Certain Classical Properties of two Variable Generalized Hypergeometric Polynomials

G. N. V. Kishore, V. S. Bhagavan, P. L. Rama Kameswarei, I.V.Ravi Kumar

Abstract: The main aim of this paper is to study some classical properties of two variable generalized hypergeometric polynomial becomes variable from generalized hypergeometric polynomials of the set In(α; β; x, y), as addition and multiplication formulas, integral representations as integral boundaries, real integrals, infinitely many infinite simple representations and duplications. Finished Integrals These results can be suitably applied to obtain many additional applications, including known and unknown hypergeometric functions.

Keywords — Special functions, two variable generalized hypergeometric polynomials $I_n(\alpha; \beta; x, y)$,

Mathematics Subject Classification: 33C45, 33C50, 33C80.

I. INTRODUCTION

The polynomial set $I_n(\alpha;\beta; x; y)$; is a product of y^n and hypergeometric function. Thus it is possible to derive many properties including the ascending descending recurrence relations which are very much essential for obtaining the generating functions by the Truesdell method. Due to the important role played by hypergeometric polynomials in the applications of applied physics and mathematics, the theory of created applications has been developed in different directions and wide applications have been obtained in various fields of analysis, such as the general theory of unlimited series, linear differential equations, statistics (various types of distributions), operations research and complex functions of variables. The hypernetric functions have also maintained their importance in science and technology.

The principle interest in our result lies in the fact that a number of special cases would yield inevitably too many new and known results of the theory of special functions. It is worth recalling that several of the fundamental identities to the two variable Laguerre polynomials are derived as the special case of our result. Moreover, a finite difference formula for $I_n(\alpha;\beta ; x; y)$; has been derived.

II. DEFINITION

The two variable generalized hypergeometric polynomial set $\{I_n(\alpha; \beta ; x; y)\}$n=0,1,2… have been defined and specified by the series as

$$(2.1)\qquad I_n(\alpha;\beta;x,y)=\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k x^n y^{n-k}}{n!k!(\beta)_k},$$

and the generating relations for $I_n(\alpha; \beta ; x; y)$ are given by

$$(2.2)\qquad \sum_{n=0}^{\infty}\frac{I_n(\alpha;\beta;x,y)t^n}{n!}=e^{yt}\,{}_1F_1\left[\alpha;\beta;-xt\right],$$

$$(2.3)\qquad \sum_{n=0}^{\infty}\frac{(\gamma)_n I_n(\alpha;\beta;x,y)t^n}{n!}=(1-yt)^{-\gamma}\,{}_2F_1\left[\gamma,\alpha;\beta;\frac{-tx}{1-yt}\right],$$

Where α is a real number and β is neither zero nor a negative integer.

III. APPLICATION

The following application has been deduced from the generating relation (2.2):

$$(2.4)\qquad \sum_{n=0}^{\infty}\frac{x^n L_n^{(\gamma)}(x,y)z^n}{(1+\gamma)_n}=exp(yz)\,{}_0F_1\left[-;1+\gamma;\frac{x}{y}\right].$$

The differential equation satisfied by the two variable generalized hyper-geometric polynomials (2VGHP) $I_n(\alpha ;\beta ; x; y)$ is

$$(2.5)\qquad \{y(x-y)D^2-[(n+\alpha-1)x-(\beta+2n-2)y]D$$
$$-n(\beta+n-1)\}\,I_n(\alpha;\beta;x,y)=0.$$

These polynomials satisfy the following differential recurrence relations:

$$(2.6)\qquad DI_n(\alpha;\beta;x,y)$$
$$=\frac{1}{y(x-y)}\left[(\beta+n)I_{n+1)}(\alpha;\beta;x,y)+[n+\alpha)x-(\beta+2n)y]\right]I_n(\alpha;\beta;x,y),$$

$$(2.7)\qquad DI_n(\alpha;\beta;x,y)=nI_{n-1}(\alpha;\beta;x,y),$$

Where D = d/dy

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G.N.V.Kishore, Department of Mathematics, Department of Mathematics, SRKR Engineering College, China Amiram, Bhimavaram, West Godavari District - 534 204, Andhra Pradesh, India.. E-mail: kishore.apr2@gmail.com

