Results in Partially Ordered Bipolar Metric Spacevia (α, ψ)-Contractive Condition

D. Ram Prasad, G.N.V.Kishore, B.Srinuvasa Rao, G. Adhi Lakshmi

***Abstract*: In this paper, we proved some Common(C) Coupled(C) Fixed (F) Point (P) Theorems in Ordered Bipolar metric spaces via (α, ψ) – contraction.**

***Index Terms*: Bipolar Metric Spaces, Partially ordered set, α-admissible, $\alpha-\psi$-Contractive mapping**

1. INTRODUCTION

In 1922, Banach [1] proved a fixed point theorem, which ensures that the existence and uniqueness of a fixed point under appropriate conditions . This result of Bänach is known as Banach fixed point theorem or contraction mapping principle. Fixed point theory is a mixture of Analysis, Topology & Geometry. Fixed Point Theory has been playing a vital role in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in a variety of fields as Chemistry, Biology, Economics, Engineering & Game theory and also in Physics. The fixed point theory has many important applications in numerical methods like Newton –Raphson Methods and establish Picard's Existence Theorem regarding existence and uniqueness of solution of first order differential equation , existence of a solution of integral equation and the system of linear equations.

Now a days, fixed point theory has been receiving the more attention on the partially ordered metric spaces - that is, the metric spaces endowed with the partial ordering .

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Turinici [2] extending the Banach contraction principles in the setting of partially ordered sets and formed the foundation a new trend in fixe d point theory. Afterward Ran and Reurings [3] developed some applications of Turinicis theorem for matrix equations and established few good results in this direction.

The results have been further extended by Nieto & Rod′rguez-L′opez [4] , [5] for non- decreasing mappings. Later on , Bhäskar and Lakshmi kantham [6], [7] introduced the new notion of coupled fixed points for mappings which were satisfying the mixed monotone property in partially ordered spaces and discussed the existence & uniqueness of a solution for the periodic boundary valued problem . Subsequently, Läkshmi kantham and ′ Ciri ′c [8] proved coupled coincidence and coupled common fixed point theorems of the non linear contractive mappings in the partially ordered complete metric spaces. Choudhury and Kundu [9] , proved the coupled coincidence result for compatible mapping s in the settings of a partially ordered metric space. Recently, Samet et al. [10] [11] have introduced notion of the α - ψ- contractive and α-admissible mapping and proved fixed point theorems for such mapping in the complete metric spaces. For more results regarding coupled fixed point in a various metric spaces, anybody can found in ([12] -[20]).

In recent time, in 2016 Mutlü and G¨urdal [21] have been introduced a notion of Bi-polar metric space, which is one of the generalization metric spaces. Also they have investigated fixed points and coupled fixed points results on this space, see ([21] , [22]).

Finally in this paper , we will continue to study the coupled fixed points in a frame of the bipolar metric space. More particularly , we will generalize the results of Mursaleenetal. [23] and Preeti, Sanjay Kumar [24] for α -ψ-contractive & α- admissible mappings using compatible mappings under α - ψ- contractions and α- admissible conditions and establish a few coupled coincidence points and the common fixed point results. Also, we gave an example to support our achieved result.

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2. Preliminaries

First we gave basic definitions.

Definition 2.1. ([21])
Let A and B be a two non − empty sets. suppose that $d: A \times B \rightarrow [0,\infty)$ *be a mapping satisfying the following properties:*

(B_0) If $d(a,b) = 0$ then $a = b$ for all $a \in A, b \in B$
(B_1) If $a = b$ then $d(a,b) = 0$ then for all $a \in A, b \in B$
(B_3) If $d(a,b) = d(b,a)$ for all $a, b \in A \cap B$
(B_4) If $d(a_1,b_2) \leq d(a_1,b_1) + d(a_2,b_1) + d(a_2,b_2)$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$

Then mapping of d is called 'Bipolar-metric' on the pair (A, B) and a triple (A, B, d) is called the Bipolar-metric space.

Example 2.2. ([21])
Let $A = (1,\infty)$ *and* $B = [-1,1]$. *Define* $d: A \times B \rightarrow [0,+\infty)$ *as* $d(a,b) = |a^2 - b^2|$, *for all* $(a,b) \in (A,B)$. *Then the triple* (A,B,d) *is a Bipolar metric space.*

Definition 2.3. ([21]) Let (A_1, B_1) and (A_2, B_2) be pairs of set and given a function $F: A_1 \cup B_1 \rightarrow A_2 \cup B_2$ is said to be a covariant map. If $F(A_1) \subseteq A_2$ & $F(B_1) \subseteq B_2$, and denote this with $F(A_1, B_1) \rightrightarrows F(A_2, B_2)$. And the mapping $F: A_1 \cup B_1 \rightarrow A_2 \cup B_2$ is said to be a contravariant map. If $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$, and write $:(A_1, B_1) \leftrightarrows (A_2, B_2)$. Particularly, if d_1 and d_2 are bipolar metric on (A_1, B_1) and (A_2, B_2), respectively, we have some time to use the notation $F: (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F: (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$.

