Complete Cototal Domination Number of Certain Graphs

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Abstract—The main purpose of this paper is to investigate Complete Cototal Domination Number of Certain Graphs. In addition its some parameters are found.

Keywords—Dominination Number, Total Domination Number, Cototal Domination Number, Complete Cototal Domination Number.

I. INTRODUCTION

A set \( D \subseteq V \) of a graph is called a total dominating set if the induced subgraph \((V - D)\) has no isolated vertices. The total domination number \( \gamma_{td}(G) \) of \( G \) is the minimum cardinality of a total dominating set of \( G \)[6].

II. COMPLETE COTOTAL DOMINATION NUMBER

Definition: 2.1

A total dominating set \( D \) is said to be a complete cototal dominating set if the induced subgraph \((V - D)\) has no isolated vertices. The complete cototal domination number \( \gamma_{cc}(G) \) of \( G \) is the minimum cardinality of a complete cototal dominating set of \( G \)[1,6].

Example: 2.2

\[
\gamma_{td}(G) = \{v_1, v_2\}
\]

\[
\gamma_{cc}(G) = \{v_1, v_2\} \cup \{v_3\}
\]

Theorem: 2.3

For a Friendship graph \( F_n, \gamma_{cc}(F_n) = 3, n \geq 2 \).

Proof. The Friendship graph \( F_n \) has \( (2n + 1) \) nodes \( v_1, v_2, \ldots, v_{2n}, u \) and \( 3n \) edges \( v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}, v_nu \), \( 1 \leq i \leq 2n \). Here \( u \) is the middle node. Let us consider the total dominating set \( \gamma_{td}(F_n) = \{v_1, v_2, v_n\} \) where \( v_n \) is an isolated node. Minimal cototal dominating set is obtained by \( (G - \{v_1, v_2, v_n\}) \cap \{y\} \) where \( y \) is an isolated node.

Hence \( \gamma_{cc}(F_n) = \{v_1, v_2\} \cup \{y\} \)

Therefore \( \gamma_{cc}(F_n) = 3 \).

Theorem: 2.4

For a Complete bipartite graph \( K_{m,n}, \gamma_{cc}(K_{m,n}) = 2, m, n \geq 2 \).

Proof. The Complete bipartite graph \( K_{m,n} \) has \( (m + n) \) nodes \( v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n \) and \( m+n \) edges \( v_iv_j, 1 \leq i \leq m \) and \( j \leq n \). Let us consider the total dominating set \( \gamma_{td}(K_{m,n}) = \{v_1, u_1\} \). Minimal cototal dominating set is obtained by \( G - \{v_1, u_1\} \).

Hence \( \gamma_{cc}(K_{m,n}) = \{v_1, u_1\} \).

Therefore \( \gamma_{cc}(K_{m,n}) = 2 \).

Theorem: 2.5

For a Fan graph \( F_n, \gamma_{cc}(F_n) = 2, n \geq 3 \).

Proof. The Fan graph \( F_n \) has \( (n + 1) \) nodes \( u, v_1, v_2, \ldots, v_n \) and \( (2n - 1) \) edges \( v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1 \), \( 1 \leq i \leq n \). Here \( u \) is the middle node. Let us consider the total dominating set \( \gamma_{td}(F_n) = \{u, v_1\} \). Minimal cototal dominating set is obtained by \( G - \{u, v_1\} \).

So \( \gamma_{cc}(F_n) = \{u, v_1\} \). Hence \( \gamma_{cc}(F_n) = 2 \).

Theorem: 2.6

For a Helm graph \( H_n, \gamma_{cc}(H_n) = n + 1 \).

Proof. The Helm graph \( H_n \) has \( (2n + 1) \) nodes \( u, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \) and \( 3n \) edges \( v_1v_2, v_2v_3, \ldots, v_nv_1, v_nu_j \), \( 1 \leq i \leq n \) and \( uv_j \), \( 1 \leq i \leq n \). Let us consider the total dominating set \( \gamma_{td}(H_n) = \{u_1, u_2, \ldots, u_n, x\} \) where \( x \) is any one of the node of cycle \( C_n \). Minimal cototal dominating set is obtained by \( G - \{u, u_1, u_2, \ldots, u_n, x\} \).

