\ \dfrac{\beta S_2 I_2}{1+\alpha I_2} + pA \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\beta\left(\dfrac{S_1 I_1}{1+\alpha I_1} - \dfrac{S_2 I_2}{1+\alpha I_2}\right) \\ \beta\left(\dfrac{S_1 I_1}{1+\alpha I_1} - \dfrac{S_2 I_2}{1+\alpha I_2}\right) \\ 0 \end{bmatrix}$$

$$|F(\phi_1) - F(\phi_2)| = \left| -\beta\left(\frac{S_1 I_1}{1+\alpha I_1} - \frac{S_2 I_2}{1+\alpha I_2}\right)\right| + \left|\beta\left(\frac{S_1 I_1}{1+\alpha I_1} - \frac{S_2 I_2}{1+\alpha I_2}\right)\right|$$

$$|F(\phi_1) - F(\phi_2)| \leq 2\beta \left| \left(\frac{S_1 I_1}{1+\alpha I_1} - \frac{S_2 I_2}{1+\alpha I_2}\right) \right|$$

$$\leq 2\beta \left| \frac{S_1 I_1 (1+\alpha I_2) - S_2 I_2 (1+\alpha I_1)}{(1+\alpha I_1)(1+\alpha I_2)} \right|$$

$$\leq 2\beta \left| S_1 I_1 - S_2 I_2 + \alpha (S_1 - S_2) I_1 I_2 \right|$$

$$\leq 2\beta |S_1 - S_2| \left(|I_1| + \alpha |I_1||I_2| \right) + 2\beta |S_2||I_1 - I_2|$$

$$\leq 2\beta \frac{A}{d} |I_1 - I_2| + 2\beta |S_1 - S_2| \left[\frac{A}{d} + \alpha \left(\frac{A}{d}\right)^2 \right]$$

$$\leq N \left[|S_1 - S_2| + |I_1 - I_2| \right]$$

where $N = \max\left[(2\beta\frac{A}{d}); (2\beta(\frac{A}{d} + \alpha\frac{A^2}{d^2})) \right]$

N is a positive constant and independent state variables $S(t)$ and $I(t) \leq \frac{A}{d}$.

Also we acquire $|K\phi_1 - K\phi_2| \leq W|\phi_1 - \phi_2|$

Where $W = \max(N, \|B\|) < \infty$

Thus it follow that the function K is uniformly Lipchitz continuous. From the definition of the control $u(t)$ and the restrictions $I(t), S(t)$ and $R(t) \geq 0$.

To locate the best possible solution, first we discover the Lagrangian and Hamiltonian for the optimal control problem (9) and (1). The lagrangian function is

$$L(S, I, R) = A_2 I(t) + A_1 S(t) + \frac{1}{2}\tau u^2(t).$$

We try to find out the minimal value of the Lagrangian, state the Hamiltonian H for the control problem as

$$H(S, I, R, u, \lambda_1, \lambda_2, \lambda_3, t) = L(S, I, u)$$
$$+\lambda_1(t)\frac{dS}{dt} + \lambda_2\frac{dI}{dt} + \lambda_3\frac{dR}{dt} \quad (12)$$

$$= A_1 S(t) + A_2 I(t) + \frac{1}{2}\tau u^2(t) + \lambda_1(t)\left((1-p)A - \frac{\beta SI}{1+\alpha I} - dS - uS\right)$$

$$+\lambda_2\left(\frac{\beta SI}{1+\alpha I} + pA - dI - \delta I - \gamma I\right) + \lambda_3\left(\gamma I - dR + uS\right)$$

Where $\lambda_2(t), \lambda_1(t)$ and $\lambda_3(t)$ are specified suitable adjoint functions.

The adjoint equations for $\lambda_1(t), \lambda_2(t)$, and $\lambda_3(t)$ are written as

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S} = -A_1 + \lambda_1\left[\frac{\beta I}{1+I\alpha} + d + u\right]$$
$$+\lambda_2\left[-\frac{\beta I}{1+\alpha I}\right] - \lambda_3 u \qquad (13)$$
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I} = -A_2 + \lambda_1\left[-\frac{\beta S}{(1+\alpha I)^2}\right]$$
$$+\lambda_2\left[-\frac{\beta S}{(1+\alpha I)^2} + d + \delta + u\right] + \lambda_3\gamma$$
$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial R} = \lambda_3 d$$

Transversality states are $\lambda_i(T) = 0, i = 1,2,3$ where $T = t_{final}$.

