Minimum Covering Gutman Energy of Unitary Cayley Graphs

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Abstract— In 2012, Chandrashekar Adiga et al. introduced the notion of a novel kind of graph energy called minimum covering energy, depending on the underlying graph as well as its particular minimum covering set C. The concept of minimum covering Gutman energy was put forward in [4]. In this paper, we bring the notion of minimum covering Gutman energy of unitary Cayley graphs, that can be defined as the absolute sum of minimum covering Gutman eigenvalues obtained from minimum covering Gutman matrix of Unitary Cayley graphs, denoted by $A_{CG}(X_n)$.

Keywords— Minimum Covering Gutman Matrix, Minimum Covering Gutman Energy, Unitary Cayley Graphs.

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I. INTRODUCTION

In this paper, we consider finite, simple connected graph. In 1930s, a method for determining approximate solutions of Schrödinger equation of unsaturated conjugated hydrocarbons was introduced by the German scholar Erich Hückel. Such a method is now termed as the Hückel Molecular Orbital (HMO) theory. For further detailed study of HMO theory, see [2]. The concept of graph energy was thus framed by I. Gutman, motivated by HMO total π-electron energy [3]. But this cannot be restricted to molecular graphs only; we can extend this to establish new mathematical results. C. Adiga et al. [1] defined the minimum covering energy of a graph.

In this paper, we determine the minimum covering Gutman matrix of unitary Cayley graphs and consequently its minimum covering Gutman energy. Detailed studies on Gutman matrix, Gutman Energy and Gutman index of Unitary Cayley graphs can be found in [4, 5, 6].

Let G be a simple connected graph with V as vertex set. A minimum covering set C is any subset of V with minimum cardinality such that at least one vertex of C is incident with every edge of G. Then the minimum covering Gutman matrix of G can be defined as

$$A_{CG}(G) = (g_{ij}),$$

where $g_{ij} =
\begin{cases}
1, & \text{if } i = j \text{ and } v_i \in C; \\
0, & \text{if } i = j \text{ and } v_i \notin C; \\
d_i d_j, & \text{otherwise},
\end{cases}
$

where $d_i$ and $d_j$ denote the degrees of vertices $v_i$ and $v_j$, and $d_i d_j$ denotes the shortest distance between the corresponding vertices $v_i$ and $v_j$. Obviously, $A_{CG}(G)$ is symmetric with the same eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. Thus the minimum covering Gutman energy of graph is the absolute sum of the eigenvalues obtained from the characteristic equation $\det(\rho I - A_{CG}(G)) = 0$.

$$i.e., GE(G) = \sum_{i=1}^{n} |\rho_i|$$

The unitary Cayley graph $X_n(n > 1)$ with $V(X_n) = \{0, 1, \ldots, (n-1)\}$. $E(X_n) = \{(a, b); gcd(a-b, n) = 1, a, b \in \mathbb{Z}_n\}$. (view [3] for more details).

In the following section, we determine the minimum covering Gutman energy of Unitary Cayley graphs $X_n$ for possible values of n taking four cases of $n$.

II. MINIMUM COVERING GUTMAN ENERGY OF UNITARY CAYLEY GRAPHS

Theorem 2.1. For a Unitary Cayley graph $X_n$ with minimum covering set $C$, the minimum covering Gutman energy is given by

1. $p(p-2)^2 + \sqrt{(p(p-2)^2 + (p-1)^2) + 4(p-1)(p^2(p-2)^2 + 2p(p-2) + 1) + \frac{(n-3)^2(n-1)^2}{2} + \frac{(n-2)^2}{2}} + 1$ if $n = 2p$, where $p \neq 3$ is prime.

2. $\left(\frac{n}{2} - 1\right)(2n^2 - 1) + 1$, if $n = 2^x \cdot \alpha > 1$.

3. $\left(\frac{n}{2} - 1\right)$ if $n \equiv 2 \pmod{4}$.

4. $4p(p-1) \cdot \left(9p - 12 + n\right)$ if $p$ is prime.

Proof. Let us prove the theorem taking 4 cases of $n$. Let $X_n$ be the unitary Cayley graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $d_i$ and $d_j$ be the degrees of vertices $v_i$ and $v_j$, respectively. Also, $d_i d_j$ denotes the shortest distance between the vertices $v_i$ and $v_j$. 

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Case 1: $n$ is prime
Let $C = \{v_1, v_2, ..., v_{n-1}\}$ be the minimum covering set. Since $n$ is prime, $p$, $X_q$ is complete. So, $d_i = d_j = p - 1$ and $d_{ij} = 1$, for all $v_i \neq v_j$. Then by the definition of minimum covering Gutman matrix, $A_{CG}(G)$ is given by

$$
(g_0) = \begin{cases} 
1, & \text{if } v_i = v_j \text{ and } v_j \in C \\
0, & \text{if } v_i = v_j \text{ and } v_j \notin C \\
(p - 1)^2, & \text{otherwise}
\end{cases}
$$

Therefore, the characteristic equation is

$$(\lambda + (p - 2)) \left( \lambda^2 - (p(p - 2) + (p - 1)) \right) = 0,$$

which gives the minimum covering Gutman eigenvalues is

$$p(p - 2) \text{ times}.$$ 

This gives the minimum covering Gutman energy, $GEC(X_a)$ is

$$\frac{(n - 1)}{2} \times (n - 1) \text{ times}$$

as a result, $GEC(X_a)$ is

$$\frac{(n - 1)}{2} \times \frac{(n - 1)}{2}.$$ 

Case 2: $n = 2\alpha, \alpha > 1$.

