(α,ψ) –Contractive results in Ordered Bipolar Metric Spaces

G. Adhi Lakshmi, G.N.V.Kishore, B.Srinuvasa Rao, D. Ram Prasad

Abstract: ***In this article, first we are introducing (α, ψ) – contraction, α - Admissible mapping. Based on these we proved some results in bipolar metric spaces.***

Index Terms: ***Bipolar metric spaces,α -admissible, (α, ψ) -Contractive mapping.***

1. INTRODUCTION

In 1922, Banach [1] proved that the fixed point theorem ensures the existence as well as the uniqueness of a fixed point under the appropriate conditions. This result of Banach Is known as Banäch fixed point theorem(or) the Contraaction-Mapping Principle. Fixed point theory is a good mixture of Analysis and Topology as well as Geometry. Fixed Point Theory is playing the vital role in the study of nonlinear phénomenä. In particular- fixed point techniques have been applied in diverse fields not only in Engineering and Physics, also in Biology, Chemistry, Economics and also in the 'Game theory'. There are many applications of fixed point theory in numerical methods – for example: Newton- Räphson Méthod and establishment of Picards Existence Theorem (regarding existence) and uniqueness in the solution of the first order differential equations and existence of solving the integral equations and even the system of linear equations.

Now, Fixed Point Théory is receiving much attention in Partially Ordered Metric Spaces.. i.e.,the metric spaces endowed with a partial order . The Turinici [2] extending of Banach contraction principle in settlement of partially ordered sets and laid the foundation for a new trend in fixed point theory.

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Subsequently, Lakshmi kantham and ´ Ciri´c [8] proved that the coupled coincidence and the coupled common fixed point theorems of nonlinear contractive mappings in the partially ordered of complete metric spaces. Kundu & Choudhury [9], proved that the coupled coincidence results for a compatible mapping in the settings of a partially ordered metric spaces. Recently, Samet_ etal [10] [11] introduced the notion of α -?? - contractive & α-admissible mapping and proved the fixed point theorems for such type of mappings in the complete metric spaces. For few more results of coupled fixed points in various metric spaces, we can found in ([12]-[20]).

Afterwards Ran and Reurings [3] developed few applications of Turinicis theorém for matrix equations and established few results in this same direction. The results have been further extended by Niéto and Rod´rguez-L´opez [4], [5] for non-decréasing mappings. Then after, Bhäskar and Läkshmi kantham [6], [7] introduced a new notion of Coupled Fixed Points for mappings which is satisfying the mixed monotone properties in the partially ordered space and discussed about the existence and an uniqueness of the solution for a periodic-boundary value problems.

Very recently, in 2016 Mutlu and G¨urdal[21] introduced a notion for Bipolar metric spaces, which is one of the generalizations metric spaces, see ([21],[22]).

In this paper, we will continue to study the Common (C) Coupled (C) Fixed (F)Point (P) theorems inOrdered Bipolar Metric spaces.

2. Preliminaries

To obtain theresults, wehave to consider the following.

Definition 2.1. ([21]) *Let A and B be a two non − empty sets. suppose that $d: A \times B \to [0,\infty)$ be a mapping satisfying the following properties:*

(B_0) If $d(a,b) = 0$ then $a = b$ for all $a \in A, b \in B$
(B_1) If $a = b$ then $d(a,b) = 0$ then for all $a \in A, b \in B$
(B_2) If $d(a,b) = d(b,a)$ for all $a, b \in A \cap B$
(B_4) If $d(a_1,b_2) \le d(a_1,b_1) + d(a_2,b_1) + d(a_2,b_2)$ for all $a_1,a_2 \in A$, $b_1,b_2 \in B$

Then d is called a Bipolarmetric and (A, B, d) is said to be Bipolar Metric Space.

Example 2.2. ([21]) *Let $A = (1,\infty)$ and $B = [-1,1]$. Define $d: A \times B \to [0,+\infty)$ as $d(a,b) = |a^2 - b^2|$, for all $(a,b) \in (A,B)$. Then the triple (A, B, d) is a Bipolar metric space.*

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Definition 2.3. ([21]) *Let* (A_1, B_1) and (A_2, B_2) as two pairs of sets and a function as $F: A_1 \cup B_1 \rightarrow A_2 \cup B_2$ is said to be a contravariant map. If $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$, and denote this with $F(A_1, B_1) \rightrightarrows F(A_2, B_2)$. And the mapping $F: A_1 \cup B_1 \rightrightarrows A_2 \cup B_2$ is said to be a contravariant map. If $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$, and write $(A_1, B_1) \leftrightarrows (A_2, B_2)$. In particular, if d_1 *and* d_2 *are bipolar metric on* (A_1, B_1) and (A_2, B_2) *respectively, we some time use the notation* $F(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F(A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$.