Definition 2.4. ([21]) Let (A, B, d) be the bipolar metric space. A point v ? A? B is called a left point if v ? A, a right point if $v \in B$ and the central point if it is both left and right points. Same way, a sequence $\{a_n\}$ on the set A is called the left sequence and the sequence $\{b_n\}$ on the set B is called right sequence. In a bipolar metric space, ä left or right sequence is called simply a sequence. A sequence $\{v_n\}$ is said to be convergent to a point v, iif $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$. A bisequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is the sequence on the set of $A \times B$. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence, if $\lim_{n,m\to\infty} d(a_n, b_m) = 0$. In the bi-polar metric space, every convergent Cauchy bisequence is convergent. A bipolar metric space is the complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.5. ([21]) Let (A_1, B_1, d_1) *and* (A_2, B_2, d_2) *be a bipolar metric spaces.*

(i) A map $F: (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ *is called left − continuous at a point* $a_0 \in A_1$, *for every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that* $d_1(a_0, b) < \delta$ *implies that* $d_2(F(a_0), F(b)) < \epsilon$ *for all* $b \in B_1$.

(ii) A map $F: (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ *is called right − continuous at a point* $b_0 \in B_1$, *for every* $\epsilon > 0$, *there is a* $\delta > 0$ *such that* $d_1(a, b_0) < \delta$ *implies that* $d_2(F(a), F(b_0)) < \epsilon$ *for all* $a \in A_1$.

(iii) F is said to be continuous, if is left continuous at each $a \in A_1$ and right continuous at each $b \in B_1$.

(iv) Contravariant map $F: (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$ is continuous iff it is continueas covariant map $F: (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$.

Now from (2.4), Covariant or Contravariant map $F: (A_1, B_1, d_1) \rightarrow (A_2, B_2, d_2)$ is continuous iff $(u_n) \rightarrow v$ *on* (A_1, B_1, d_1) *implies* $F((u_n)) \rightarrow F(v)$ *on* (A_2, B_2, d_2).

Definition 2.6. Let (A, B, ≤) be ordered partial set and F: (A, B) $\rightrightarrows (A,B)$ *be a covariant mapping, we say that F is a non − decreasing with respect to* $\leq$ *if* $a, b \in A \cup B$, $a \leq$ *implies* $F(a) \leq F(b)$, *and similarly, a non − increasing mapping is defined.*

Definition 2.7. Let (A, B, ≤) be ordered partial set, F: $(A^2, B^2) \rightrightarrows (A,B)$ *be a covariant mapping. The map F has the mixed monotone property, if* $F(a,b)$ *is monotone non − decreasing in a and is monotone non − increasing in b, that is, for any* $(a,b) \in A^2 \cup B^2$

$(a_1, a_2) \in A^2, a_1 \leq a_2 \Rightarrow F(a_1, b) \leq F(a_2, b)$

$(b_1, b_2) \in B^2, b_1 \leq b_2 \Rightarrow F(a, b_1) \geq F(a, b_2)$

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Definition 2.8. *Let F:* $(A^2,B^2) \rightrightarrows (A,B)$ *be a covariant map, an element* $(a,b) \in A^2 \cup B^2$ *is called couple fixed point of F if* $F(a,b) = a$, *and* $F(b,a) = b$.

Definition 2.9. Let $(A, B, \leq)$ be a partial ordered set and F: $(A, B) \rightrightarrows (A,B)$ and g: $(A, B) \rightrightarrows (A,B)$ be two covariant maps. We say that F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $(a,b) \in A^2 \cup B^2$.

$$(a_1,a_2) \in A^2, ga_1 \leq ga_2 \Rightarrow F(a_1,b) \leq F(a_2,b)$$

$$(b_1,b_2) \in B^2, gb_1 \leq gb_2 \Rightarrow F(a,b_1) \geq F(a,b_2)$$

Definition 2.10. Let F: $(A^2,B^2) \rightrightarrows (A,B)$ and g: (A, B) $\rightrightarrows (A,B)$ be two covariant maps.

(i) an element $(a,b) \in A^2 \cup B^2$ *is called coupled coincidence point of F and g if* $F(a,b) = g(a)$, *and* $F(b,a) = g(b)$

(ii) and an element $(a,b) \in A^2 \cup B^2$ *is called* common (C) coupled (C) fixed (F) point (P) of F and g if F(a, b) = g(a) = a & F(b, a) = g(b) = b

Definition 2.11. *The covariant mappings F:* $(A^2,B^2) \rightrightarrows (A,B)$ *and g: (A, B)* $\rightrightarrows (A,B)$ *are said to compatible if,*

$$\lim_{n\to\infty} d\left(g\left(F(a_n,b_n)\right), F\left(g(p_n),g(q_n)\right)\right) = 0 = \lim_{n\to\infty} d\left(F\left(g(a_n),g(b_n)\right), g\left(F(p_n,q_n)\right)\right),$$

$$\lim_{n\to\infty} d\left(g\left(F(b_n,a_n)\right), F\left(g(q_n),g(p_n)\right)\right) = 0 = \lim_{n\to\infty} d\left(F\left(g(b_n),g(a_n)\right), g\left(F(q_n,p_n)\right)\right)$$

where $(\{a_n\},\{p_n\})$ *and* $(\{b_n\},\{q_n\})$ *are bisequence in* (A,B) *such that*

$$\lim F_{n\to\infty}(a_n,b_n) = \lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} F(p_n,q_n) = \lim_{n\to\infty} g(p_n)$$

And

$$\lim_{n\to\infty} F(b_n,a_n) = \lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} F(q_n,p_n) = \lim_{n\to\infty} g(q_n)$$

Definition 2.12.[23] *Let* $\boldsymbol{\Psi} = \{\psi: \psi: [0,+\infty) \to [0,+\infty)\}$ *be a family non – decreasing functions, such that* $\sum_{n=0}^{\infty}\psi_n(t) < t$ *for all* $t > 0$, *where* $\psi(t) < t$ *for all* $t > 0$, *where* ψ_n *is the* n^{th} *iterate of* ψ *satisfying*

(i) $\psi^{-1}\{0\} = \{0\}$

(ii) $\psi(t) < t$ *for all* $t > 0$

(iii) $\lim_{r\to t^+}\psi(r) < 1$ *for all* $t > 0$

Lemma 2.13. [23]

If $\psi: [0,+\infty) \to [0,+\infty)$ *is non – decreasing and right continuous, then* $\psi_n \to 0$ *as* $n \to \infty$ *for all* $t \geq 0$ if and only if $\psi(t) < t$ *for all* $t > 0$.

