Minimum Covering Gutman Energy of Unitary Cayley Graphs

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***Abstract---** In 2012, ChandrashekarAdiga et.al introduced the notion of a novel kind of graph energy called minimum covering energy, depending on the underlying graph aswell as its particular minimum covering set C. The concept of minimum covering Gutman energy was put forward in [4]. In this paper, we bring the notion of minimum covering Gutman energy of unitary Cayley graphs, that can be defined as the absolute sum of minimum covering Gutman eigenvalues obtained from minimum covering Gutman matrix of Unitary Cayley graphs, denoted by $A_{Cg}(X_n)$.*

***Keywords---** Minimum Covering Gutman Matrix, Minimum Covering Gutman Energy,Unitary Cayley Graphs.*

2010 AMS Classification--- 05C50, 05C78

I. INTRODUCTION

In this paper, we consider finite, simple connected graph. In 1930s, a methodfor determiningapproximatesolutionsofSchro¨dingerequation of unsaturated conjugated hydrocarbons was introduced by the German scholar Erich Hu¨ckel.Such a method is now termed as the Hu¨ckel Molecular Orbital (HMO) theory. For further detailed study of HMO theory,see [2]. The concept of graph energy was thus framed by I.Gutman, motivated by HMO total π- electron energy [3]. But this cannot be restricted to molecular graphs only; we can extend this to establish new mathematical results.C.Adiga et.al.[1] defined the minimum covering energy of a graph.

In this paper, we determine the minimum covering Gutman matrix of unitary Cayley graphs and consequently its minimum covering Gutman energy. Detailed studies onGutman matrix,Gutman Energy andGutman index of Unitary Cayley graphs can be found in [4,5,6].

Let G be simple connected graph with V as vertex set. A minimum covering set C is any subset of V with minimum cardinality such that atleastonevertexofCisincidentwitheveryedgeofG.Thenthe minimum covering GutmanmatrixofGcanbedefinedasann $\times$n matrix.

$$A_{Cg}(G) = (g_{ij}),\ \text{where}\ g_{ij} = \begin{cases} 1, \text{if } i = j \text{ and } v_i \in C \\ 0, \text{if } i = j \text{ and } v_i \notin C \\ d_i d_j d_{ij}, \text{otherwise} \end{cases}$$

where$d_i$andd_j denotethedegreesofvertices$v_i$and$v_j$andd_{ij}denotetheshortestdistancebetween the vertices$v_i$andv_j.Obviously, $A_{Cg}(G)$ issymmetricwithitsrealeigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$.ThustheminimumcoveringGutmanenergyistheabsolutesumoftheeigenvaluesobtainedfromthecharacteristic equation $\det(\rho I - A_{C_g}(G)) = 0$.

i.e.,$GE_C(G) = \sum_{i=1}^{n} |\rho_i|$.

The unitary Cayley graph $X_n(n>1)$ with $V(X_n) = Z_n = \{0,1,\ldots,(n-1)\}$.

$E(X_n) = \{(a,b); gcd(a-b,n)=1,\ a,b \in Z_n\}$.(view [3] for more details).

In thefollowingsection,wedeterminetheminimum coveringGutman energyofunitaryCayleygraphs X_n forpossiblevaluestakingfourcases ofn (i)nisprime(ii)n=2^α,α>1(iii)nisevenandhasanoddprimedivisor (iv) n is p^2 and 3p,p is prime.

II. MINIMUM COVERING GUTMAN ENERGY OF UNITARY CAYLEY GRAPHS

Theorem 2.1. For a Unitary Cayley graph X_n with minimum covering set C, the minimum covering Gutman energy is given by

1. $p(p-2)^2 + \sqrt{(p(p-2)^2 + (p-1))^2 + 4(p-1)(p^2(p-2)^2 + 2p(p-2)+1)}$ if n is prime p.
2. $(\frac{n}{2} - 1)(2n^2-1) + 1$, if n = 2^α ,α>1.
3. $(\frac{n}{2} - 1)$ $\sqrt{(n-3)^2(n-1)^2 + 4\left(\frac{(n-3)^2(n-1)+1}{2}\right)} + \left(\frac{(n-2)^2}{2} + 1\right)$, if n = 2p, where p$\neq$ 3 is prime.
4. (a) $4p(p-1)\ \phi(n)^2 - p^2 + 3p - 2$, if n is p^2 where p is prime.