∴ \( \gamma_{cc}(H_n) = \{u_1, u_2, \ldots, u_n, x\} \) so that \( \gamma_{cc}(H_n) = n + 1 \).

Theorem: 2.7

For a Comb graph \( P_2 \circ K_1, \gamma_{cc}(P_2 \circ K_1) = 2n, n \geq 2 \).

Proof. The Comb graph \( P_2 \circ K_1 \) has \( 2n \) nodes \( v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n \) and \( (2n - 1) \) edges \( v_iv_{i+1}, 1 \leq i \leq n - 1 \) and \( v_1u_1, 1 \leq i \leq n \). Let \( v_1, v_2, v_n \) be the nodes of \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the pendant nodes. Let us consider the total dominating set \( \gamma_{td}(P_2 \circ K_1) = \{v_1, v_2, \ldots, v_n\} \) where \( y \) is an isolated node. Minimal cototal dominating set is obtained by \( (G - \{v_1, v_2, \ldots, v_n\}) \cap \{y\} \) where \( y \) is an isolated node so that \( \gamma_{cc}(P_2 \circ K_1) = \{v_1, v_2, \ldots, v_n\} \) and \( \gamma_{cc}(P_2 \circ K_1) = 2n \).

Theorem: 2.8

For a Ladder graph \( L_n, \gamma_{cc}(L_n) = n, n \geq 2 \).

Proof. The Ladder graph \( L_n \) has \( 2n \) nodes \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) and \( (3n - 2) \) edges \( x_1y_1, x_2y_2, \ldots, x_ny_n \), \( 1 \leq i \leq n \). Let us consider the total dominating set \( \gamma_{td}(L_n) = \{x_1, x_2, \ldots, x_n\} \) Minimal cototal dominating set is obtained by \( G - \{x_1, x_2, \ldots, x_n\} \).

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So $\gamma_{cctd}(L_n) = \{x_1, x_2, \ldots, x_n\}$. Hence $\gamma_{cctd}(L_n) = n$.

**Theorem 2.9** For a $n$-sunlet graph, $\gamma_{cct}(n$ – sunlet $) = n + 1, n \geq 3$.

**Proof.** The $n$-sunlet graph has $2n$ nodes $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and $2n$ edges $u_1v_1, u_1v_2, \ldots, u_nv_n$, and $u_nv_i, 1 \leq i \leq n$ and $v_iu_j, 1 \leq i, j \leq n$. Let us consider the total dominating set $\gamma_{cctd}(n$ – sunlet $) = \{u_1, u_2, \ldots, u_n, x\}$ where $x$ belongs to the node of cycle $C_n$. Minimal cototal dominating set is obtained by $G - \{u_1, u_2, \ldots, u_n, x\}$. Hence $\gamma_{cct}(L_n) = n + 1$.

**Theorem 2.10** For a Bistar graph $B_{m,n}$, $\gamma_{cct}(B_{m,n}) = m + n + 2$.

**Proof.** The Bistar graph $B_{m,n}$ has $(m + n + 2)$ nodes $u, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, and $(m + n + 1)$ edges $uu_j, uu_i$ and $v_1v_n$. Let us consider the total dominating set $\gamma_{cctd}(B_{m,n}) = \{u, v_1, v_2, \ldots, v_n\}$. Minimal cototal dominating set is obtained by $G - \{u, v_1, v_2, \ldots, v_n\}$.

**Theorem 2.11** For a Corona graph $P_n \odot K_2, x_{cct}(P_n \odot K_2) = n + 2$.

**Proof.** The Corona graph $P_n \odot K_2$ has $3n$ nodes $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ and $(4n - 1)$ edges. Let $v_1, v_2, \ldots, v_n$ be the nodes of $P_n$ and $u_1, u_2, \ldots, u_n$ be the nodes of $K_2$. Let us consider the total dominating set $\gamma_{cctd}(P_n \odot K_2) = \{v_1, v_2, \ldots, v_n\}$. Minimal cototal dominating set is obtained by $G - \{v_1, v_2, \ldots, v_n\}$.