V.S.Bhagavan, Department of Mathematics, Department of Mathematics, KL University Vaddeswaram, Guntur, India-522502.
Email: drvsb002@gmail.com

P.L.Rama Kameswari., Department of Mathematics., Swarnandhra college of Engineering and Technology., Narsapuram – 534280, West Godavari District., Andhra Pradesh., INDIA
Email: drilrk@gmail.com

Mr. I.V.Ravi Kumar, Department of ECE., Swarnandhra college of Engineering and Technology., Narsapuram – 534280, West Godavari District., Andhra Pradesh., INDIA.
Mail : ravi.ivrk@gmail.com

1. CLASS $I_n(\alpha;\beta;x,y)$ IES OF

Hence an attempt to obtain some classical properties of $I_n(\alpha;\beta;x;y)$, such as "finite

difference formula, Multiplication and Addition formulae and integral representations of various types."

1. *Formulae:*

To obtain the addition and multiplication formulae, we have used the well-known results

$$(3.1.1)\qquad D^n(u,v)=\sum_{k=0}^{n}\binom{n}{k}(D^{n-k}u)(D^k v),$$

$$(3.1.2)\qquad f(x+y)=\sum_{n=0}^{\infty}\frac{f^n(x)y^n}{n!}$$

and

$$(3.1.3)\qquad f(xy)=\sum_{n=0}^{\infty}\frac{(y-1)^n x^n f^n(x)}{n!},$$

To derive the finite difference formula, we have used the operator $\Delta\alpha$, where

$$(3.1.4)\qquad \Delta_\alpha f(\alpha)=f(\alpha+1)-f(\alpha),$$

Which is further generalization gives

$$(3.1.5)\qquad \Delta_\alpha^n f(\alpha)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(\alpha+k).$$

To obtain the different integrals, we have used the following well known results

$$(3.1.6)\qquad \int_0^\infty\int_0^\infty \phi(x+y)x^\alpha y^\beta\,dx\,dy=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\int_0^\infty \phi(z)z^{\alpha+\beta+1}dz,$$

$$(3.1.7)\qquad \int_0^\infty(1-ax^h)^{-v}x^{s-1}dx=h^{-1}a^{-\frac{s}{h}}\beta\left(\frac{s}{h},v-\frac{s}{h}\right),$$

$$|arg\alpha|<\pi,h>0,0<Re(s)<hRe(v).$$

$$(3.1.8)\qquad \sqrt{\pi}\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right),$$

$$(3.1.9)\qquad \Gamma(z)\Gamma(1-z)=\frac{\pi}{sin\pi z},$$

$$(3.1.10)\qquad \Gamma\left(\rho-k+\tfrac{1}{2}\right)=\int_{-\infty}^{\infty}exp(-t^2)t^{2(\rho-k)}dt,$$

$$(3.1.11)\qquad \frac{(a)_k}{(b)_k}=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a+k-1}(1-t)^{b-a-1}dt.$$

2.Addition and Multiplication Formula

Theorem. Prove that

$$(3.2.1)\qquad I_n(\alpha;\beta;x,y+z)=\sum_{k=0}^{\infty}\binom{n}{k}I_{n-k}(\alpha;\beta;x,y)(z)^k$$

and

$$(3.2.2)\qquad I_n(\alpha;\beta;x,yz)=\sum_{k=0}^{\infty}\binom{n}{k}I_{n-k}(\alpha;\beta;x,y)(z-1)^k(y)^k$$