Definition 2.14.

Let (A,B,d) *be a partially ordered bipolar metric space and* F: $(A^2,B^2) \rightrightarrows (A,B)$ *then F is said to be* α *– contractive if there exist two functions* $\alpha: (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0,+\infty)$ *and* $\psi \in \Psi$ *such that*

$$\alpha\left((a,b),(p,q)\right) d\left(F(a,b),F(p,q)\right) \leq \psi\left(\frac{d(a,p)+d(b,q)}{2}\right)$$

for all $a,b \in A$ *and* $p,q \in B$ *with* $a \geq p$ *and* $b \leq q$.

Definition 2.15. Let F: $(A^2,B^2) \rightrightarrows (A,B)$ *be a covariant mapping and let* $\alpha: (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0,+\infty)$ *be a mappings. Then F is said to be* α *– admissible if* $\alpha\left((a,b),(p,q)\right) \geq 1$ *implies*

$$\alpha\left(\left(F(a,b),F(b,a)\right),\left(F(p,q),F(q,p)\right)\right) \geq 1$$

for all $a,b \in A$ *and* $p,q \in B$.

Definition 2.16.

Let (A,B,d) *be a partially ordered bipolar metric spaces and* $F: (A^2,B^2) \rightrightarrows (A,B)$ & *g : (A, B)* $\rightrightarrows (A,B)$ *be two covariant mappings. Then the mappings F and g are said to be* (α,ψ) *– contractive if there exist two functions* $\alpha: (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0,+\infty)$ *and* $\psi \in \Psi$ *such that*

$$\alpha\left(\left(g(a),g(b)\right),\left(g(p),g(q)\right)\right) d\left(F(a,b),F(p,q)\right) \leq \psi\left(\frac{d\left(g(a),g(p)\right)+d\left(g(b),g(q)\right)}{2}\right)$$

for all $a,b \in A$ *and* $p,q \in B$ *with* $g(a) \geq g(p)$ *and* $g(b) \leq g(q)$.

Definition 2.17. *Let F:* $(A^2,B^2) \rightrightarrows (A,B)$, *g : (A, B)* $\to (A,B)$ *and* $\alpha: (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0,+\infty)$ *be mappings. Then F and g are said to be* α *– admissible if* $\alpha\left(\left(g(a),g(b)\right),\left(g(p),g(q)\right)\right) \geq 1$ *implies*

$$\alpha\left(\left(F(a,b),F(b,a)\right),\left(F(p,q),F(q,p)\right)\right) \geq 1$$

for all $a,b \in A$ *and* $p,q \in B$.

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3. Main Results

Theorem 3.1.

Let $(A, B, \leq)$ *be a partially ordered set and there exist a bipolar metric* d *on* (A, B) *such that* (A, B, d) *is complete bipolar metric space. Let* $F: (A^2, B^2) \rightrightarrows (A, B)$ *and* $g: (A, B) \to (A, B)$ *be two covariant maps. Suppose* F *has* g *– mixed monotone property and there exists* $\psi \in \Psi$ *and*

$\alpha : (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0, +\infty)$

$$\alpha((g(a), g(b)), (g(p), g(q)))\, d(F(a, b), F(p, q)) \leq \psi\left(\frac{d(g(a), g(p)) + d(g(b), g(q))}{2}\right) \quad (3.1)$$

for all $a, b \in A$ *and* $p, q \in B$ *with* $g(a) \geq g(p)$ *and* $g(b) \leq g(q)$.

Suppose also that

(3.1.1) F and g are α – *admissible*

(3.1.2) There exist $a_0, b_0 \in A \cup B$ *such that*

$\alpha((F(g(a_0), g(b_0)), F(a_0, b_0)), (F(b_0, a_0))) \geq 1$ *and*

$\alpha((g(b_0), g(a_0)), F(b_0, a_0), F(a_0, b_0))) \geq 1$

(3.1.3) $F(A^2 \cup B^2) \subseteq g(A \cup B)$ *and* g *is complete.*

(3.1.4) {F, g} is compatible in $A \cup B$.

If there exist $a_0, b_0 \in A \cup B$ *such that* $g(a_0) \leq F(a_0, b_0)$ *and* $g(b_0) \geq F(b_0, a_0)$.

Then F and g has coincidence point that is there exist $a, b \in A \cup B$ *such that* $F(a, b) = g(a) = a$ *and* $F(b, a) = g(b) = b$

Proof. *Let* $a_0, b_0 \in A$ *and* $p_0, q_0 \in B$ *be such that*

$\alpha((g(a_0), g(b_0)), (F(a_0, b_0, F(b_0, a_0))) \geq 1,$

$\alpha((g(b_0), g(a_0)), F(b_0, a_0), F(a_0, b_0))) \geq 1$

And

$\alpha((g(p_0), g(q_0)), (F(p_0, q_0, F(q_0, p_0))) \geq 1,$

$\alpha((g(q_0), g(p_0)), F(q_0, p_0), F(p_0, q_0))) \geq 1$

and $g(a_0) \leq F(a_0, b_0) = g(a_1)$, $g(b_0) \geq F(b_0, a_0) = g(b_1)$

and $g(p_0) \leq F(p_0, q_0) = g(p_1), gq_0 \geq F(q_0, p_0) = g(q_1)$.