Using the Hamiltonian equation (12) established the adjoin equations and transversality form. Set $I(t) = I^*(t), S(t) = S^*(t), R(t) = R^*(t)$, and differentiate (12) w.r.to $I, S, and\ R$ respectively. We get (10).

The optimal condition we have

$$\frac{\partial H}{\partial u} = \tau u^*(t) - S^*(t)\lambda_1(t) + S^*(t)\lambda_3(t) = 0, \quad \text{at}$$
$u = u^*(t)$.

$$u^*(t) = \frac{S^*(t)}{\tau}\left(\lambda_1(t) - \lambda_3(t)\right)$$

F

$$u^*(t) = \begin{cases} 0, if \dfrac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau} \le 0 \\ \dfrac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau}, if 0 < \dfrac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau} < u_{max} \\ u_{max}, if \dfrac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau} \ge u_{max} \end{cases}$$

Theorem: Let the optimal control variable $u^*(t)$ connected with optimal state solutions of (1) and (9) at $S^*(t), I^*(t)$, and $R^*(t)$ then there exists adjoint variables $\lambda_1(t), \lambda_2(t)$, and $\lambda_3(t)$

That satisfies

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial S} = -A_1 + \lambda_1\left[\frac{\beta I^*}{1+\alpha I^*} + d + u\right]$$
$$+\lambda_2\left[-\frac{\beta I^*}{1+\alpha I^*}\right] - \lambda_3 u$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial I} = -A_2 + \lambda_1\left[-\frac{\beta S^*}{(1+\alpha I^*)^2}\right]$$
$$+\lambda_2\left[-\frac{\beta S^*}{(1+\alpha I^*)^2} + d + \delta + u\right] + \lambda_3\gamma$$
$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial R} = \lambda_3 d$$

s

$$u^*(t) = \left\{\max\left\{\min\left(\frac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau}, u_{\max}\right), 0\right\}\right.$$

Now to get optimal point we have to solve

$$\frac{dS^*}{dt} = (1-p)A - \frac{\beta S^*}{1+\alpha I^*} - dS^*$$
$$-S^*\left(\left\{\max\left\{\min\left(\frac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau}, u_{\max}\right), 0\right\}\right)\right.$$

$$\frac{dI^*}{dt} = \frac{\beta S^* I^*}{1+\alpha I^*} + pA - (d+\delta+\gamma)I^*$$
$$\frac{dR^*}{dt} = \gamma I^* - dR^*$$
$$+S^*\left\{\max\left\{\min\left(\frac{(\lambda_1(t)-\lambda_3(t))S^*(t)}{\tau}, u_{\max}\right), 0\right\}\right. \text{Wi}$$

th Hamiltonian H^* at

$$H^*\left(S^*, I^*, R^*, u^*, \lambda_1, \lambda_2, \lambda_3, t\right)$$
$$= A_1 S^* + A_2 I^*$$

$$+\frac{1}{2}\left[\tau S^{*}\left(\left\{\max\left\{\min\left(\frac{\left(\lambda_1(t)-\lambda_3(t)\right)S^{*}(t)}{\tau},u_{\max}\right),0\right\}\right)^{2}\right]$$

$$+\lambda_1(t)\qquad\left[(1-p)A-\frac{\beta S^{*}}{1+\alpha I^{*}}-dS^{*}-S^{*}\right.$$

$$\left.\left(\left\{\max\left\{\min\left(\frac{\left(\lambda_1(t)-\lambda_3(t)\right)S^{*}(t)}{\tau},u_{\max}\right),0\right\}\right)\right]$$

$$+$$

$$\lambda_3(t)\left[\gamma I^{*}-dR^{*}+S^{*}\left(\left\{\max\left\{\min\left(\frac{\left(\lambda_1(t)-\lambda_3(t)\right)S^{*}(t)}{\tau},u_{\max}\right),0\right\}\right)\right]$$

