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On Direct Product of a Fuzzy Subgroup with an Anti-fuzzy Subgroup

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Abstract: ***We have introduced and analysed some new refreshing concepts in the field of fuzzy abstract algebra. The main contributions of this paper are fivefold: (1) we have introduced the notion of dual-fuzzy subgroup, (2) we have defined the direct product of a fuzzy subgroup with an anti-fuzzy subgroup, (3) Furthermore, we have defined mixed level subset and mixed level subgroup, (4) we have also developed some new theories as well as propositions based on these newly defined notions and lastly (5) we have redefined these notions using general T-norm and T* conorm.***

Index Terms: ***Fuzzy subgroup, Anti-fuzzy subgroup, Dual-fuzzy subgroup, Mixed level subgroup.***

I. INTRODUCTION

Crisp set theory has certain drawbacks. It is quite insufficient in case of handling real-life problems. Fuzzy set theory[*1*] is more reliable in tackling such scenarios. Since the very beginning of fuzzy set theory, many researchers have carried out that perception on various realistic problems like in optimization technique[*2*], game theory[*3*], statistical analysis[*4*], uncertainty theory[*5,6,7,8*], database[*9*], etc. Presently, neutrosophic set theory[*10*] has been developed which is nothing but generalization of fuzzy set theory. Some researchers have implemented this new concept in various fields like the shortest path problem[*11,12,13,14*], image processing[*15*], decision making problems[*16*], etc. Rosenfeld[*17*] introduced the notion of the fuzzy subgroup (FS) with respect to the T-norm T_1 ($T_1(m,u) = min\{m,u\}$). Later on, Biswas[*18*] introduced the notion of anti-fuzzy subgroup (AFS), which was based on T-conorm T_1^* ($T_1^*(m,u) = max\{m,u\}$). He proved that an AFS on the basis of T_1^* is nothing but the complement of a FS on the basis of T_1. Anthony and Sherwood[*19*] redefined the notion of FS based on general T-norm. By redefining the notion of AFS Gayen et al[*20*] introduced some new notions like subgroup generated AFS and function generated AFS. Later on, some pioneers have worked on different algebraic structures of FS as well as AFS. They have also implemented these concepts in various building blocks of abstract algebra, like ideal[*17*], ring, field, etc.

The direct product of fuzzy subgroups was introduced by Sherwood[*21*]. He defined the direct product of two FSs on the basis of general T-norm. Osman[*22*] redefined the direct product of two FS which was based on the redefined notion of FS by Anthony and Sherwood[*19*]. Later on, Ray[*23*]suggested some important theories regarding the direct product of two fuzzy subgroups. But, his version of the direct product was based on T-norm T_1. Also, he has given some important theories on normal forms and conjugacy class of FS. Aktaş and Çağman[*24*] have generalized Ray's results and defined t-fuzzy subgroup. Ray[*25*]introduced the concept of the free product of two FSs. He proposed that the free product induces a FS on a quotient group. Sharma[*26*]introduced direct product of intuitionistic fuzzy subgroups which was based on intuitionistic fuzzy sets. The direct product of two AFS was first proposed by Dong[*27*]. He proposed some important theories regarding the product of two AFSs, conjugacy class of AFS and normal AFS. Furthermore, he has discussed some other results in the direct product of AFS[*28*]. The direct product of a FS and an AFS is still undefined. Till now this topic is unexplored and it may become a fruitful research field. But, before defining the direct product of a FS with an AFS firstly, one needs to consider the product of two fuzzy sets depending on a particular T-norm or T-conorm. But, this may arise certain difficulties. In this paper, we have taken some effective steps to bypass those difficulties and have defined the desired notions. Also, there exist some fuzzy sets which have both the characteristics of a FS as well as an AFS. We have defined this type of special subgroups. Furthermore, we have developed and proved some theories based on them which give some insights regarding their algebraic structures.

The major contributions of this article are:

- We have introduced a new concept known as dual FS (DFS) and proved some theories regarding DFS.
- We have also developed a new notion, which is the direct product of a FS and an AFS.
- Also, for the first time, we have defined mixed level sets and proved some theories regarding FS, AFS and mixed level set.
- Again, for the first time, we have defined the mixed level subgroup (MLS) which is a direct product of two different groups.
- Furthermore, we have mentioned some new results and theories based on these newly defined notions.
- We have also redefined and generalized some notions using general T-norm and T-conorm.