Let $a_2, b_2 \in A$ *and* $p_2, q_2 \in B$ *be such that*

$F(a_1, b_1) = g(a_2), F(b_1, a_1) = g(b_2)$ *and*

$F(p_1, q_1) = g(p_2), F(q_1, p_1) = g(q_2)$

continuing this process, we can construct two bisequences $(\{a_n\}, \{p_n\})$ *and* $(\{b_n\}, \{q_n\})$ *in* (A, B) *as follows. For all* $n \geq 0$,

$F(a_n, b_n) = g(a_{n+1}), F(b_n, a_n) = g(b_{n+1})$ *and*

$F(p_n, q_n) = g(p_{n+1}), F(q_n, p_n) = g(q_{n+1})$

Now we will show that

$g(a_n) \leq g(a_{n+1}), g(b_n) \geq g(b_{n+1})$ *and*

$g(p_n) \leq g(p_{n+1}), g(q_n) \geq g(q_{n+1})$ *for all* $n \geq 0$

(3.2)

For $n = 0$, *since,* $g(a_0) \leq F(a_0, b_0) = g(a_1)$,

$g(b_0) \geq F(b_0, a_0) = g(b_1)$

and $g(p_0) \leq F(p_0, q_0) = g(p_1)$,

$g(q_0) \geq F(q_0, p_0) = g(q_1)$.

That is we have $g(a_0) \leq g(a_1), g(b_0) \geq g(b_1)$ *and* $g(p_0) \leq g(p_1), g(q_0) \geq g(q_1)$.

Thus (3.2) *holds for* $n = 0$.

Now suppose that (3.2) *holds for some fixed point* $n \geq 0$.

Then since,

$g(a_n) \leq g(a_{n+1}), g(b_n) \geq g(b_{n+1})$ *and* $g(p_n) \leq g(p_{n+1}), g(q_n) \geq g(q_{n+1})$

Therefore, by g – *mixed monotone property of* F, *we have*

$g(a_{n+2}) = F(a_{n+1}, b_{n+1}) \geq F(a_n, b_{n+1}) \geq F(a_n, b_n) = g(a_{n+1})$,

$g(b_{n+2}) = F(b_{n+1}, a_{n+1}) \leq F(b_n, a_{n+1}) \leq F(b_n, a_n) = g(b_{n+1})$

and

$g(p_{n+2}) = F(p_{n+1}, q_{n+1}) \geq F(p_n, q_{n+1}) \geq F(p_n, q_n) = g(p_{n+1})$,

$g(q_{n+2}) = F(q_{n+1}, p_{n+1}) \leq F(q_n, p_{n+1}) \leq F(q_n, p_n) = g(q_{n+1})$.

from above, we conclude that

$g(a_{n+1}) \leq g(a_{n+2}), g(b_{n+1}) \geq g(b_{n+2})$ *and*

$g(p_{n+1}) \leq g(p_{n+2}), g(q_{n+1}) \geq g(q_{n+2})$

From data F , g are α – Admissible. From (3. 1. 2), we get

$\alpha((g(a_0), g(b_0)), (g(a_1), g(b_1))) =$

$\alpha((g(a_0), g(b_0)), (F(a_0, b_0), F(b_0, a_0))) \geq 1$

It follows that

$\alpha((F(a_0, b_0), F(b_0, a_0)), (F(a_1, b_1), F(b_1, a_1))) =$

$\alpha((g(a_1), g(b_1)), (g(a_2), g(b_2))) \geq 1$

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And $\alpha\left(\left(g(p_0), g(q_0)\right), \left(g(p_1), g(q_1)\right)\right) =$

$\alpha\left(\left(g(p_0), g(q_0)\right), \left(F(p_0, q_0), F(q_0, p_0)\right)\right) \geq 1$

$\alpha\left(\left(F(p_0, q_0), F(q_0, p_0)\right), \left(F(p_1, q_1), F(q_1, p_1)\right)\right) =$

$\alpha\left(\left(g(p_1), g(q_1)\right), \left(g(p_2), g(q_2)\right)\right) \geq 1$

Thus by mathematical induction, we have

$\alpha\left(\left(g(a_n), g(b_n)\right), \left(g(a_{n+1}), g(b_{n+1})\right)\right) \geq 1,$

$\alpha\left(\left(g(p_n), g(q_n)\right), \left(g(p_{n+1}), g(q_{n+1})\right)\right) \geq 1$ (3.3)

Similarly, for all $n \in N$ we have

$\alpha\left(\left(g(b_n), g(a_n)\right), \left(g(b_{n+1}), g(a_{n+1})\right)\right) \geq 1,$

$\alpha\left(\left(g(q_n), g(p_n)\right), \left(g(q_{n+1}), g(p_{n+1})\right)\right) \geq 1$ (3.4)

From (3.1) and conditions (3.1.1) and (3.1.2) of hypothesis, we get

$d\left(g(a_n), g(p_{n+1})\right) = d\left(F(a_{n-1}, b_{n-1}), F(p_n, q_n)\right)$

$\leq \alpha\left(\left(g(a_{n-1}), g(b_{n-1})\right), \left(g(p_n), g(q_n)\right)\right)$