Définition 2.4. ([21]) *Let* (A, B, d) *be a bipolar metric space. A point* $v \in A \cup B$ *is called a left point if* $v \in A$, *a right point if* $v \in B$ *and a central point if both left and right point. Similarly, a sequence* $\{a_n\}$ *on the set* A *is called a left sequence and a sequence* $\{b_n\}$ *on the set* B *is called right sequence. In a bipolar metric space, a left or right sequence is called simply a sequence. A sequence* $\{v_n\}$ *is said to be a convergent to a point* v, *iff* $\{v_n\}$ *is a left sequence,* v *is a right point and* $\lim_{n \to \infty} d(v_n, v) = 0$; *or* $\{v_n\}$ *is a right sequence,* v *is a left point and* $\lim_{n \to \infty} d(v, v_n) = 0$. *A bisequence* $(\{a_n\}, \{b_n\})$ *on* (A, B, d) *is sequence on the set* $A X B$. *If the sequence* $\{\{a_n\}$ *and* $\{b_n\}$ *are convergent, then the bisequence* $(\{a_n\}, \{b_n\})$ *is said to be convergent.* $(\{a_n\}, \{b_n\})$ *is Cauchy sequence, if* $\lim_{n,m \to \infty} d(a_n, b_m) = 0$. *In a bipolar metric space, every convergent Cauchy bisequence is a biconvergent. A bipolar metric space is called complete, if every cauchy bisequence is convergent. Hence, biconvergent.*

Definition 2.5. Let (A, B, ≤) be partial ordered set, F: (A, B) $\rightrightarrows (A, B)$ *be a covariant mapping, we say that F is a non-decreasing with respect to* $\leq$ *if* $a, b \in A \cup B$, $a \leq b$ *implies* $F(a) \leq F(b)$, *and similarly, a non-increasing mapping is defined.*

Definition 2.6. Let (A, B, ≤) be ordered set, F: $(A^2, B^2) \rightrightarrows (A, B)$ *be a covariant mapping.* The *The map F has the mixed monotone property, if* $F(a, b)$ *is monotone non-decreasing in a and is monotone non-increasing in b, that is, for any* $(a, b) \in A^2 \cup B^2$

$$(a_1, a_2) \in A^2, a_1 \leq a_2 \Rightarrow F(a_1, b) \leq F(a_2, b)$$

$$(b_1, b_2) \in B^2, b_1 \leq b_2 \Rightarrow F(a, b_1) \geq F(a, b_2)$$

Definition 2.7. *Let F:* $(A^2, B^2) \rightrightarrows (A, B)$ *be a covariant map, an element* $(a, b) \in A^2 \cup B^2$ *is called couple fixed point of F if* $F(a, b) = a$, *and* $F(b, a) = b$.

Definition 2. 8. Let (A, B, ≤) be a partial ordered set and F: (A, B) $\rightrightarrows (A, B)$ and g: (A, B) $\rightrightarrows (A, B)$ be two covariant maps. We can say that F has the mixed g-monotone property if F is monotoné g-non-decreasing in its first ärgument & is monotone g-non-increasing in the second argument that is for any $(a, b) \in A^2 \cup B^2$,

$$(a_1, a_2) \in A^2, ga_1 \leq ga_2 \Rightarrow F(a_1, b) \leq F(a_2, b)$$

$$(b_1, b_2) \in B^2, gb_1 \leq gb_2 \Rightarrow F(a, b_1) \geq F(a, b_2)$$ $i\ g$

if $F(a, b) = ga$, *and* $F(b, a) = gb$

(ii) and an element $(a, b) \in A^2 \cup B^2$ *is called* common coupledfixed point of F and g if F(a, b) = g(a) = a and F(b, a) = g(b) = b

Definition 2.10. ([23])