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This paper has been arranged as the following: In Section II we have included some preliminary definitions and theorems. In Section III we have proposed notions like DFS, the product of a FS and an AFS, mixed level set and MLS. In Section IV we have generalized some notions using general T-norm and T-conorm. Finally, in Section V we have concluded that our proposed notions are new, effective and will generate more scopes of future researches.

II. PRELIMINARIES

Definition II.1[1] A fuzzy subset σ of a crisp set U is a function from U to $[0,1]$ i.e. $\sigma: U \to [0,1]$.

Definition II.2[1] Let α be a fuzzy subset of U. Then $\forall t \in [0,1]$ the sets $\alpha_t = \{x \in U : \alpha(x) \geq t\}$ are called level subsets (t -level subsets) of α .

Definition II.3[18] Let β be a fuzzy subset of U. Then $\forall t \in [0,1]$ the sets $\overline{\beta}_t = \{x \in U : \beta(x) \leq t\}$ are called lower level subsets (t -lower level subsets) of β .

Definition II.4[29]A function $T^*:[0,1] \to [0,1]$ is termed as T-normiff $\forall m,u,t \in [0,1]$ the subsequentconditions are fulfilled:

(i) $T(m,1) = m$

(ii) $T(m,u) = T(u,m)$

(iii) $T(m,u) \leq T(t,u)$ if $m \leq t$

(iv) $T(m,T(u,t)) = T(T(m,u),t)$

Definition II.5[29]A function $T^*:[0,1] \to [0,1]$ is termed as T-conorm iff $\forall m,u,t \in [0,1]$ the subsequent conditions are fulfilled:

(i) $T^*(m,0) = m$

(ii) $T^*(m,u) = T^*(u,m)$

(iii) $T^*(m,u) \leq T^*(t,u)$ if $m \leq t$

(iv) $T^*(m,T^*(u,t)) = T^*(T^*(m,u),t)$

Definition II.6[17]A fuzzy subset α of a group P is termed as a fuzzy subgroup of P iff $\forall m,u \in P$, the subsequent conditions are fulfilled:

(i) $\alpha(mu) \geq min\{\alpha(m), \alpha(u)\}$

(ii) $\alpha(m^{-1}) \geq \alpha(m)$.

Here $\alpha(m^{-1}) = \alpha(m)$ and $\alpha(m) \leq \alpha(e)$, where e represents the neutral element of P .

Theorem II.1[17] α is a FS of P iff $\forall m,u \in P$

$$\alpha(mu^{-1}) \geq min\{\alpha(m), \alpha(u)\}.$$

Definition II.7[18] A fuzzy subset β of a group U is termed as an AFS of U if $\forall m,u \in U$, the subsequent conditions are fulfilled:

(i) $\beta(mu) \leq max\{\beta(m), \beta(u)\}$,

(ii) $\beta(m^{-1}) \leq \beta(m)$.

Here observe that $\beta(m^{-1}) = \beta(m)$, $\beta(m) \geq \alpha(e')$, where e' represents the neutral element of U.

Theorem II.2[18] β is an AFS of U iff $\forall m,u \in U$

$$\beta(mu^{-1}) \leq max\{\beta(m), \beta(u)\}$$

Definition II.8[30] Let α be a FS of a group P. Then $\forall t \in [0,1]$ and $\alpha(e) \geq t$ the subgroups α_t are called level subgroups of α .

Definition II.9[18] Let β be an AFS of a group U. Then $\forall t \in [0,1]$ and $\beta(e) \leq t$ the subgroups $\overline{\beta}_t$ are called lower-level subgroups of β .

Definition II.10[23] Let α and β be two fuzzy subsets of P and U respectively. The product of α and β denoted as $\alpha \times \beta$ is defined as

$$(\alpha \times \beta)(m,u) = min\{\alpha(m), \beta(u)\}.$$

Notice that inDefinition II.10T-norm as $T(m,u) = min\{m,u\}$ has been used.