$d\left(F(a_{n-1}, b_{n-1}), F(p_n, q_n)\right)$

$\leq \psi\left(\frac{d(g(a_{n-1}), g(p_n)) + d(g(b_{n-1}), g(q_n))}{2}\right)$ (3.5)

And

$d\left(g(b_n), g(q_{n+1})\right) = d\left(F(b_{n-1}, a_{n-1}), F(q_n, p_n)\right)$

$\leq \alpha\left(\left(g(b_{n-1}), g(a_{n-1})\right), \left(g(q_n), g(p_n)\right)\right)$

$d\left(F(b_{n-1}, a_{n-1}), F(q_n, p_n)\right)$

$\leq \psi\left(\frac{d(g(b_{n-1}), g(q_n)) + d(g(a_{n-1}), g(p_n))}{2}\right)$ (3.6)

Add (3.5) & (3.6), we have

$\frac{d(g(a_n), g(p_{n+1})) + d(g(b_n), g(q_{n+1}))}{2} \leq \psi\left(\frac{d(g(a_{n-1}), g(p_n)) + d(g(b_{n-1}), g(q_n))}{2}\right)$ (3.7)

Similarly, we have

$\frac{d(g(a_n), g(p_{n+1})) + d(g(b_n), g(q_{n+1}))}{2} \leq \psi^n\left(\frac{d(g(a_0), g(p_1)) + d(g(b_0), g(q_1))}{2}\right)$ (3.8)

On the other hand

$d\left(g(a_{n+1}), g(p_n)\right) = d\left(F(a_n, b_n), F(p_{n-1}, q_{n-1})\right)$

$\leq \alpha\left(\left(g(a_n), g(b_n)\right), \left(g(p_{n-1}), g(q_{n-1})\right)\right)$

$d\left(F(a_n, b_n), F(p_{n-1}, q_{n-1})\right)$

$\leq \Psi\left(\frac{d(g(a_n), g(p_{n-1})) + d(g(b_n), g(q_{n-1}))}{2}\right)$ (3.9)

And

$d\left(g(b_{n+1}), g(q_n)\right) = d\left(F(b_n, a_n), F(q_{n-1}, p_{n-1})\right)$

$\leq$

$\alpha\left(\left(g(b_n), g(a_n)\right), \left(g(q_{n-1}), g(p_{n-1})\right)\right) d\left(F(b_n, a_n), F(q_{n-1}, p_{n-1})\right)$

$\leq \Psi\left(\frac{d(g(b_n), g(q_{n-1})) + d(g(a_n), g(p_{n-1}))}{2}\right)$ (3.10)

Add (3.9) &(3.10), we get

$\frac{d(g(a_{n+1}), g(p_n)) + d(g(b_{n+1}), g(q_n))}{2} \leq \Psi\left(\frac{d(g(a_n), g(p_{n-1})) + d(g(b_n), g(q_{n-1}))}{2}\right)$ (3.11)

Repeat the above procedure, we get

$\frac{d(g(a_{n+1}), g(p_n)) + d(g(b_{n+1}), g(q_n))}{2} \leq \Psi^n\left(\frac{d(g(a_1), g(p_0)) + d(g(b_1), g(q_0))}{2}\right)$ (3.12)

Moreover

$d\left(g(a_n), g(p_n)\right) = d\left(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1})\right)$

$\leq \alpha\left(\left(g(a_{n-1}), g(b_{n-1})\right), \left(g(p_{n-1}), g(q_{n-1})\right)\right)$

$d\left(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1})\right)$

$\leq \Psi\left(\frac{d(g(a_{n-1}), g(p_{n-1})) + d(g(b_{n-1}), g(q_{n-1}))}{2}\right)$ (3.13)

And

$d\left(g(b_n), g(q_n)\right) = d\left(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1})\right)$

$\leq \alpha\left(\left(g(b_{n-1}), g(a_{n-1})\right), \left(g(q_{n-1}), g(p_{n-1})\right)\right)$

$d\left(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1})\right)$

$\leq \Psi\left(\frac{d(g(b_{n-1}), g(q_{n-1})) + (g(a_{n-1}), g(p_{n-1}))}{2}\right)$ (3.14)

Add (3.13) & (3.14) we get

$\frac{d(g(a_n), g(p_n)) + d(g(b_n), g(q_n))}{2} \leq \Psi\left(\frac{d(g(a_{n-1}), g(p_{n-1})) + d(g(b_{n-1}), g(q_{n-1}))}{2}\right)$ (3.15)

Similarly, we have

$\frac{d(g(a_n), g(p_n)) + d(g(b_n), g(q_n))}{2} \leq \Psi^n\left(\frac{d(g(a_0), g(p_0)) + (g(b_0), g(q_0))}{2}\right)$ (3.16)

For an arbitrary $\epsilon > 0$, there exist $n(\epsilon) \in N$ such that

$\sum_{n \geq n(\epsilon)} \Psi^n\left(\frac{d(g(a_0), g(p_1)) + (g(b_0), g(q_1))}{2}\right) + \sum_{n \geq n(\epsilon)} \Psi^n\left(\frac{d(g(a_0), g(p_0)) + (g(b_0), g(q_0))}{2}\right) < \frac{\epsilon}{2}$

And

$\sum_{n \geq n(\epsilon)} \Psi^n\left(\frac{d(g(a_1), g(p_0)) + (g(b_1), g(q_0))}{2}\right) + \sum_{n \geq n(\epsilon)} \Psi^n\left(\frac{d(g(a_0), g(p_0)) + (g(b_0), g(q_0))}{2}\right) < \frac{\epsilon}{2}$