Proof: we have

$$I_n(\alpha;\beta;x,y+z)=\sum_{k=0}^{\infty}\frac{\frac{d^k}{dy^k}(I_n(\alpha;\beta;x,y)z^k}{k!}$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n}\frac{(-n)_p(\alpha)_p x^p z^k}{p!k!(\beta)_p}\frac{d^k}{dy^k}(y^{n-p})$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n}\frac{(-n)_p(\alpha)_p(n-p)!z^k x^p y^{n-p-k}}{k!p!(n-p-k)!(\beta)_p}$$
$$=\sum_{k=0}^{\infty}\binom{n}{k}I_{n-k}(\alpha;\beta;x,y)(z)^k,$$

Which is (3.2.1)

$$(3.2.2)\qquad I_n(\alpha;\beta;x,yz)=\sum_{k=0}^{\infty}\frac{(z-1)^k y^k\frac{d^k}{dy^k}[I_n(\alpha;\beta;x,y)]}{k!}$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n}\frac{(-n)_p(\alpha)_p x^p(z-1)^k y^k}{k!p!(\beta)_p}\frac{d^k}{dy^k}(y^{n-p})$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k}{k!}I_{n-k}(\alpha;\beta;x,y)(z-1)^k(y)^k.$$
$$=\sum_{k=0}^{\infty}\binom{n}{k}I_{n-k}(\alpha;\beta;x,y)(z-1)^k(y)^k.$$

Theorem peovwd.

3.Finite Difference Formula

Theorem. Recommend

$$(3.3.1)\qquad I_n(\alpha+\lambda;\beta+\lambda;x,y)=\frac{(-1)^n\Gamma(\beta+\lambda)x^{-\lambda}y^{n+\lambda}}{\Gamma(\alpha+\lambda)}\Delta_\lambda^n\left[\frac{\Gamma(\alpha+\lambda}{\Gamma(\beta+\lambda)}x^\lambda y^{-\lambda}\right],$$

$$\text{where}\qquad \Delta_\lambda f(\lambda)=f(\lambda+1)-f(\lambda)$$

$$\text{and}\qquad \Delta_\lambda^n f(\lambda)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(\lambda+k).$$

Proof: we know that

$$I_n(\alpha;\beta;x,y)=\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k}{(\beta)_k k!}x^k y^{n-k}$$
$$=\frac{(-1)^n\Gamma(\beta)}{\Gamma(\alpha)}\sum_{k=0}^{n}\frac{(-1)^{n-k}\binom{n}{k}\Gamma(\alpha+k)x^k y^{n-k}}{\Gamma(\beta+k)}$$

Now, writing α +λ and β + λ for α and β respectively, we have

$$I_n(\alpha+\lambda;\beta+\lambda;x,y)=\frac{(-1)^n\Gamma(\beta+\lambda)x^{-\lambda}y^{n+\lambda}}{\Gamma(\alpha+\lambda)}\sum_{k=0}^{n}\frac{(-1)^{n-k}\binom{n}{k}\Gamma(\alpha+\lambda+k)x^{\lambda+k}y^{-\lambda-k}}{\Gamma(\beta+\lambda+k)}$$
$$=\frac{(-1)^n\Gamma(\beta+\lambda)x^{-\lambda}y^{n+\lambda}}{\Gamma(\alpha+\lambda)}\Delta_\lambda^n\left[\frac{\Gamma(\alpha+\lambda}{\Gamma(\beta+\lambda)}x^\lambda y^{-\lambda}\right].$$

Make a theorem

2. *Integral Representations*

Integral representations of polynomial set $I_n(\alpha;\beta;x;y)$ are

(i). Contour Integral representation

From (2.3)

$$(3.4.1)\qquad \sum_{n=0}^{\infty}\frac{(\gamma)_n I_n(\alpha;\beta;x,y)t^n}{n!}=(1-yt)^{-\gamma}\,{}_2F_1\left[\gamma;\alpha;\beta;\frac{-tx}{1-yt}\right].$$

Let us write

$$(3.4.2)\qquad f(t)=(1-yt)^{-\gamma}\,{}_2F_1\left[\gamma;\alpha;\beta;\frac{-tx}{1-yt}\right]$$