Let $\boldsymbol{\Psi} = \{\psi: \psi: [0, +\infty) \rightarrow [0, +\infty)\}$ *be a family non-decreasing functions, such that* $\sum_{n=0}^{\infty} \psi_n < t$ *for all* $t > 0$, *where* $\psi(t) < t$ *for all* $t > 0$,

where ψ_n *is the* n^{th} *iterate of* ψ *satisfying*

(i) $\psi^{-1}\{0\} = \{0\}$

(ii) $\psi(t) < t$ *for all* $t > 0$

(iii) $\lim_{r \to t^+} \psi(r) < 1$ *for all* $t > 0$

Lemma 2.11. [23]

If $\psi: [0, +\infty) \rightarrow [0, +\infty)$ *is non-decreasing and right continuous, then* $\psi_n \rightarrow 0$ *as* $n \rightarrow \infty$ *for all* $t \geq 0$ if and only if $\psi(t) < t$ *for all* $t > 0$.

Definition 2.12:

Let (A, B, d) *be a partially ordered bipolar metric space and* F: $(A^2, B^2) \rightrightarrows (A, B)$ *then*

F is said to be α*-contractive if there exist two functions* $\alpha: A^2 \times B^2 \rightarrow [0, +\infty)$ *and* $\psi \in \Psi$ *such that*

$$\alpha((a, b), (p, q))\, d(F(a, b), F(p, q)) \leq \psi\left(\frac{d(a, p) + d(b, q)}{2}\right)$$ *for all* $a, b \in A$ *and* $p, q \in B$ *with* $a \geq p$ *and* $b \leq q$.

Definition 2.13. *Let F:* $(A^2, B^2) \rightrightarrows (A, B)$ *be a covariant mapping and let* $\alpha: A^2 \times B^2 \rightarrow [0, +\infty)$ *be a mappings.*

Then F is said to be α*-admissible if*

$\alpha((a, b), (p, q)) \geq 1$ *implies*

$$\alpha\left((F(a, b), F(b, a)), (F(p, q), F(q, p))\right) \geq 1$$

for all $a, b \in A$ and $p, q \in B$.

Definition 2.14. Let (A, B, d) be a partially ordered bipolar metric spaces and $F(A^2, B^2) \rightrightarrows (A, B)$ and g : (A, B) $\rightrightarrows (A, B)$ be two covariant mappings. Then the mappings F and g are said to be (α, ψ) − contractive if there exist two functions $\alpha: A^2 \times B^2 \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha((ga, gb), (gp, gq))d(F(a, b), F(p, q)) \leq \psi\left(\frac{d(ga, pg) + d(gb, gq)}{2}\right)$$ for all $a, b \in A$ and $p, q \in B$ with $ga \geq pg$ and $gb \leq gq$.

Definition 2.15. Let F: $(A^2, B^2) \rightrightarrows (A, B)$, g : (A, B) $\to (A, B)$ and $\alpha : A^2 \times B^2 \to [0, +\infty)$ be mappings. Then F and g are said to be α − admissible if $\alpha((ga, gb), (gp, gq)) \geq 1$ implies

$$\alpha\left((F(a, b), F(b, a)), (F(p, q), F(q, p))\right) \geq 1$$

for all $a, b \in A$ and $p, q \in B$.

3. MAIN RESULTS

Theorem 3.1. Let $(A, B, \leq)$ be a partially ordered set and there exist a bipolar metric d on (A, B) such that (A, B, d) is complete bipolar metric space. Let $F: (A^2, B^2) \rightrightarrows (A, B), g: (A, B) \to (A, B)$ be two covariant maps. Suppose F has g − mixed monotone property and there exists $\psi \in \Psi$ and $\alpha : A^2 \times B^2 \to [0, +\infty)$

$$\alpha((ga, gb), (gp, gq))d(F(a, b), F(p, q)) \leq \psi\left(\frac{d(ga, pg) + d(gb, gq)}{2}\right) \quad (3.1)$$

for all $a, b \in A$ and $p, q \in B$ with $ga \geq pg$ and $gb \leq gq$.

Suppose also that

(3.1.1) F and g are α − admissible

(3.1.2) There exist $a_0, b_0 \in A \cup B$ such that

$$\alpha\left((F(ga_0, gb_0), F(a_0, b_0)), (F(b_0, a_0))\right) \geq 1$$

and $\alpha((gb_0, ga_0), F(b_0, a_0), F(a_0, b_0))) \geq 1$

(3.1.3) $F(A^2 \cup B^2) \subseteq g(A \cup B)$ and $g(A \cup B)$ is complete.