**Theorem 2.12** For a Coconut tree graph $(m, n), x_{cct}(CT(m, n)) = m + n$, where $m \geq 2$.

**Proof.** The Coconut tree graph $CT(m, n)$ has $(m + n)$ nodes $u, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$ and $(m + n - 1)$ edges $u_iu_j, uu_i$ and $v_1v_n$. Let us consider the total dominating set $\gamma_{cctd}(CT(m, n)) = \{u, v_1, v_2, \ldots, v_n\}$ where $x$ belongs to the node of path $P_n$ with $u$ as an isolated node. Let us consider the total dominating set $\gamma_{cctd}(CT(m, n)) = \{u, v_1, v_2, \ldots, v_n\}$.

**Observation 2.13** For a Fan graph $F_{n,1}$, $x_{cct}(F_{n,1}) = n - 1$, where $n \geq 3$.

## III. BOUNDS FOR $x_{cctd}(G)$

**Theorem 3.1** Let $G$ be a connected graph, then $x_{cctd}(G) \geq \left\lfloor \frac{n}{\Delta(G)} \right\rfloor$.

**Proof.** Let $S \subseteq V(G)$ be a complete cototal dominating set in $G$. Every node in $S$ dominates at most $\Delta(G) - 1$ vertices of $V(G) - S$ and dominate at least one of the vertices in $S$. Hence, $|S| \leq (\Delta(G) - 1) + |S| > n$. Since $S$ is an arbitrary complete cototal dominating set, then $x_{cctd}(G) \geq \left\lfloor \frac{n}{\Delta(G)} \right\rfloor$.

**Theorem 3.2** If $G$ is a connected graph with the girth of length $g(G) \geq 3$ and $\delta(G) \geq 2$, then $x_{cctd}(G) > n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Proof.** Let $G$ be a connected graph with $g(G) \geq 3$ and let $C$ be a cycle of length $g(G)$. Remove the form to get a graph $G'$. Suppose an arbitrary node $v \in V(G)$. Since $\delta(G) \geq 2$, $v$ has at least two neighbors say $x$ and $y$. Let $x, y \in C$. If $d(x, y) \leq 2$, then replacing the path from $x$ to $y$ with the path $x, v, y$, which reduces the girth of $G$. If $d(x, y) \geq 3$, then $x, y$ are on either $C_2$ or $C_4$ in $G$. Contradicting the hypothesis that $g(G) \geq 3$. Hence, no node in $G$ has two or more neighbors. Since $\delta(G) \geq 2$, the graph $G$ has no isolated node. Hence, $x_{cctd}(G) > n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Theorem 3.3** If $G$ is a complete bipartite graph with the girth of length $g(G) \geq 4$ and $\delta(G) \geq 2$, then $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Proof.** Let $G$ be a Complete bipartite graph with $g(G) \geq 4$ and let $C$ be a cycle of length $g(G)$. Remove the form to get a graph $G'$. Suppose an arbitrary node $v \in V(G)$. Since $\delta(G) \geq 2$, $v$ has at least two neighbors say $x$ and $y$. If $d(x, y) \leq 2$, then replacing the path from $x$ to $y$ with the path $x, v, y$ which reduces the girth of $G$. If $d(x, y) \geq 3$, then $x, y$ are on either $C_2$ or $C_4$ in $G$. Contradicting the hypothesis that $g(G) \geq 4$. Hence, no node in $G$ has two or more neighbors. Since $\delta(G) \geq 2$, the graph $G$ has no isolated node. Hence, $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Theorem 3.4** If $G$ is a Fan graph with the girth of length $g(G) \geq 3$ and $\delta(G) \geq 2$, then $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Proof.** Let $G$ be a Fan graph with $g(G) \geq 3$ and let $C$ be a cycle of length $g(G)$. Remove the form to get a graph $G'$. Suppose an arbitrary node $v \in V(G)$. Since $\delta(G) \geq 2$, $v$ has at least two neighbors say $x$ and $y$. Let $x, y \in C$. If $d(x, y) \geq 3$, then replacing the path from $x$ to $y$ with the path $x, v, y$ which reduces the girth of $G$. A contradiction. If $d(x, y) \leq 2$, then $x, y$ are on either $C_2$ or $C_4$ in $G$. Contradicting the hypothesis that $g(G) \geq 4$. Hence, no node in $G$ has two or more neighbors. Since $\delta(G) \geq 2$, the graph $G$ has no isolated node. Hence, $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$.