Using property (B_4), for each $n, m \in N$ *such that* $n(\epsilon) < n < m$. then, from (3.8),(3.12) and (3.16), we have

$\frac{d(g(a_n), g(p_m)) + d(g(b_n), g(q_m))}{2}$

$\leq \frac{(d(g(a_n), g(p_{n+1})) + d(g(b_n), g(q_{n+1})))}{2} + \frac{(d(g(a_{n+1}), g(p_{n+1})) + d(g(b_{n+1}), g(q_{n+1})))}{2} + \cdots + \frac{(d(g(a_{m-1}), g(p_{m-1})) + d(g(b_{m-1}), g(q_{m-1})))}{2}$

$+ \frac{d(g(a_{m-1}), g(p_m)) + d(g(b_{m-1}), g(q_m))}{2}$

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$$\leq \psi^{n}\left(\frac{d(g(a_0),g(p_1))+(g(b_0),g(q_1))}{2}\right)+ \psi^{n+1}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)+\cdots+ \psi^{m-1}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)+ \psi^{m-1}\left(\frac{d(g(a_0),g(p_1))+(g(b_0),g(q_1))}{2}\right)$$

$$\leq \sum_{k=n}^{m-1}\psi^{k}\left(\frac{d(g(a_0),g(p_1))+(g(b_0),g(q_1))}{2}\right)+ \sum_{k=n+1}^{m-1}\psi^{k}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)$$

$$\leq \sum_{n\geq n(\epsilon)}\psi^{n}\left(\frac{d(g(a_0),g(p_1))+(g(b_0),g(q_1))}{2}\right)+ \sum_{n\geq n(\epsilon)}\psi^{n}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)<\frac{\epsilon}{2} \quad (3.17)$$

And

$$\frac{d(g(a_m),g(p_n))+d(g(b_m),g(q_n))}{2}$$

$$\leq \frac{(d(g(a_m),g(p_{m-1}))+d(g(b_m),g(q_{m-1})))}{2}+ \frac{(d(g(a_{m-1}),g(p_{m-1}))+d(g(b_{m-1}),g(q_{m-1})))}{2}+\cdots+ \frac{(d(g(a_{n+1}),g(p_{n+1}))+d(g(b_{n+1}),g(q_{n+1})))}{2} + \frac{d(g(a_{n+1}),g(p_n))+d(g(b_{n+1}),g(q_n))}{2}$$

$$\leq \psi^{m-1}\left(\frac{d(g(a_1),g(p_0))+(g(b_1),g(q_0))}{2}\right)+ \psi^{m-1}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)+\cdots+ \psi^{n+1}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)+ \psi^{n}\left(\frac{d(g(a_1),g(p_0))+(g(b_1),g(q_0))}{2}\right)$$

$$\leq \sum_{k=n}^{m-1}\psi^{k}\left(\frac{d(g(a_1),g(p_0))+(g(b_1),g(q_0))}{2}\right)+ \sum_{k=n+1}^{m-1}\psi^{k}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)$$

$$\leq \sum_{n\geq n(\epsilon)}\psi^{n}\left(\frac{d(g(a_1),g(p_0))+(g(b_1),g(q_0))}{2}\right)+ \sum_{n\geq n(\epsilon)}\psi^{n}\left(\frac{d(g(a_0),g(p_0))+(g(b_0),g(q_0))}{2}\right)<\frac{\epsilon}{2} \quad (3.18)$$

From (3.17) and (3.18), we have

$d(g(a_n),g(p_m))+d(g(b_n),g(q_m))<\epsilon$, $d(g(a_m),g(p_n))+d(g(b_m),g(q_n))<\epsilon$,

Hence $(g(a_n),g(p_n))$ and $(g(a_m),g(p_n))$ are Cauchy bisequences in triplet (A,B,d), also it is complete, therefore $(g(a_n),g(p_n))$ and $(g(a_m),g(p_n))$ are convergent.

Thusthere exist $a,b\in A$ and $p,q\in B$with

$\lim_{n\to\infty}F(a_n,b_n)=\lim_{n\to\infty}g(a_{n+1})=p$,

$\lim_{n\to\infty}F(b_n,a_n)=\lim_{n\to\infty}g(b_{n+1})=q$

$\lim_{n\to\infty}F(p_n,q_n)=\lim_{n\to\infty}g(p_{n+1})=a$,

$\lim_{n\to\infty}F(q_n,p_n)=\lim_{n\to\infty}g(q_{n+1})=b$

Since g is continuous we have

$\lim_{n\to\infty}F(a_n,b_n)=\lim_{n\to\infty}g^2(a_{n+1})=g(p)$,

$\lim_{n\to\infty}F(b_n,a_n)=\lim_{n\to\infty}g^2(b_{n+1})=g(q)$

$\lim_{n\to\infty}F(p_n,q_n)=\lim_{n\to\infty}g^2(p_{n+1})=g(a)$,

$\lim_{n\to\infty}F(q_n,p_n)=\lim_{n\to\infty}g^2(q_{n+1})=g(b)$

Since $\{f,g\}$ is compatible mapping , so, we have

$\lim_{n\to\infty}F(g(a_n),g(b_n))=g(p)$, $\lim_{n\to\infty}F(g(b_n),g(a_n))=g(q)$ (3.19)

And

$\lim_{n\to\infty}F(g(p_n),g(q_n))=g(a)$, $\lim_{n\to\infty}F(g(q_n),g(p_n))=g(b)$ (3.20)

Now we will prove g(a)=p, g(b)=q and g(p)=a, g(q)=b.