If there exist $a_0, b_0 \in A \cup B$ such that $g(a_0) \leq F(a_0, b_0)$ and $g(b_0) \geq F(b_0, a_0)$.

Then F and g has coincidence point that is there exist $a, b \in A \cup B$ such that $F(a, b) = g(a)$ and $F(b, a) = g(b)$.

Proof. Let $a_0, b_0 \in A$ and $p_0, q_0 \in B$ be such that

$$\alpha\left((ga_0, gb_0), (F(p_0, q_0, F(q_0, p_0))\right) \geq 1$$

and $\alpha((gb_0, ga_0), F(q_0, p_0), F(p_0, q_0))) \geq 1$

and $ga_0 \leq F(p_0, q_0) = gp_1, gb_0 \geq F(q_0, p_0) = gq_1$

and $gp_0 \leq F(a_0, b_0) = ga_1, gq_0 \geq F(b_0, a_0) = gb_1$.

Let $a_2, b_2 \in A$ and $p_2, q_2 \in B$ be such that $F(a_1, b_1) = ga_2$, $F(b_1, a_1) = gb_2$ and $F(p_1, q_1) = gp_2, F(q_1, p_1) = gq_2$

continuing this process, we can construct two bisequences $(\{a_n\}, \{p_n\})$ and $(\{b_n\}, \{q_n\})$ in (A, B) as follows. For all $n \geq 0, F(a_n, b_n) = ga_{n+1}, F(b_n, a_n) = gb_{n+1}$ and $F(p_n, q_n) = gp_{n+1}, F(q_n, p_n) = gq_{n+1}$

Now we will show that $ga_n \leq gp_{n+1}, gb_n \geq gq_{n+1}$ and $gp_n \leq ga_{n+1}, gq_n \geq gb_{n+1}$ for all $n \geq 0$ (3.2)

For $n = 0$, since, $ga_0 \leq F(p_0, q_0) = gp_1, gb_0 \geq F(q_0, p_0) = gq_1$ and $gp_0 \leq F(a_0, b_0) = ga_1$, $gq_0 \geq F(b_0, a_0) = gb_1$.

That is we have $ga_0 \leq gp_1, gb_0 \geq gq_1$ and $gp_0 \leq ga_1, gq_0 \geq gb_1$.

Thus (3.2) holds for $n = 0$.

Now suppose that (3.2) holds for some fixed point $n > 0$.

Therefore, by g − mixed monotone property of F, we have

$$gp_{n+2} = F(p_{n+2}, q_{n+2}) \geq F(p_{n+1}, q_{n+2}) \geq F(p_{n+1}, q_{n+1}) \geq ga_{n+1} = F(a_n, b_n).$$

$$gq_{n+2} = F(q_{n+2}, p_{n+2}) \geq F(q_{n+1}, p_{n+2}) \geq F(q_{n+1}, p_{n+1}) \geq gb_{n+1} = F(b_n, a_n).$$

and

$$ga_{n+2} = F(a_{n+2}, b_{n+2}) \geq F(a_{n+1}, b_{n+2}) \geq F(a_{n+1}, b_{n+1}) \geq gp_{n+1} = F(p_n, q_n).$$

$$gb_{n+2} = F(b_{n+2}, a_{n+2}) \geq F(b_{n+1}, a_{n+2}) \geq F(b_{n+1}, a_{n+1}) \geq gq_{n+1} = F(q_n, p_n).$$

from above, we conclude that

$ga_{n+1} \leq gp_{n+2}, gb_{n+1} \geq gq_{n+2}$

and $gp_{n+1} \leq ga_{n+2}, gq_{n+1} \geq gb_{n+2}$

Since F and g are α – admissible, from (3.2), we have

$$\alpha((ga_0, gb_0), (gp_1, gq_1)) = \alpha\left((ga_0, gb_0), (F(p_0, q_0), F(q_0, p_0))\right) \geq 1$$

implies

$$\alpha\left((F(a_0, b_0), F(b_0, a_0)), (F(p_1, q_1), F(q_1, p_1))\right) = \alpha((ga_1, gb_1), (gp_2, gq_2)) \geq 1$$

Thus by mathematical induction, we have

$$\alpha((ga_n, gb_n), (gp_{n+1}, gq_{n+1})) \geq 1 \quad (3.3)$$

Similarly, for all $n \in N$ we have

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$$\alpha\big((gb_n, ga_n), (gq_{n+1}, gp_{n+1})\big) \geq 1 \quad (3.4)$$