By Maclaurin's theorem

$$(3.4.3)\qquad f(t)=\sum_{n=0}^{\infty}\frac{f^n(0)t^n}{n!}.$$

where $f^n(0)$, n = 0,1,2….are given by the means of integral

$$(3.4.4)\qquad f^n(0)=\frac{n!}{2\pi i}$$

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Thus from (3.4.1) and (3.4.4), we arrive at the following theorem

Theorem

If $(1-yt)^{-\gamma}\ {}_2F_1\left[\gamma;\alpha;\beta;\frac{-tx}{1-yt}\right]=\sum_{n=0}^{\infty}\frac{(\gamma)_n I_n(\alpha;\beta;x,y)t^n}{n!}$

then

$$(3.4.5)\ I_n(\alpha;\beta;x,y)=\frac{n!}{2\pi i(\gamma)_n}\int^{(0+)}t^{-n-1}(1-yt)^{-\gamma}\ {}_2F_1\left[\gamma;\alpha;\beta;\frac{-tx}{1-yt}\right]dt,$$

Where the counter integration encircles the origin of the t- plane in positive direction

(ii). Real Integral Representation

If , in the equation(3.4.5), we replace the contoure $e^{i\theta}$ then we get

$$I_n(\alpha;\beta;x,y)=\frac{n!}{2\pi(\gamma)_n}\int_0^{2\pi}e^{in\theta}(1-ye^{i\theta})^{-\gamma}\ {}_2F_1\left[\gamma;\alpha;\beta;-\frac{e^{i\theta}x}{1-ye^{i\theta}}\right]d\theta$$

$$=\frac{n!}{2\pi(\gamma)_n)}\sum_{k=0}^{\infty}\frac{(\gamma)_k(\alpha)_k(-x)^k}{k!(\beta)_k}\int_0^{2\pi}e^{-in\theta}(1-ye^{i\theta})^{-\gamma}d\theta$$

$$=\frac{n!}{2\pi(\gamma)_n)}\sum_{k=0}^{\infty}\frac{(-1)^k x^k(\gamma)_k(\alpha)_k}{k!(\beta)_k}\int_0^{2\pi}e^{(k-n)i\theta}(1-ye^{i\theta})^{-\gamma-k}d\theta$$

$$=\frac{n!}{2\pi(\gamma)_n)}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k x^k(\gamma)_k(\gamma+k)_s(\alpha)_k}{k!s!(\beta)_k}\int_0^{2\pi}e^{(k-n)i\theta}(ye^{i\theta})^s d\theta$$

$$=\frac{n!}{2\pi(\gamma)_n)}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k x^k y^s(\gamma)_{k+s}(\alpha)_k}{k!s!(\beta)_k}\int_0^{2\pi}e^{(k-n+s)i\theta}d\theta$$

$$=\frac{n!}{2\pi(\gamma)_n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k x^k y^s(\gamma)_{k+s}(\alpha)_k}{k!s!(\beta)_k}\int_0^{2\pi}Cis(k-n+s)\theta d\theta.$$

where $Cis\phi=cos\phi+isin\phi$

Consequently, we get

Theorem

$$(3.4.6)\ I_n(\alpha;\beta;x,y)=\frac{n!}{2\pi(\gamma)_n}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k x^k y^s(\gamma)_{k+s}(\alpha)_k}{k!s!(\beta)_k}\int_0^{2\pi}Cis(k-n+s)\theta d\theta.$$

(iii). Infinite single integral representation.