Now consider,

$$d(g^2(a_{n+1}),g(p_{n+1}))=d(F(ga_n,gb_n),F(p_n,q_n))$$
$$\leq \alpha((g^2(a_n),(g^2(b_n)),(g(p_n),g(q_n))d(F(ga_n,gb_n),F(p_n,q_n))$$

$$\leq \Psi\left(\frac{d((g^2(a_n),g(p_n))+d((g^2(b_n),g(q_n))}{2}\right) \quad (3.21)$$

And

$$d(g^2(b_{n+1}),g(q_{n+1}))=d(F(gb_n,ga_n),F(q_n,p_n))$$
$$\leq \alpha((g^2(b_n),(g^2(a_n)),(g(q_n),g(p_n))d(F(gb_n,ga_n),F(q_n,p_n))$$

$$\leq \Psi\left(\frac{d((g^2(b_n),g(q_n))+d((g^2(a_n),g(p_n))}{2}\right) \quad (3.22)$$

Combining (3.21) and (3.22)

$$\frac{d((g^2(a_{n+1}),g(p_{n+1}))+d((g^2(b_{n+1}),g(q_{n+1}))}{2}\leq \Psi\left(\frac{d((g^2(a_n),g(p_n))+d((g^2(b_n),g(q_n))}{2}\right)$$

$$<\frac{d((g^2(a_n),g(p_n))+d((g^2(b_n),g(q_n))}{2}$$

Letting $\to\infty$, we get $\frac{d(g(p),a)+d(g(q),b)}{2}<\frac{d(g(p),a)+d(g(q),b)}{2}$

Which yields that

$d(g(p),a)+d(g(q),b)=0$ *implies* $g(p)=a, g(q)=b$.

similarly, we

Can prove that g(a)=p,g(b)=q. now we will show that $F(a,b)=p, F(b,a)=q$ & $F(p,q)=a, F(q,p)=b$

for all $n\geq 0$, by using (3.1) and property (B_4), since g is continuous, we have

$$d(F(a,b),p)\leq d(F(a,b),g^2(p_{n+1}))+ d(g^2(a_{n+1}),g^2(p_{n+1}))+d(g^2(a_{n+1}),p)$$
$$=d\left(F(a,b),F(g(p_n),g(q_n))\right)+d(g^2(a_{n+1}),g^2(p_{n+1})) +d(g^2(a_{n+1}),g(p))\ )$$

$$\leq \alpha((g(a),g(b),(g^2(p_n),(g^2(q_n))) d\left(F(a,b),F(g(p_n),g(q_n))\right)+ d(g^2(a_{n+1}),g^2(p_{n+1}))+d(g^2(a_{n+1}),g(p))$$

$$\leq \Psi\left(\frac{d(g(a),g^2(p_n))+d(g(b),g^2(q_n))}{2}\right)+ d(g^2(a_{n+1}),g^2(p_{n+1}))+d(g^2(a_{n+1}),g(p))\to 0 \text{ as } n\to\infty.$$

Therefore,

$d(F(a,b),p)=0$ *implies* $F(a,b)=p$

similarly, we can show

$F(b,a) = q$ and $F(p,q) = a,$ $F(q,p) = b$
Therefore,
$F(a,b) = p = g(a), F(b,a) = q = g(b)$ and $F(p,q) = a = g(p), F(q,p) = b = g(q).$

On the other hand,
$d(a,p) = d\left(\lim_{n\to\infty} g(p_n), \lim_{n\to\infty} g(a_n)\right)$
$= \lim_{n\to\infty} d\big(g(a_n), g(p_n)\big) = 0$
$d(b,q) = d\left(\lim_{n\to\infty} g(q_n), \lim_{n\to\infty} g(b_n)\right)$
$= \lim_{n\to\infty} d\big(g(b_n), g(q_n)\big) = 0$
Therefore
$a = p$ and $b = q$ and $hence$ $F(a,b) = g(a)$ and $F(b,a) = g(b) = b$
now we will prove the uniqueness, we begin by taking $(a^*, b^*) \in A^2 \cup B^2$ be other coupled fixed point of F and g.
If $(a^*, b^*) \in A^2$, then
$d(a^*, a) = d\big(g(a^*), g(a)\big) = d(F((a^*, b^*), F(a,b))$
$\leq \alpha\Big(\big(g(a^*), g(b^*).\big(g(a), g(b)\big)\Big) d(F(a^*, b^*), F(a,b))$
$\leq \Psi\left(\frac{d(g(a^*),g(a))+d(g(b^*),g(b))}{2}\right)$
$< \frac{d(g(a^*),g(a))+d(g(b^*),g(b))}{2}$
$< \frac{d(a^*,a)+d(b^*,b)}{2}$
And
$d(b^*, b) = d\big(g(b^*), g(b)\big) = d(F((b^*, a^*), F(b,a))$
$\leq \alpha\Big(\big(g(b^*), g(a^*).\big(g(b), g(a)\big)\Big) d(F(b^*, a^*), F(b,a))$
$\leq \Psi\left(\frac{d(g(b^*),g(b))+d(g(a^*),g(a))}{2}\right)$
$< \frac{d(g(b^*),g(b))+d(g(a^*),g(a))}{2}$
$< \frac{d(b^*,b)+d(a^*,a)}{2}$
Therefore
$\frac{d(a^*,a)+d(b^*,b)}{2} < \frac{d(a^*,a)+d(b^*,b)}{2}$
Which yields that
$d(a^*, a) + d(b^*, b) = 0$ $implies$ $d(a^*, a) = 0, d(b^*, b) = 0$
that is $a = a^*$ and $b = b^*.$
Hence (a, b) is Unique common coupled fixed point.