Definition II.11[27] Let α and β respectively be two fuzzy subsets of P and U. The direct product of α and β denoted as $\alpha \times \beta$ is defined as

$$(\alpha \times \beta)(m,u) = max\{\alpha(m), \beta(u)\}.$$

Notice that inDefinition II.11T-conorm as $T^*(m,u) = max\{m,u\}$ has been used.

Theorem II.3[21] Let α and α' be FS of two groups P and U respectively. Then $\alpha \times \alpha'$ is a FS of $P \times U$.

Theorem II.4[27] Let β and β' be two AFS of two groups P and U respectively. Then $\beta \times \beta'$ is an AFS of $P \times U$.

Theorem II.5[23] Let α and α' be fuzzy subsets of P and U respectively. Then $(\alpha \times \alpha')_t = \alpha_t \times \alpha'_t$ $\forall t \in [0,1]$.

In a similar way, many other researchers have chosen eitherDefinition II.10orDefinition II.11 according to their topic requirements.
For instance,Definition II.10has been chosen by Ray[23] for his research article, whereasDefinition II.11has been chosen by Dong[27]. In this paper, we need to consider both the definitions to define the direct product of a FS and an AFS.

A. *A List of Abbreviations Used Throughout This Paper*

FS stands for “Fuzzy subgroup”.

AFS stands for “Anti-fuzzy subgroup”.

DFS stands for “Dual-fuzzy subgroup”.

MLS stands for “Mixed level subgroup”.

$\boldsymbol{T}$ stands for “General T-norm”.

$\boldsymbol{T^*}$ stands for “General T-conorm”.

$\boldsymbol{T_1}$ stands for “T-norm

$T_1(m,u) = min\{m,u\}$ ”.

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T_1^* stands for "T-conorm $T_1^*(m,u) = max\{m,u\}$".

$K \lesssim H$ stands for "K is a subgroup of H".

III. SOME PROPOSED NOTIONS

A. *DFS of a Crisp Group*

Definition III.1A fuzzy subset α of a group U is termed as a DFS of P if $\forall\, m,u \in P$, the subsequent conditions are fulfilled:

(i) $min\{\alpha(m),\alpha(u)\} \le \alpha(mu) \le max\{\alpha(m),\alpha(u)\}$

(ii) $\alpha(m^{-1}) = \alpha(m)$

Example III.1 Let $P = \{e, a\}$ (order of a is 2) and $\alpha = \{(e,\ 0.3),\ (a, 0.3)\}$. Here α is a DFS of P.

Theorem III.1Complement of a DFS is also a DFS.

Proof: Let α be a DFS of a group P. Then $m,u \in P$

$$min\{\alpha(m),\alpha(u)\} \le \alpha(mu) \le max\{\alpha(m),\alpha(u)\}$$

$$\text{Or, } 1 - min\{\alpha(m),\alpha(u)\} \ge 1 - \alpha(mu) \ge 1 - max\{\alpha(m),\alpha(u)\}$$

$$\text{Or, } 1 - max\{\alpha(m),\alpha(u)\} \le \alpha^c(mu) \le 1 - min\{\alpha(m),\alpha(u)\}$$

$$\text{Or, } min\{\alpha^c(m),\alpha^c(u)\} \le 1 - max\{\alpha(m),\alpha(u)\} \le \alpha^c(mu) \le 1 - min\{\alpha(m),\alpha(u)\} \le max\{\alpha^c(m),\alpha^c(u)\}$$

$$\text{Or, } min\{\alpha^c(m),\alpha^c(u)\} \le \alpha^c(mu) \le max\{\alpha^c(m),\alpha^c(u)\} \qquad (1)$$

Hence from(1) complement of a DFS is also a DFS.

Theorem III.2Product of two DFS is also a DFS.

Proof: To prove this one we need to consider both Definition II.10as well as Definition II.11. Let α and β be two DFS of a group P. Then

$$\begin{aligned}(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1} &= (\alpha\times\beta)(m_1m_2^{-1},\ u_1u_2^{-1})\\ &= min\{\alpha(m_1m_2^{-1}),\beta(u_1u_2^{-1})\} && \text{(by Definition II.10)}\\ &\ge min\{min\{\alpha(m_1),\alpha(m_2)\}, min\{\beta(u_1),\beta(u_2)\}\} && \text{(As } \alpha \text{ and } \beta \text{ are FS of } P\text{)}\\ &= min\{min\{\alpha(m_1),\beta(u_1)\}, min\{\alpha(m_2),\beta(u_2)\}\}\\ &= min\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\} && \text{(by Definition II.10)} \qquad (2)\end{aligned}$$

Hence $\alpha\times\beta$ is a FS of $P\times P$.