   (b) $\phi(n)^2 (9p - 12 + n) + 2$,n is 3p where p is prime.

PROOF. Let us prove the theorem taking 4 cases of n. Let $X_n$be the unitary Cayley graph with vertex set V = $\{v_1,v_2,\ldots,v_n\}$. Let d_i and d_j be the degrees of vertices v_i and v_j respectively. Also,d_{ij}denotes the shortest distance between the vertices v_i and v_j.

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Case 1: *n* is prime

Let $C = \{v_1, v_2, \ldots, v_{n-1}\}$ be the minimum covering set. Since nis prime *p*, $X_n$is complete. So, $d_i = d_j = p-1$ and $d_{ij} = 1$, for all $v_i \neq v_j$. Then by the definition of minimum covering Gutman matrix, $A_{Cg}(G)$ is given by

$$(g_{ij}) = \begin{cases} 1, if\ v_i = v_j\ and\ v_i \in C \\ 0, if\ v_i = v_j\ and\ v_i \notin C \\ (p-1)^2, otherwise \end{cases}$$

Thereforethe characteristic equation is

$(\lambda + p(p-2))(\lambda^2 - (p(p-2)^2 + (p-1))\lambda - (p-1)(p(p-2)(p(p-2)+2)+1) = 0,$

whichgives the minimum covering Gutman eigenvalues is p(p-2) ((p-2) times) and

$$\frac{(p(p-2)^2+(p-1)) \pm \sqrt{(p(p-2)^2+(p-1))^2+4(p-1)(p^2(p-2)^2+2p(p-2)+1)}}{2}$$

This gives the minimum covering Gutman energy , $GE_C(X_n)$ is

(p-2)|p(p-2)|+|

$$\frac{(p(p-2^2+(p-1)) \pm \sqrt{(p(p-2)^2+(p-1))^2+4(p-1)(p^2(p-2)^2+2p(p-2)+1)}}{2}$$

|.=p(p-2)^2+

$$\sqrt{(p(p-2)^2+(p-1))^2+4(p-1)(p^2(p-2)^2+2p(p-2)+1}$$

Case 2: n = 2^{α} ,α>1.

This condition of n gives a vertex partitionto the vertex set $V(X_n) = A \cup B = \{v_1, v_2, \ldots v_{n-1}\} \cup \{v_2, v_4, \ldots, v_n\}$.Further, X_n is complete bipartite. So, for $v_i \neq v_j$,$d_{ij} = 1$ when $v_i \in A$, $v_j \in B$ orvice versa and $d_{ij} = 2$ when both v_i and v_j are in either A or B. Then the minimum covering Gutman matrix, $A_{Cg}(G)$ is given by

$$(g_{ij}) = \begin{cases} 1, if\ v_i = v_j\ and\ v_i \in C \\ 0, if\ v_i = v_j\ and\ v_i \notin C \\ \frac{n^2}{4}, if\ v_i \neq v_j\ and\ d_{ij} = 1 \\ \frac{n^2}{4}, if\ v_i \neq v_j\ and\ d_{ij} = 2 \end{cases}$$

Therefore the characteristic equation is

$$\left(\left(\lambda + \frac{n^2}{2}\right)^{\frac{n}{2}-1}\right) \cdot \left(\left(\lambda + \frac{n^2}{2} - 1\right)^{\frac{n}{2}-1}\right) \cdot$$

$$\left(\lambda^2 - (n^2\left(\frac{n}{2}-1\right)+1)\lambda + \frac{n^4}{64}(n-4)(3n-4) + \frac{n^2}{2}(2\alpha-3)\right) = 0.$$