From (3.1) and conditions (3.1.1) and (3.1.2) of hypothesis, we get

$$d(ga_n, gp_{n+1}) = d\big(F(a_{n-1}, b_{n-1}), F(p_n, q_n)\big)$$
$$\leq \alpha\big((ga_{n-1}, gb_{n-1}), (gp_n, gq_n)\big) d\big(F(a_{n-1}, b_{n-1}), F(p_n, q_n)\big)$$
$$\leq \psi\left(\frac{d(ga_{n-1}, gp_n) + d(gb_{n-1}, gq_n)}{2}\right) \quad (3.5)$$

And

$$d(gb_n, gq_{n+1}) = d\big(F(b_{n-1}, a_{n-1}), F(q_n, p_n)\big)$$
$$\leq \alpha\big((gb_{n-1}, ga_{n-1}), (gq_n, gp_n)\big) d\big(F(b_{n-1}, a_{n-1}), F(q_n, p_n)\big)$$
$$\leq \psi\left(\frac{d(gb_{n-1}, gq_n) + d(ga_{n-1}, gp_n)}{2}\right) \quad (3.6)$$

From (3.5) and (3.6), we have

$$\frac{d(ga_n, gp_{n+1}) + d(gb_n, gq_{n+1})}{2} \leq \psi\left(\frac{d(ga_{n-1}, gp_n) + d(gb_{n-1}, gq_n)}{2}\right) \quad (3.7)$$

Similarly, we have

$$\frac{d(ga_n, gp_{n+1}) + d(gb_n, gq_{n+1})}{2} \leq \psi^n\left(\frac{d(ga_{n-1}, gp_n) + d(gb_{n-1}, gq_n)}{2}\right) \quad (3.8)$$

On the other hand

$$d(ga_{n+1}, gp_n) = d\big(F(a_n, b_n), F(p_{n-1}, q_{n-1})\big)$$
$$\leq \alpha\big((ga_n, gb_n), (gp_{n-1}, gq_{n-1})\big) d\big(F(a_n, b_n), F(p_{n-1}, q_{n-1})\big)$$
$$\leq \psi\left(\frac{d(ga_n, gp_{n-1}) + d(gb_n, gq_{n-1})}{2}\right) \quad (3.9)$$

And

$$d(gb_{n+1}, gq_n) = d\big(F(b_n, a_n), F(q_{n-1}, p_{n-1})\big)$$
$$\leq \alpha\big((gb_n, ga_n), (gq_{n-1}, gp_{n-1})\big) d\big(F(b_n, a_n), F(q_{n-1}, p_{n-1})\big)$$
$$\leq \psi\left(\frac{d(gb_n, gq_{n-1}) + d(ga_n, gp_{n-1})}{2}\right) \quad (3.10)$$

On adding (3.9) and (3.10), we get

$$\frac{d(ga_{n+1}, gp_n) + d(gb_{n+1}, gq_n)}{2} \leq \psi\left(\frac{d(ga_n, gp_{n-1}) + d(gb_n, gq_{n-1})}{2}\right) \quad (3.11)$$

Repeating the above process, we get

$$\frac{d(ga_{n+1}, gp_n) + d(gb_{n+1}, gq_n)}{2} \leq \psi^n\left(\frac{d(ga_1, gp_0) + d(gb_1, gq_0)}{2}\right) \quad (3.12)$$

Moreover,

$$d(ga_n, gp_n) = d\big(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1})\big)$$
$$\leq \alpha\big((ga_{n-1}, gb_{n-1}), (gp_{n-1}, gq_{n-1})\big) d\big(F(a_{n-1}, b_{n-1}), F(p_{n-1}, q_{n-1})\big)$$
$$\leq \psi\left(\frac{d(ga_{n-1}, gp_{n-1}) + d(gb_{n-1}, gq_{n-1})}{2}\right) \quad (3.13)$$

And

$$d(gb_n, gq_n) = d\big(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1})\big)$$
$$\leq \alpha\big((gb_{n-1}, ga_{n-1}), (gq_{n-1}, gp_{n-1})\big) d\big(F(b_{n-1}, a_{n-1}), F(q_{n-1}, p_{n-1})\big)$$
$$\leq \psi\left(\frac{d(gb_{n-1}, gq_{n-1}) + d(ga_{n-1}, gp_{n-1})}{2}\right) \quad (3.14)$$