This condition of $n$ gives a vertex partition of the vertex set $V(X_a) = A \cup B = \{v_1, v_2, ..., v_{2\alpha - 1}\} \cup \{v_{2\alpha}, v_{2\alpha + 1}, ..., v_n\}$. Further, $X_a$ is complete bipartite. So, for $v_i, v_j \in A$, $d_{ij} = 2$ when both $v_i$ and $v_j$ are in either $A$ or $B$. Then the minimum covering Gutman matrix, $A_{CG}(G)$ is given by

$$
(g_0) = \begin{cases} 
1, & \text{if } v_i = v_j \text{ and } v_j \in C \\
0, & \text{if } v_i = v_j \text{ and } v_j \notin C \\
\frac{n^2}{4}, & \text{if } v_i \neq v_j \text{ and } d_{ij} = 1 \\
\frac{n^2}{4}, & \text{if } v_i \neq v_j \text{ and } d_{ij} = 2
\end{cases}
$$

Therefore, the characteristic equation is

$$\lambda \left( \lambda + \frac{n^2}{4} \right)^{\frac{n^2}{4} - 1} \left( \lambda + \frac{n^2}{4} \right)^{\frac{n^2}{4} - 1} = 0.$$

This gives the minimum covering Gutman eigenvalues is

$$\frac{n^2}{2} \left( \frac{n}{2} \right)^{\frac{n}{2} \times (n - 1) \text{ times}}.$$
In particular, when \( n = 2p \) (\( p \neq 3 \)), the characteristic equation is
\[
\left[ \left( \frac{1}{2} + (n-1) \left( \frac{1}{2} + \frac{2}{(n-2)(n-8)+16} \right) \right) \right]^{n-1} + \left( \frac{1}{2} + \frac{2}{(n-2)(n-8)+16} \right) \left( \frac{1}{2} + \frac{2}{(n-2)(n-8)+16} \right) = \frac{n}{2} - 1
\]
0.

Then the minimum covering Gutman eigenvalues obtaining are
\[
- \frac{n-3}{2} (n-1) \pm \sqrt{\frac{n-3}{2} (n-1)^2 + 4 \left( \frac{n-2}{2} (n-1) + 1 \right)}
\]
\[
\frac{n}{2} - 1 \text{ times and}
\]
\[
\left( \frac{n-2}{2} + 1 \right) \pm \sqrt{\left( \frac{n-2}{2} + 1 \right)^2 - 4 \left( \frac{n-2}{2} + 1 \right) \left( \frac{3n-2}{2} (n-8)+16 \right)}
\]
Hence \( \text{GE}(X_{2p}) \) is
\[
\frac{n}{2} - 1 \left( \frac{n-3}{2} (n-1)^2 + 4 \left( \frac{n-2}{2} (n-1) + 1 \right) \right) + \left( \frac{n-2}{2} + 1 \right)
\]
\[
(p=3).
\]

Case 4: \( n \) is odd but not prime.

Let \( C = \{ \{ v_i \in U \} \}
\]
Here, for \( v_i \neq v_j \),
\[
d(v_i,v_j) = \begin{cases} 1 & \text{when } \text{gcd}(i-j,n)=1 \\ 2 & \text{when } \text{gcd}(i-j,n) \neq 1 \end{cases}
\]

Therefore, the minimum covering Gutman matrix \( A_{C}(G) \) follows:
\[
\begin{align*}
1. & \text{ if } v_i = v_j \text{ and } v_i \in C \\
0. & \text{ if } v_i = v_j \text{ and } v_i \notin C \\
(g_0) = & \begin{cases} \phi(n)^2, & \text{if } v_i \neq v_j \text{ and } dij = 1 \\ 2\phi(n)^2, & \text{if } v_i \neq v_j \text{ and } dij = 2 \end{cases}
\end{align*}
\]

Particularly, let us consider two conditions here (a) \( n = p^2 \) and (b) \( n = 3p \).

(a) For \( n = p^2 \), the spectrum is
\[
\begin{cases} -2\phi(n)^2, & \frac{1}{p-1} \end{cases}
\]
\[\left( p-2\right)^2 \frac{1}{p-1} \text{ and }
\]
\[1 + \left( p-2\right)(p-1)\phi(n)^2.
\]

(b) Similarly when \( n = 3p \), the spectrum is
\[
\begin{cases} -3\phi(n)^2, & \frac{1}{p} \end{cases}
\]
\[\left( p-2\right)^2 \frac{1}{p-1} \text{ and }
\]
\[1 + \left( p-3\right)\phi(n)^2.
\]

Finally, the minimum covering Gutman energy, \( \text{GE}(X_{2p}) \) is
\[
-2\phi(n)^2 \left( \frac{1}{p-1} \right)^2 + \left( 1-2 \phi(n)^2 \right) + | \left( p-2 \right) \phi(n)^2 | + | \left( p-2 \right) | (1+p) \phi(n)^2 | + \left( 1+\left( p+2 \right) (p-1) \phi(n)^2 \right) = 4p(p-1) \phi(n)^2 - p^2 + 3p - 2 \text{ and analogously, } \text{GE}(X_{3p}) \text{ is } \phi(n)^2 (9p-12+n) + 2.
\]

### III. CONCLUSION

In this paper, we observed minimum covering Gutman matrix and minimum covering Gutman energy which we presented in a conference. With these ideas, we determined here the minimum covering Gutman energy of Unitary Cayley graphs \( X_n \) for possible values of \( n \).

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### REFERENCES