Again,

$$\begin{aligned}(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1} &= (\alpha\times\beta)(m_1m_2^{-1},\ u_1u_2^{-1})\\ &= max\{\alpha(m_1m_2^{-1}),\beta(u_1u_2^{-1})\} && \text{(by Definition II.11)}\\ &\le max\{max\{\alpha(m_1),\alpha(m_2)\}, max\{\beta(u_1),\beta(u_2)\}\} && \text{(As } \alpha \text{ and } \beta \text{ are AFS of } P\text{)}\\ &= max\{max\{\alpha(m_1),\beta(u_1)\}, max\{\alpha(m_2),\beta(u_2)\}\}\\ &= max\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\} && \text{(by Definition II.11)} \qquad (3)\end{aligned}$$

Hence $\alpha\times\beta$ is an AFS of $P\times P$.

Clearly from (2) and (3) $\alpha\times\beta$ is a DFS of $P\times P$.

Definition III.2Let α and β be fuzzy subsets of P and U, respectively. Then the set $\alpha_t \times \overline{\beta_t} = \{(p,q) \in P\times U : \alpha(p) \ge t\ \&\beta(q) \le t\}$ is called a mixed level subset (mixed t-level subset) of $P\times U$.

B. *Direct Product of a FS and an AFS.*

Let α be a FS and β be an AFS of the groups P and U respectively. Then the direct product of α and β denoted as $\alpha\times\beta$ gives four consequences:

- $\alpha\times\beta$ is a FS of $P\times U$
- $\alpha\times\beta$ is an AFS of $P\times U$
- $\alpha\times\beta$ is a DFS of $P\times U$
- $\alpha\times\beta$ is neither a FS nor an AFS of $P\times U$

Proposition III.1Let α be a FS of a group P and β be an AFS of a group U. Then $\alpha\times\beta$ is a FS of $P\times U$ iff $\forall(m_1,u_1),(m_2,u_2)\in P\times U$

$(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}\geq min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$ (In this case Definition II.10has been considered) and $\alpha\times\beta$ is an AFS of $P\times U$ iff $\forall(m_1,u_1),(m_2,u_2)\in P\times U$

$(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}\leq max\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$ (Here Definition II.11has been considered).

Proof: Let $\alpha\times\beta$ is a FS of $P\times U$.

Then $\forall(m_1,u_1),(m_2,u_2)\in P\times U$,

$$\begin{aligned}(\alpha\times\beta)&(m_1,u_1)(m_2,u_2)^{-1}\\ &\geq min\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\}\\ &= min\{min\{\alpha(m_1),\beta(u_1)\},min\{\alpha(m_2),\beta(u_2)\}\}\\ &= min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\} \qquad (4)\end{aligned}$$

Conversely, let

$(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}\geq min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$.

Then

$$\begin{aligned}(\alpha\times\beta)&(m_1,u_1)(m_2,u_2)^{-1}\\ &\geq min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}\\ &= min\{min\{\alpha(m_1),\beta(u_1)\},min\{\alpha(m_2),\beta(u_2)\}\}\\ &= min\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\} \qquad (5)\end{aligned}$$

Similarly, for the second part let $\alpha\times\beta$ is an AFS of $P\times U$.

Then

$$\begin{aligned}(\alpha\times\beta)&(m_1,u_1)(m_2,u_2)^{-1}\\ &\leq max\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\}\\ &= max\{max\{\alpha(m_1),\beta(u_1)\},max\{\alpha(m_2),\beta(u_2)\}\}\\ &= max\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\} \qquad (6)\end{aligned}$$

Conversely, let

$(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}\leq max\{\alpha(n_1),\alpha(n_2),\beta(v_1),\beta(v_2)\}$.