**Result 3.5** The above bound is sharp for $f_3$ since $x_{cctd}(f_3) = 2$.

**Theorem 3.6** If $G$ is a Fan graph with the girth of length $g(G) \geq 3$ and $\delta(G) \geq 2$, then $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$. 

**Proof.** Let $G$ be a Fan graph with $g(G) \geq 3$ and $\delta(G) \geq 2$. Then $x_{cctd}(G) < n - \frac{\lfloor g(G) \rfloor}{2} + 1$. 

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Proof: Let \( G \) be a connected graph and a Pan graph with \( g(G) \geq 3 \) and let \( C \) be a cycle of length \( g(G) \). Remove \( C \) from \( G \) to form a graph \( G' \). Suppose an arbitrary node \( v \in V(G') \). Since \( \delta(G) \geq 2 \), \( v \) has at least two neighbors say \( x \) and \( y \). Let \( x, y \in C \). If \( d(x, y) \geq 3 \), then replacing the path from \( x \) to \( y \) by the path \( x, v, y \) which reduces the girth of \( G \), a contradiction. If \( d(x, y) \leq 2 \), then \( x, y \) are on either \( C_3 \) or \( C_4 \) in \( G \), contradicting the hypothesis that \( g(G) \geq 3 \). Hence, no node in \( G \) has two or more neighbors on \( C \). Since \( \delta(G) \geq 2 \), the graph \( G \) has minimum degree at least \( \delta(G) - 1 \geq 1 \). Then \( G \) has no isolated node. Now let \( S \) be a \( y_{ct} \)-set for \( C \). Then \( S = S' \cup V(G') \) is a complete cototal dominating set for \( G \). Hence, \( y_{ct}(G) \leq n - \left\lceil \frac{g(G)}{2} \right\rceil + 1 \).

Result 3.7: The above bound is sharp for \( 3 - \text{pan} \) and \( 4 - \text{pan} \) since \( y_{ct}(3 - \text{pan}) = 2 \) and \( y_{ct}(4 - \text{pan}) = 3 \).

Theorem 3.8: Let \( G \) be a graph with \( \text{diam}(G) \geq 1 \), then \( y_{ct}(G) \geq \delta(G) + 1 \).

Proof: Let \( x \in V(G) \) and \( \text{deg}(x) = \delta(G) \). Since \( \text{diam}(G) \geq 1 \), then \( N(x) \) is a total dominating set for \( G \). Now \( S = N(x) \cup \{x\} \) is a complete cototal dominating set for \( G \) and \( |S| = \delta(G) + 1 \). Hence, \( y_{ct}(G) \geq \delta(G) + 1 \).

Result 3.9: The above bound is sharp for \( F_5 \) since \( y_{ct}(F_5) = 3 \) and also \( \delta(F_5) = 2 \).

Theorem 3.10: Let \( G \) be a graph with \( \text{diam}(G) \geq 1 \), then \( y_{ct}(G) < \delta(G) + 1 \), if \( G \) is either complete bipartite graph or Pan graph or ladder graph or corona graph.

Proof: Let \( x \in V(G) \) and \( \text{deg}(x) = \delta(G) \). Since \( \text{diam}(G) \geq 1 \), then \( N(x) \) is a total dominating set for \( G \). Now \( S = N(x) \cup \{x\} \) is a complete cototal dominating set for \( G \) and \( |S| = \delta(G) + 1 \). Hence, \( y_{ct}(G) < \delta(G) + 1 \).

IV. CONCLUSION

In this paper, Complete Cototal Domination Number of Certain Graphs and some bounds were studied.

REFERENCES