Byädding (3.13) and (3.14), we get

$$\frac{d(ga_n, gp_n) + d(gb_n, gq_n)}{2} \leq \psi\left(\frac{d(ga_{n-1}, gp_{n-1}) + d(gb_{n-1}, gq_{n-1})}{2}\right) \quad (3.15)$$

Repeating the above process, we get

$$\frac{d(ga_n, gp_n) + d(gb_n, gq_n)}{2} \leq \psi^n\left(\frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2}\right) \quad (3.16)$$

For an arbitrary $\epsilon > 0$, there exist $n(\epsilon) \in N$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n\left(\frac{d(ga_0, gp_1) + d(gb_0, gq_1)}{2}\right) + \sum_{n \geq n(\epsilon)} \psi^n\left(\frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2}\right) < \frac{\epsilon}{2}$$

And

$$\sum_{n \geq n(\epsilon)} \psi^n\left(\frac{d(ga_1, gp_0) + d(gb_1, gq_0)}{2}\right) + \sum_{n \geq n(\epsilon)} \psi^n\left(\frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2}\right) < \frac{\epsilon}{2}$$

Using the property (B_4)

$$d(ga_n, gp_m) \leq d(ga_n, gp_{n+1}) + d(ga_{n+1}, gp_{n+1}) + \cdots + d(ga_{m-1}, gp_m)$$

$$d(gb_n, gq_m) \leq d(gb_n, gq_{n+1}) + d(gb_{n+1}, gq_{n+1}) + \cdots + d(gb_{m-1}, gq_m) \quad (3.17)$$

And

$$d(ga_m, gp_n) \leq d(ga_m, gp_{m-1}) + d(ga_{m-1}, gp_{m-1}) + \cdots + d(ga_{n+1}, gp_n)$$

$$d(gb_m, gq_n) \leq d(gb_m, gq_{m-1}) + d(gb_{m-1}, gq_{m-1}) + \cdots + d(gb_{n+11}, gq_n) \quad (3.18)$$

For each $n, m \in N$ be such that $n(\epsilon) < n < m$, then from (3.8), (3.12),(3.16),(3.17) and (3.18), we have

$$\frac{d(ga_n, gp_m) + d(gb_n, gq_m)}{2} \leq \psi^n\left(\frac{d(ga_0, gp_1) + d(gb_0, gq_1)}{2}\right) + \psi^{n+1}\left(\frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2}\right) + \cdots + \psi^{m-1}\left(\frac{d(ga_0, gp_0) + d(gb_0, gq_0)}{2}\right) + \psi^{m-1}\left(\frac{d(ga_0, gp_1) + d(gb_0, gq_1)}{2}\right)$$

$\leq \sum_{k=n}^{m-1} \psi^k \left(\frac{d(ga_0, gp_1)+d(gb_0, gq_1)}{2}\right) +$
$\sum_{k=n+1}^{m-1} \psi^k \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right)$

$\leq \sum_{n\geq n(\epsilon)} \psi^n \left(\frac{d(ga_0, gp_1)+d(gb_0, gq_1)}{2}\right) +$
$\sum_{n\geq n(\epsilon)} \psi^n \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right) < \frac{\epsilon}{2}$ (3.19)

And

$\frac{d(ga_m, gp_n)+d(gb_m, gq_n)}{2}$
$\leq \frac{d(ga_m, gp_{m-1})+d(gb_m, gq_{m-1})}{2} +$
$\frac{d(ga_{m-1}, gp_{m-1})+d(gb_{m-1}, gq_{m-1})}{2} + \cdots +$
$\frac{d(ga_{n+1}, gp_{n+1})+d(gb_{n+1}, gq_{n+1})}{2} + \frac{d(ga_{n+1}, gp_n)+d(gb_{n+1}, gq_n)}{2}$

$\leq \psi^{m-1} \left(\frac{d(ga_1, gp_0)+d(gb_1, gq_0)}{2}\right) +$
$\psi^{m-1} \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right) + \cdots +$
$\psi^{n+1} \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right) + \psi^n \left(\frac{d(ga_1, gp_0)+d(gb_1, gq_0)}{2}\right)$

$\leq \sum_{k=n}^{m-1} \psi^k \left(\frac{d(ga_1, gp_0)+d(gb_1, gq_0)}{2}\right) +$
$\sum_{k=n+1}^{m-1} \psi^k \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right)$