Then

$$\begin{aligned}(\alpha\times\beta)&(m_1,u_1)(m_2,u_2)^{-1}\\ &\leq max\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}\\ &= max\{max\{\alpha(m_1),\beta(u_1)\},max\{\alpha(m_2),\beta(u_2)\}\}\\ &= max\{(\alpha\times\beta)(m_1,u_1),(\alpha\times\beta)(m_2,u_2)\} \qquad (7)\end{aligned}$$

Hence, from(4), (5), (6) and (7)Proposition III.1 can be easily concluded.

Definition III.3Let α be a FS of a group P and β be an AFS of a group U. Then $\alpha\times\beta$ is a DFS of $P\times U$ iff $\forall(m_1,u_1),(m_2,u_2)\in P\times U$

$$min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}\leq(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}$$
$$\leq max\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$$

Note that to prove $\alpha\times\beta$ is neither a FS nor an AFS of $P\times U$ we need to consider both Definition II.10and Definition II.11. There exist some examples for which $\alpha\times\beta$ is neither a FS nor an AFS. With the following examples, we have justified our claim.

Example III.2Let $P=\{e,a\}$ (order of a is 2) and $U=\{e',m,\ u,mu\}$ (Klein 4-group). Let $\alpha=\{(e,0.3),(a,0.2)\}$ is a FS of P and $\beta=\{(e',0.8),\ (m,0.6),(u,0.6),\ (mu,0.2)\}$ is an AFS of U. Let the following Table III.1 and Table III.2represent $\alpha\times\beta$ according to Definition II.10 and Definition II.11respectively.

Table III.1: Representation of $\alpha\times\beta$ by Definition II.10

Minimum	β	e'	m	u	mu
α		0.8	0.6	0.6	0.2
e	0.3	0.3	0.3	0.3	0.2
a	0.2	0.2	0.2	0.2	0.2

Table III.2: Representation of $\alpha\times\beta$ by Definition II.11

Maximum	β	e'	m	u	mu
α		0.8	0.6	0.6	0.2
e	0.3	0.8	0.6	0.6	0.3
a	0.2	0.8	0.6	0.6	0.2

Here observe that

$$\begin{aligned}(\alpha\times\beta)(e,m)(e,u)^{-1} &= (\alpha\times\beta)(e,m)(e^{-1},u^{-1})\\ &= (\alpha\times\beta)(e,m)(e,u)\\ &= (\alpha\times\beta)(e,mu)\\ &= 0.2 \text{ (by Table III.1)}\\ &\ngeq 0.3\\ &= min\{0.3,0.6,0.6\}\\ &= min\{\alpha(e),\ \beta(m),\ \beta(u)\} \qquad (8)\end{aligned}$$

From (8) it is evident that $\alpha\times\beta$ is a not a FS of $P\times U$.

Again

$$\begin{aligned}(\alpha\times\beta)(e,m)(a,m)^{-1} &= (\alpha\times\beta)(e,m)(a^{-1},m^{-1})\\ &= (\alpha\times\beta)(e,m)(a,m)\\ &= (\alpha\times\beta)(a,e')\end{aligned}$$

$= 0.8$ (by Table III.2)

$\nleq 0.6$

$= \max\{0.3, 0.2, 0.6\}$

$= max\{\alpha(e),\ \alpha(a),\ \beta(m)\}$ (9)

Evidently from (9) $\alpha\times\beta$ is a not an AFS of $P\times U$. So, $\alpha\times\beta$ is a neither a FS nor an AFS of $P\times U$.

Example III.3Let $U=\{1,-1,i,-i\}$ **be a group with regular multiplication. Let** α **is defined as** $\alpha: U\to[0,1]$ **by**

$$\alpha(p)=\begin{cases}1 & if\ p=1\\ 0.5 & if\ p=-1\\ 0 & if\ p=i,-i\end{cases} \text{ and } \alpha^c(p)=\begin{cases}0 & if\ p=1\\ 0.5 & if\ p=-1\\ 1 & if\ p=i,-i\end{cases}$$

Notice that here α is a FS and α^c is an AFS of U. Let the following Table III.3 and Table III.4 represent $\alpha\times\alpha^c$ according to Definition II.10and Definition II.11respectively.