VI. NUMERICAL SIMULATIONS

For the values $A=1.2$, $d=0.118$, $p=0.32$, $\beta=0.56$, $\alpha=0.000015$, $\gamma=0.75$, $u=0.301$, $\delta=0.1916$, $R_0=1.1556$

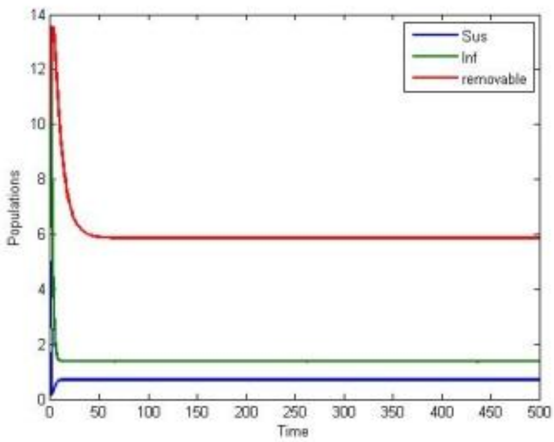

Fig.1

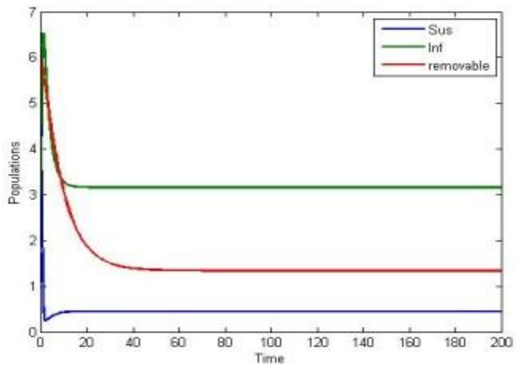

Fig. 2

From Fig.1 we detected that the recovered population raised because the disease is being composed by vaccination of the susceptible population. The infected population decreases with time and so it is feasible to decrease or eradicate the dominance of the disease in the population in the presence of intervention. In the Fig.1 the dispersal of the population with time is shown in all classes where it is found that, the susceptible populace and the infected population decreases with time. Fig.2 shows that without control strategies in the absence of intervention, the susceptible population initially decreases at low level and then kept constant to its carrying capacity. The infectious population keeps on increasing as time increases, which lead to many susceptible individuals, get infection due to the inflow of infected immigrants and absence of control strategies. It shows that the recovered population reduces as time goes on which means the natural resistance is possible to fade. Thus it can be concluded that, as long as there are no control strategies, the disease will invade the population since $R_0>1$.

General variations of Infective population for different values of p , it is examined that as the infected immigrants' p decrease with time in the population, the infected population also decreases. This shows that the disease decreases as well and is slowly been controlled or abolished in the population. Therefore the government should take more control strategies to diminish the infections which are spread by infected immigrants in order to avert the host population opposed to epidemic and prevalence of diseases.

Numerical simulation of optimal control and state variables are solved by using forward and backward R-K 4th order method. The optimality system (13) has 6 differential equations. The state equations are solved by a forward R-K 4th order method in time. That states come in handy for the adjoint equations with a back ward 4th order R-K method because of the transversal conditions.