Theorem3.2.
let $(A, B, \leq)$ *be a partially ordered set and thereexist bipolar metric* d *on* (A, B) *such that* (A, B, d) *is complete bipolar metric space.*
Let $F: (A^2, B^2) \rightrightarrows (A, B)$ *be covariant map. suppose* F *has a mixed monotone property and* $\exists \Psi \in$ $\boldsymbol{\Psi}$ **and** $\alpha: A^2 \times B^2 \to [0, +\infty)$

$$\alpha\big((a,b),(p,q)\big) d(F(a,b), F(p,q)) \leq \Psi\left(\frac{d(a,b)+d(b,q)}{2}\right) \quad (3.23)$$

for all $a, b \in A$ and $p, q \in B$ $with$ $a \geq p$ and $b \leq q.$
suppose also that (3.1.1) F is $\alpha - admissible$
(3.1.2) there exist $a_0, b_0 \in A \cup B$ such that
$\alpha\big((a_0, b_0)\big), (F(a_0, b_0), F(b_0, a_0))) \geq 1$&
$\alpha\big((b_0, a_0)\big), (F(b_0, a_0), F(a_0, b_0))) \geq 1$
If suppose $a_0, b_0 \in A \cup B$ with
$a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F have coupled fixed point

Example 3.3. .let $A = \{U_m(R) / U_m(R)$ is upper triangular matrices over $R\}$ and $B = \{L_m(R) / L_m(R)$ is lower triangular matrices over $R\}$ with the bipolar metric
$d(P, Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|$
For all
$P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. on the set (A, B) we consider the following relation:
$P, Q \in A^2 \cup B^2, P \preccurlyeq Q \Leftrightarrow p_{ij} \leq q_{ij}$
Where $\leq$ is usual ordering. Then clearly, (A, B, d, $\leq$) Ordered Bipolar Metric Space. Define $F: (A^2, B^2) \rightrightarrows (A, B)$ by
$F(P, Q) = \left(\frac{p_{ij}+q_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m}$
$\forall$ $(P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m})) \in A^2 \cup B^2$
And $g: A \cup B \to A \cup B$ by $g(P) = (p_{ij})_{m\times m}$ and let $\Psi: [0,1) \to [0,1)$ by $\Psi(t) = \frac{2t}{5}$ for $t \in [0,1)$
Let the bisequence
(P_n, Q_n) in (A, B) $such$ $that$ $\lim_{n\to\infty} F(P_n, Q_n) = \lim_{n\to\infty} F\, g(P_n) = X$
and
$\lim_{n\to\infty} F(Q_n, P_n) = \lim_{n\to\infty} F\, g(Q_n) = Y$ then obviously
,$X = Y = \frac{2}{5}(I_{ij})_{m\times m}$
So that
$\lim_{n\to\infty} d(g(F(P_n, Q_n)), F(g(P_n), g(Q_n))) = 0$
$\lim_{n\to\infty} d(g(F(Q_n, P_n)), F(g(Q_n), g(P_n))) = 0$
Also choose $\alpha: (A^2 \cup B^2) \times (A^2 \cup B^2) \to [0, \infty)$ with
$\alpha\big((g(P), g(Q)), (g(R), g(S))\big) =$
$\begin{cases} (I_{ij})_{m\times m} & if\ P \succeq Q, R \preccurlyeq S \\ (O_{ij})_{m\times m} & Otherwise \end{cases}$

Then obviously, F has the g-mixed monotone property , also there exist $P_0 = (O_{ij})_{m\times m}$ and $Q_0 = (I_{ij})_{m\times m}$ such that
$F\left((O_{ij})_{m\times m}, (I_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m} \succeq (O_{ij})_{m\times m}$
And
$F\left((I_{ij})_{m\times m}, (O_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m} \preccurlyeq (I_{ij})_{m\times m}$

Taking
$(P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m}), (R = (r_{ij})_{m\times m}, S = (s_{ij})_{m\times m}) \in A^2 \cup B^2$
with $P \succeq Q, R \preccurlyeq S$ that is $p_{ij} \geq r_{ij}$, $q_{ij} \leq s_{ij}$ we have
$d(F(P, Q), F(R, S)) = d\left(\frac{p_{ij}+q_{ij}}{5} + \frac{2}{5}(I_{ij}), \frac{r_{ij}+s_{ij}}{5} + \frac{2}{5}(I_{ij})\right)$
$= \frac{1}{5} \sum_{i,j=1}^{m} |(p_{ij} + q_{ij}) - (r_{ij} + s_{ij})|$
$\leq \frac{1}{5}\left(\sum_{i,j=1}^{m} |(p_{ij} - r_{ij})| + \sum_{i,j=1}^{m} |(q_{ij} - s_{ij})|\right)$
$\leq \frac{1}{5}(d(gP, gR) + d(gQ, gS))$
$\leq \frac{2}{5}\left(\frac{d(gP,gR)+d(gQ,gS)}{2}\right)$
Therefore, all the conditions of Theorem 3. 1. are hold.
Also$\left(\left(\frac{2}{5}I_{ij}\right)_{m\times m}, \left(\frac{2}{5}I_{ij}\right)_{m\times m}\right)$ is the coupled coincidence point of g and F.

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