Table III.3:Representation of $\alpha\times\alpha^c$ **byDefinition II.10**

Minimum	α	1	-1	i	-i
α^c		1	0.5	0	0
1	0	0	0	0	0
-1	0.5	0.5	0.5	0	0
i	1	1	0.5	0	0
-i	1	1	0.5	0	0

Table III.4:Representation of $\alpha\times\alpha^c$ **byDefinition II.11**

Maximum	α	1	-1	i	-i
α^c		1	0.5	0	0
1	0	1	0.5	0	0
-1	0.5	1	0.5	0.5	0.5
i	1	1	1	1	1
-i	1	1	1	1	1

Note that if $\alpha\times\alpha^c$ is a FS of $U\times U$ then

$(\alpha\times\alpha^c)(1,1)\geq(\alpha\times\alpha^c)(m,u)\ \forall(m,u)\in\alpha\times\alpha^c$.

Clearly, from Table III.3

$(\alpha\times\alpha^c)(1,1)=0\leq(\alpha\times\alpha^c)(m,u)\ \forall(m,u)\in\alpha\times\alpha^c$.

So, $\alpha\times\alpha^c$ is not a FS of $U\times U$.

Similarly, if $\alpha\times\alpha^c$ is an AFS of $U\times U$. Then

$(\alpha\times\alpha^c)(1,1)\leq(\alpha\times\alpha^c)(m,u)\ \forall(m,u)\in\alpha\times\alpha^c$.

Evidently, from Table III.4

$(\alpha\times\alpha^c)(1,1)=1\geq(\alpha\times\alpha^c)(m,u)\ \forall(m,u)\in\alpha\times\alpha^c$.

Hence $\alpha\times\alpha^c$ is not an AFS of $U\times U$.

So, $\alpha\times\alpha^c$ is neither a FS nor an AFS of $U\times U$.

Again, there exist some examples which are DFS. The following is one of them:

Example III.4Let $U=\{e,a\}$ **(order of** α **is 2) and** $\alpha=\{(e,0.3),(a,0.2)\}$ **. Here** α **is a FS and** α^c **is an AFS of** U**. Let the followingTable III.5 and Table III.6 represent** $\alpha\times\alpha^c$ **according to Definition II.10 and Definition II.11respectively.**

Table III.5:Representation of $\alpha\times\alpha^c$ **byDefinition II.10**

Minimum	α	e	a
α^c		0.3	0.2
e	0.7	0.3	0.2
a	0.8	0.3	0.2

Table III.6:Representation of $\alpha\times\alpha^c$ **byDefinition II.11**

Maximum	α	e	a
α^c		0.3	0.2
e	0.7	0.7	0.7
a	0.8	0.8	0.8

FromTable III.5 and Table III.6notice that

$\forall(m_1,u_1),\ (m_2,u_2)\in U\times U$

$min\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$

$\leq(\alpha\times\beta)(m_1,u_1)(m_2,u_2)^{-1}$

$\leq max\{\alpha(m_1),\alpha(m_2),\beta(u_1),\beta(u_2)\}$

Hence $\alpha\times\alpha^c$ a dual FS of $U\times U$.

Theorem III.3Let β and β' be fuzzy subsets of G and H, respectively. Then $\forall t\in[0,1]$ $\overline{(\beta\times\beta')}_t=\overline{\beta}_t\times\overline{\beta}'_t$.

Proof: Let $(m,u)\in\overline{(\beta\times\beta')}_t$. Then

$(\beta\times\beta'_t)(m,u)\leq t$

Or, $\max\{\beta(m),\beta'_t(u)\}\leq t$ (by Definition II.11)

Or, $\beta(m)\leq t$ and $\beta'_t(u)\leq t$

Or, $m\in\overline{\beta}_t$ and $u\in\overline{\beta}'_t$

Or, $(m,u)\in\overline{\beta}_t\times\overline{\beta}'_t$

Or, $\overline{(\beta\times\beta')}_t\subseteq\overline{\beta}_t\times\overline{\beta}'_t$ (10)

Again, let $(m,u)\in\overline{\beta}_t\times\overline{\beta}'_t$. Then

$m\in\overline{\beta}_t$ and $u\in\overline{\beta}'_t$

Or, $\beta(m)\leq t$ and $\beta'_t(u)\leq t$

Or, $\max\{\beta(m),\beta'_t(u)\}\leq t$

Or, $(\beta\times\beta'_t)(m,u)\leq t$ (by Definition II.11)

Or, $(m,u)\in\overline{(\beta\times\beta')}_t$

Or, $\overline{\beta}_t\times\overline{\beta}'_t\subseteq\overline{(\beta\times\beta')}_t$ (11)

From (10) and(11) $\forall t\in[0,1]$

$\overline{(\beta\times\beta')}_t=\overline{\beta}_t\times\overline{\beta}'_t$.