For the parameter values

$A=0.4, \beta=0.624, \delta=0.01, \gamma=0.2, d=0.3, \alpha=0.56,$

$p=0.218, u=0.01, R_0=1.2346,$

Equilibrium point

(S_1, I_1, R_1) is $(0.4131, 0.1585, 0.1056)$

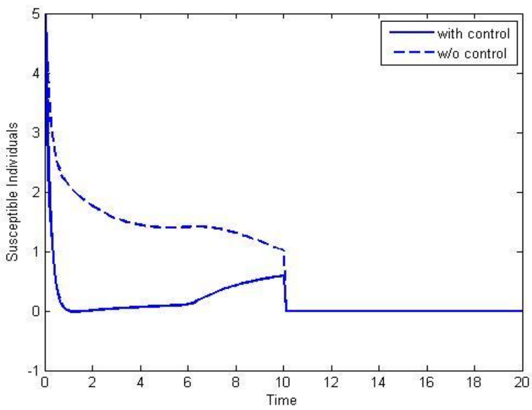

Fig.3

Susceptible population with and without control

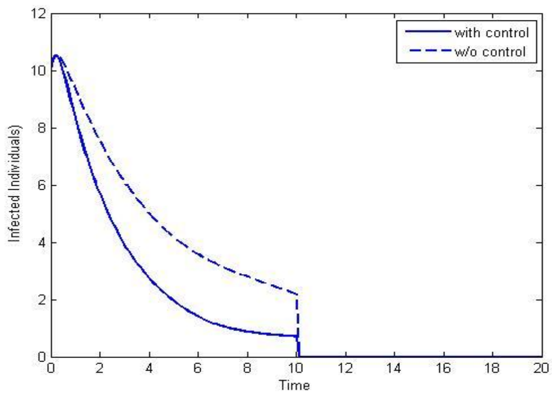

Fig. 4

Infected population with and without control

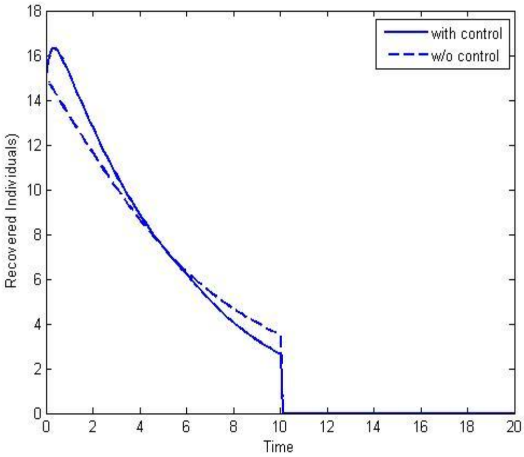

Fig. 5

Recovered population with and without control

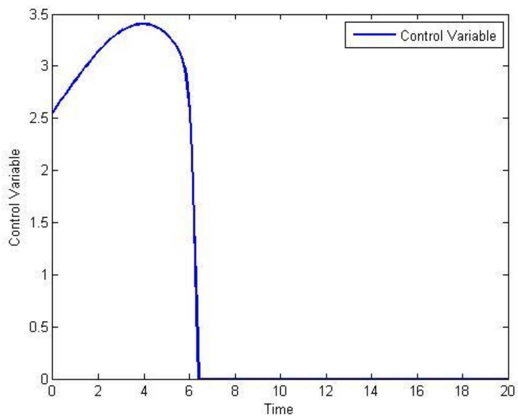

Fig. 6

Control variable

The susceptible individuals having control parameter and without control, are illustrated. From Fig.3 the populace of susceptible with control decreases slowly to least minimum. From Fig.4 the rate of contaminated individuals decrease after control and maximum number of individuals recovered. From the Fig.5 recovered individuals are increased by following the vaccination policy in huge population and shows that some infected individuals also there in the population. This proves that to control infections, educational campaigns are also required. The control individuals view as solid line and without control individual's appeared by dashed line. From Fig.6 identify that by using vaccination, after 6 weeks disease controlled limited number of susceptible and infected individuals is there in the populace. From initial time to at the end of 10 weeks population maintain a steady stable state.

V. CONCLUSION

The problem related to SIR model having optimal control with infected immigrants has been investigated analytically and numerically. The conditions for stability states of the system equilibria are found, that is if $R_0<1$ DFE is locally asymptotically stable and globally stable, If $R_0>1$ then EE is unstable. In this work varying population size is considered and vaccination is proved to be an effective control strategy against the diseases (like measles and other childhood diseases) in the presence of inflow of infected immigrants. Optimal control problem is formulated subjected to SIR model and PMP applied to describe the control and derive the optimality system. An evaluation between with control and without control is examined in the population. It is observed that if there is no intervention put in place, revealed that the disease might be endemic and it would lead to more cases of disease in the population. By using effective vaccination as control, infective individuals are minimized and recovered population is maximized, which is shown in the above Fig.4&Fig.5. Results of the numerical simulations for different values of parameters shown that after 6 weeks diseased is controlled and it has taken 10 weeks time to achieve steady stable state.

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