Take that

$$I_n(\alpha;\beta;x,y)=\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k}{k!(\beta)_k}x^k y^{(n-k)}$$

$$=\sum_{k=0}^{n}\frac{(-n)_k\Gamma(\alpha+k)}{k!(\beta)_k\Gamma(\alpha)}x^k y^{(n-k)}$$

$$=\sum_{k=0}^{n}\frac{(-n)_k\Gamma(\alpha+k-\frac{1}{2}+\frac{1}{2})}{k!(\beta)_k\Gamma(\alpha)}x^k y^{(n-k)}$$

$$=\sum_{k=0}^{n}\frac{(-n)_k x^k y^{n-k}}{k!(\beta)_k\Gamma(\alpha)}\int_{-\infty}^{\infty}exp(-t^2)t^{2(\alpha+k-\frac{1}{2}}dt$$

(by using (3.1.10))

$$I_n(\alpha;\beta;x,y)=\frac{(y)^n}{\Gamma(\alpha)}\int_{-\infty}^{\infty}exp(-t^2)t^{2(\alpha-1)}\ {}_1F_1\left[-n;\beta;\frac{t^2x}{y}dt\right]$$

Thus we have

Theorem

If Re(α) > ½ , then

$$I_n(\alpha;\beta;x,y)=\frac{y^n}{\Gamma(\alpha)}\int_{-\infty}^{\infty}exp(-t^2)t^{2(\alpha-1)}\ {}_1F_1\left[-n;\beta;\frac{t^2x}{y}dt\right]$$

(iv). Finite single integral representation.

Take that

$$I_n(\alpha;\beta;x,y)=\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k x^k y^{n-k}}{(\beta)_k k!}$$

$$=\frac{\Gamma(\beta)}{\Gamma(\alpha)(\Gamma(\beta-\alpha)}\sum_{k=0}^{n}\frac{(-n)_k x^k y^{n-k}}{k!}\int_0^1 t^{\alpha+k-1}(1-t)^{\beta-\alpha-1}dt$$

$$=\frac{\Gamma(\beta)y^n}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\left(1-\frac{xt}{y}\right)^n dt$$

$$=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}(y-xt)^n dt.$$

We conclude

Theorem. If Re(α) > 0 and Re(β – α) > 0, then

$$(4.9)\qquad I_n(\alpha;\beta;x,y)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}(y-xt)^n dt.$$

(v). Finite double integral representation

From Srivastava,H.M. and Karlsson,P.w. [11;P.275] that if Re(a) > 0; Re(b) > 0 and Re(c) > 0, then

$$(4.10)\ \int\int_D u^{a-1}v^{b-1}(1-u-v)^{c-1}dudv=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$$

Where D is the bounded by the lines $u\geq 0, v\geq 0$ and $u+v\leq 1.$

We have

$$I_n(\alpha;\beta;x,y)=\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k x^k y^{n-k}}{(\beta)_k k!}$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)}\sum_{k=0}^{n}\frac{(-n)_k\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha+k)\Gamma(\beta)(\gamma)_k x^k y^{n-k}}{k!\Gamma(\gamma+k)\Gamma(\beta+k)}$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)}\sum_{k=0}^{n}\frac{(-n)_k(\gamma)_k x^k y^{n-k}}{k!\Gamma(\beta+k)}\int\int_D u^{\alpha+k-1}v^{\beta-1}(1-u-v)^{\gamma-\alpha-\beta-1}dudv$$

[using (4.9)]

$$=\frac{\Gamma(\gamma)y^n}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\int\int_D u^{\alpha-1}v^{\beta-1}(1-u-v)^{\gamma-\alpha-\beta-1}\ {}_2F_1\left[-n;\gamma;\beta;\frac{xu}{y}\right]dudv$$

$$=\frac{\Gamma(\gamma)y^n}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\int\int_D u^{\alpha-1}v^{\beta-1}(1-u-v)^{\gamma-\alpha-\beta-1}I_n\left(\gamma;\beta;x,\frac{y}{u}\right)$$

Thus we arrive at

Theorem. Re(α), Re(β) > 0 and Re(γ– α –β) > 0, then

$$I_n(\alpha;\beta;x,y)=\frac{\Gamma(\gamma)y^n}{\Gamma(\gamma-\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)}\int\int_D u^{\alpha-1}v^{\beta-1}(1-u-v)^{\gamma-\alpha-\beta-1}I_n\left(\gamma;\beta;x,\frac{y}{u}\right)$$

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