$\sum_{n\geq n(\epsilon)} \psi^n \left(\frac{d(ga_1, gp_0)+d(gb_1, gq_0)}{2}\right) +$
$\sum_{n\geq n(\epsilon)} \psi^n \left(\frac{d(ga_0, gp_0)+d(gb_0, gq_0)}{2}\right)$
$<\frac{\epsilon}{2}$ (3.20)

From (3.19) & (3.20), we have
$d(ga_n, gp_m) + d(gb_n, gq_m) < \epsilon, d(ga_m, gp_n) + d(gb_m, gq_n) < \epsilon,$
Hence $(\{ga_n\}, \{gp_n\}$ and $\{gb_n\}, \{gq_n\})$ are Cauchy bisequences in $(g(A), g(B), d)$.
Therefore $\lim_{n\to\infty} (ga_n, gp_n) = \lim_{n\to\infty} (ga_n, gp_n) = 0$
Since, $g(A \cup B)$ complete subspace of (A, B, d), therefore $(\{ga_n\}, \{gp_n\})$ and $(\{gb_n\}, \{gq_n\})$ are convergent in $(g(A), g(B))$. There exist, $a, b \in A$ and $p, q \in B$ such that
$\lim_{n\to\infty} F(a_n, b_n) = \lim_{n\to\infty} ga_{n+1} = gp$,
$\lim_{n\to\infty} F(b_n, a_n) = \lim_{n\to\infty} gb_{n+1} = gq$
$\lim_{n\to\infty} F(p_n, q_n) = \lim_{n\to\infty} gp_{n+1} = ga$,
$\lim_{n\to\infty} F(q_n, p_n) = \lim_{n\to\infty} gq_{n+1} = gb$
Now we will show that $F(a, b) = gp, F(b, a) = gq$ and $F(p, q) = ga, F(q, p) = gb$
for all $n \geq 0$
By using (3.1) and property (B_4) we have
$d(F(a, b), gp) \leq d(F(a, b), gp_{n+1}) + d(ga_{n+1}, gp_{n+1}) + d(ga_{n+1}, gp)$

$= d(F(a, b), F(p_n, q_n)) + d(ga_{n+1}, gp_{n+1}) + d(ga_{n+1}, gp)$

$\leq \alpha((ga, gb), (gp_n, gq_n)) d(F(a, b), F(p_n, q_n)) + d(ga_{n+1}, gp_{n+1}) + d(ga_{n+1}, gp)$

$\leq \Psi\left(\frac{d(ga, gp_n)+d(gb, gq_n)}{2}\right) + d(ga_{n+1}, gp_{n+1}) + d(ga_{n+1}, gp) \to 0$ as $n \to \infty$.

Therefore $d(F(a, b), gp) = 0$ implies $F(a, b) = gp$. similarly, we can show $F(b, a) = gq$ and $F(p, q) = ga, F(q, p) = gb$.
On the other hand,
$d(ga, gp) = d\left(\lim_{n\to\infty} gp_n, \lim_{n\to\infty} ga_n\right)$
$= \lim_{n\to\infty} d(ga_n, gp_n) = 0$
And
$d(gb, gq) = d\left(\lim_{n\to\infty} gq_n, \lim_{n\to\infty} gb_n\right)$
$= \lim_{n\to\infty} d(gb_n, gq_n) = 0$
Therefore
$ga = gp$ and $gb = gq$ and hence $F(b, a) = gq$ and $F(p, q) = ga, F(q, p) = gb$.
Therefore F, g have coupled (C) coincidence (C) point (P).

Theorem 3.2. Let $(A, B, \leq)$ be a partially ordered set and there exist a bipolar metric d on (A, B) such that (A, B, d) is complete bipolar metric space. Let $F: (A^2, B^2) \rightrightarrows (A, B)$ be a covariant map. Suppose F has mixed monotone property and there exists ?? $\in \Psi$ and $\alpha: A^2 \times B^2 \to [0, +\infty)$
$\alpha((a, b), (p, q)) d(F(a, b), F(p, q)) \leq \Psi\left(\frac{d(a,b)+d(b,q)}{2}\right)$
(3.21)
For all $a, b \in A$ and $p, q \in B$ with $a \geq p$ and $b \leq q$.
suppose also that (3.1.1) F is $\alpha - admissible$
(3.1.2) there exist $a_0, b_0 \in A \cup B$ such that
$\alpha((a_0, b_0)), (F(a_0, b_0), F(b_0, a_0))) \geq 1$ and
$\alpha((b_0, a_0)), (F(b_0, a_0), F(a_0, b_0))) \geq 1$
If there exist $a_0, b_0 \in A \cup B \ni a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$ then F have coupled fixed point, i.e., $\exists\, a, b \in A \cup B$ such that $F(a, b) = a$ and $F(b, a) = b$