Theorem III.4Let α be a FS and β be an AFS of the groups P and U respectively. Also, let $\forall t \in [0,1]$ $\alpha(e) \geq t$, $\beta(e') \leq t$, α_t be level sets of P and $\overline{\beta_t}$ be lower level sets of U, where e and e' are neutral elements of P and U respectively. Then $\alpha_t \times \overline{\beta_t} \lesssim P \times U$.

Proof: As $\alpha(e) \geq t$, and $\beta(e') \leq t$, $(e,e') \in \alpha_t \times \overline{\beta_t}$ i.e. $\alpha_t \times \overline{\beta_t} \neq \phi$. Let $a_1 = (m_1,u_1) \in \alpha_t \times \overline{\beta_t}$ and $a_2 = (m_2,u_2)$, $\in \alpha_t \times \overline{\beta_t}$. To show that $a_1a_2^{-1} \in \alpha_t \times \overline{\beta_t}$

Here
$$a_1a_2^{-1} = (m_1,u_1)(m_2,u_2)^{-1} = (m_1,u_1)(m_2^{-1},u_2^{-1}) = (m_1m_2^{-1},u_1u_2^{-1})$$

Since α is a FS of P we will have $\alpha(m_1m_2^{-1}) \geq \min\{\alpha(m_1),\alpha(m_2)\} \geq t$ i.e. $m_1m_2^{-1} \in \alpha_t$. Again, as β is an AFS of U we know that $\beta(u_1u_2^{-1}) \leq max\{\beta(u_1),\beta(u_2)\} \leq t$ i.e. $u_1u_2^{-1} \in \overline{\beta_t}$. So, $a_1a_2^{-1} = (m_1m_2^{-1},u_1u_2^{-1}) \in \alpha_t \times \overline{\beta_t}$. So, $\alpha_t \times \overline{\beta_t} \lesssim P \times U$.

Theorem III.5Let α and β be fuzzy subsets of the groups P and U respectively. Also, let $\forall t \in [0,1]$ $\alpha(e) \geq t$, $\beta(e') \leq t$ (where e and e' are neutral elements of P and U respectively). Again, let α_t be level sets of P, $\overline{\beta_t}$ be lower level sets of U and $\alpha_t \times \overline{\beta_t} \lesssim P \times U$. Then α is a FS of P and β is an AFS of U.

Proof: Let $a_1 = (m_1,u_1) \in P \times U$ and $a_2 = (m_2,u_2) \in P \times U$. So, $m_1, m_2 \in P$ and $u_1,u_2 \in U$. Again, let $\alpha(m_1) = t_1$ and $\alpha(m_2) = t_2$ for some $t_1,t_2 \in [0,1]$ .Then $m_1 \in \alpha_{t_1}$ and $m_2 \in \alpha_{t_2}$. Suppose that $t_1 < t_2$. Then $\alpha_{t_2} \subseteq \alpha_{t_1}$. So, $m_2 \in \alpha_{t_1}$ and hence both $m_1, m_2 \in \alpha_{t_1}$. Again, as $\alpha_{t_1} \times \overline{\beta_{t_1}} \lesssim P \times U$ $\alpha_{t_1} \lesssim P$ and $\overline{\beta_{t_1}} \lesssim U$ Thus, for $m_1, m_2 \in \alpha_{t_1}$, $m_1m_2^{-1} \in \alpha_{t_1}$, wherefrom

$$\alpha(m_1m_2^{-1}) \geq t_1 = min\{\alpha(m_1),\alpha(m_2)\} .$$

So, α is a FS of P.