Example 3.3. .let $A = \{U_m(R) / U_m(R)$ is upper triangular matrices over $R\}$ and $B = \{L_m(R) / L_m(R)$ is lower triangular matrices over $R\}$ with the bipolar metric $d(P, Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|$
For all $P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. on the set (A, B) we consider the following relation:
$P, Q \in A^2 \cup B^2, P \preccurlyeq Q \Leftrightarrow p_{ij} \leq q_{ij}$
Where $\leq$ is usual ordering. Then clearly, (A, B, d) is complete bipolar metric space and $(A, B, \preccurlyeq)$ is a partially ordered set. Let $F: (A^2, B^2) \rightrightarrows (A, B)$ Be defined as
$F(P, Q) = \left(\frac{p_{ij}+q_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m}$
$\forall\ (P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m})) \in A^2 \cup B^2$
And $g: A \cup B \to A \cup B$ by $g(P) = (p_{ij})_{m\times m}$ and let $\Psi: [0,1) \to [0,1)$ by $\Psi(t) = \frac{2t}{5}$ for $t \in [0,1)$
Let the bisequence
(P_n, Q_n) in (A, B) such that $\lim_{n\to\infty} F(P_n, Q_n) = \lim_{n\to\infty} F\, g(P_n) = X$
and
$\lim_{n\to\infty} F(Q_n, P_n) = \lim_{n\to\infty} F\, g(Q_n) = Y$ then obviously ,$X = Y = \frac{2}{5}(I_{ij})_{m\times m}$
Consider the mapping $\alpha: A^2 \times B^2 \to [0, \infty)$ be such that

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$$\alpha((gP,gQ),(gR,gS)) = \begin{cases} (I_{ij})_{m\times m} & if\ P \succeq Q, R \preccurlyeq S \\ (O_{ij})_{m\times m} & Otherwise \end{cases}$$

Then obviously, F has the g-mixed monotone property , also there exist $P_0 = (O_{ij})_{m\times m}$ and $Q_0 = (I_{ij})_{m\times m}$ such that

$$F\left((O_{ij})_{m\times m},(I_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m} \succeq (O_{ij})_{m\times m}$$

And

$$F\left((I_{ij})_{m\times m},(O_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} + \frac{2}{5}(I_{ij})_{m\times m} \succeq (I_{ij})_{m\times m}$$

Taking

$(P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m}), (R = (r_{ij})_{m\times m}, S = (s_{ij})_{m\times m}) \in A^2 \cup B^2$

with $P \succeq Q, R \preccurlyeq S$ that is $p_{ij} \geq r_{ij}$, $q_{ij} \leq s_{ij}$ we have

$$d(F(P,Q),F(R,S)) = d\left(\frac{p_{ij}+q_{ij}}{5} + \frac{2}{5}(I_{ij}), \frac{r_{ij}+s_{ij}}{5} + \frac{2}{5}(I_{ij})\right)$$

$$= \frac{1}{5}\sum_{i,j=1}^{m}|(p_{ij}+q_{ij}) - (r_{ij}+s_{ij})|$$

$$\leq \frac{1}{5}\left(\sum_{i,j=1}^{m}|(p_{ij}-r_{ij})| + \sum_{i,j=1}^{m}|(q_{ij}-s_{ij})|\right)$$

$$\leq \frac{1}{5}(d(gP,gR) + d(gQ,gS))$$

$$\leq \frac{2}{5}\left(\frac{d(gP,gR)+d(gQ,gS)}{2}\right)$$

Therefore, all the conditions of Theorem (3.1) are holds$\left(\left(\frac{2}{5}I_{ij}\right)_{m\times m}, \left(\frac{2}{5}I_{ij}\right)_{m\times m}\right)$ is the Coupled Coincidence (CC) point of F and g.

4. CONCLUSIONS

In this paper, we proved some coupled (C) Fixed (F) Point (P) theorems by Using (α, ψ) – contraction and we gave suitable examples to supporting our main results.

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