Similarly, let $\beta(u_1) = s_1$ and $\beta(u_2) = s_2$ for some $s_1,s_2 \in [0,1]$. Then $u_1 \in \overline{\beta_{s_1}}$ and $u_2 \in \overline{\beta_{s_2}}$ .Suppose that $s_1 < s_2$. Then $\overline{\beta_{s_1}} \subseteq \overline{\beta_{s_2}}$. So, $u_1 \in \overline{\beta_{s_2}}$ and hence both $u_1,u_2 \in \overline{\beta_{s_2}}$. Again, as $\alpha_{s_2} \times \overline{\beta_{s_2}} \lesssim P \times U$ $\alpha_{s_2} \lesssim P$ and $\overline{\beta_{s_2}} \lesssim U$. Thus, for $u_1,u_2 \in \overline{\beta_{s_2}}$, $u_1u_2^{-1} \in \overline{\beta_{s_2}}$, wherefrom

$$\beta(u_1u_2^{-1}) \leq s_2 = max\{\beta(u_1),\beta(u_2)\} .$$

So, β is an AFS of U.

Definition III.4Let α be a FS and β be an AFS of the groups P and U respectively. Then $\forall t \in [0,1]$ the subgroups $\alpha_t \times \overline{\beta_t}$ with $\alpha(e) \geq t$ and $\beta(e') \leq t$ (e and e' are neutral elements of P and U respectively) are called mixed level subgroups (mixed t-level subgroup) of $P \times U$.

Notice that in all the notions discussed so far, we have used T_1 and T_1^* . Using T and T^* we can redefine and generalize those notions. In the following section we have done that.

IV. SOME REDEFINED AND GENERALIZED NOTIONS

Definition IV.1A fuzzy subset α of a group P is termed as a DFS of P if $\forall m, u \in P$, the subsequent conditions are fulfilled:

(i) $T(\alpha(m),\alpha(u)) \leq \alpha(mu) \leq T^*(\alpha(m),\alpha(u))$

(ii) $\alpha(m^{-1}) = \alpha(m)$

Definition IV.2Let α be a FS of a group P and β be an AFS of a group U. Then $\alpha \times \beta$ is a FS of $P \times U$ iff $\forall (m_1,u_1), (m_2,u_2) \in P \times U$,

$$(\alpha \times \beta)(m_1,u_1)(m_2,u_2)^{-1} \geq T(T(\alpha(m_1),\beta(u_1)),T(\alpha(m_2),\beta(u_2)))$$

For the above definition product of two fuzzy sets will be taken as $(\alpha \times \beta)(m,u) = T(\alpha(m),\beta(u))$.

Definition IV.3Let α be a FS of a group P and β be an AFS of a group U. Then $\alpha \times \beta$ is an AFS of $P \times U$ iff $\forall (m_1,u_1),(m_2,u_2) \in P \times U$

$$(\alpha \times \beta)(m_1,u_1)(m_2,u_2)^{-1} \leq T^*(T^*(\alpha(m_1),\beta(u_1)),T^*(\alpha(m_2),\beta(u_2)))$$

For the above definition product of two fuzzy sets will be taken as $(\alpha \times \beta)(m,u) = T^*(\alpha(m),\beta(u))$.

Definition IV.4Let α be a FS of a group P and β be an AFS of a group U. Then $\alpha \times \beta$ is a FS of $P \times U$ iff $\forall (m_1,u_1),$

$(m_2,u_2) \in P \times U$

$$T(T(\alpha(m_1),\beta(u_1)),T(\alpha(m_2),\beta(u_2))) \leq (\alpha \times \beta)(m_1,u_1)(m_2,u_2)^{-1} \leq T^*(T^*(\alpha(m_1),\beta(u_1)),T^*(\alpha(m_2),\beta(u_2))).$$

V. CONCLUSION

We have proposed some new notions, like DFS, direct product of a FS with an AFS, mixed level subset, MLS, etc. To the best of our knowledge, these concepts have been introduced for the first time in this paper. We have also discussed some new interesting theories and propositions regarding these notions.

These newly defined perceptions will surely generate some scopes of future researches.

For instance, there are still some unexplored aspects on the algebraic structures of DFS as well as a direct product of a FS with an AFS. One can also modify these ideas of AFS, DFS as well as direct product of a FS with an AFS and implement them in different algebras, like BH/BE/CI/BCK/BCH/BCI/BF, etc, which are extensively used in pattern recognition problems, medical science, control theory, decision making, robotics, artificial intelligence and also numerous other applied fields.

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