This gives the minimum covering Gutman eigenvalues is $\frac{n^2}{2}\left(\frac{n}{2}-1\ times\right)$, $\left(\frac{n^2}{2}-1\right)$ $\left(\frac{n}{2}-1\ times\right)$ and

$$\frac{\left(n^2\left(\frac{n}{2}-1\right)+1\right) \pm \sqrt{\left(n^2\left(\frac{n}{2}-1\right)+1\right)^2 - 4\left(\frac{n^4}{64}(n-4)(3n-4) + \frac{n^2}{2}(2n-3)\right)}}{2}$$

As a result, $GE_C(X_n)$ is

$$\left(\frac{n}{2}-1\right)\left|\frac{n^2}{2}\right| + \left(\frac{n}{2}-1\right)\left|\frac{n^2}{2}-1\right| +$$

$$\left|\frac{\left(n^2\left(\frac{n}{2}-1\right)+1\right) \pm \sqrt{\left(n^2\left(\frac{n}{2}-1\right)+1\right)^2 - 4\left(\frac{n^4}{64}(n-4)(3n-4) + \frac{n^2}{2}(2n-3)\right)}}{2}\right|$$

$= \left(\frac{n}{2}-1\right)(2n^2 - 1) + 1.$

Case 3:*n* is even and has an odd prime divisor.

This case also gives us the bipartite graph by the property of n. Then the vertexset$V(X_n)$hasthepartition $A \cup B = \{v_1, v_3, \ldots, v_{n-1}\} \cup \{v_2, v_4, \ldots, v_n\}$. Obviously, $d(v_i, v_j) = 1$ for $\phi(n) = \left(\frac{n}{2}-1\right)$ neighbours v_j to v_i. Since X_n is bipartite, $d(v_i, v_j) = 2$ as there exists common neighbours between some v_i and v_j. For the reason that X_n is not complete, there is the possibility of $d(v_i, v_j) = 3$.

Claim $d(v_i, v_j) = 3$

Consider $B = X \cup Y$ such that $X = \{v_j \in B$ where $v_i v_j \in E(X_n)\}$ and

$Y = \{v_j \in B$ where $v_i v_j \notin E(X_n)\}$. We know that $\phi(n) = \left(\frac{n}{2}-1\right)$ neighbours of v_i are in B with $d(v_i, v_j) = 1$. Let $x \in X, v_j \in Y$. Then v_i is adjacent to x. Since $x \in X$ and $v_j \in Y$,there should exists a common neighbour z of x and v_j so that $d(v_i, v_j) = 3$.

Thus, in general,for $v_i \neq v_j$,$d_{ij} = 1$ when both v_i and v_j are either in A or in B,$d_{ij} = 2$ when gcd(i-j,n) = 1 and $d_{ij} = 3$ when gcd(i-j,n) $\neq$ 1. Then our minimumcovering Gutman matrix $A_{Cg}(G)$ follows:(g_{ij})

$$= \begin{cases} 1, if\ v_i = v_j\ and\ v_i \in C \\ 0, if\ v_i = v_j\ and\ v_i \notin C \\ \left(\frac{n}{2}-1\right)^2, if\ v_i \neq v_j\ and\ d_{ij} = 1 \\ 2.\left(\frac{n}{2}-1\right)^2, if\ v_i \neq v_j\ and\ d_{ij} = 2 \\ 3.\left(\frac{n}{2}-1\right)^2, if\ v_i \neq v_j\ and\ d_{ij} = 3 \end{cases}$$

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In particular,when n =2p (p ≠3), the characteristic equation is

$$\left(\left(\lambda^2+(n-3)\left(\frac{(n-3)^2(n-1)+1}{2}\right)\right)^{\frac{n}{2}-1}\right)\left(\lambda^2-\left(\left(\frac{n-2}{2}\right)^2+1\right)\lambda+\frac{(n-2)^3}{64}(3n(n-2)(n-8)+16)\right)=0.$$

Then the minimum covering Gutman eigenvalues obtaining are

$$\frac{-(n-3)(n-1)\pm\sqrt{(n-3)^2(n-1)^2+4\left(\frac{(n-2)^2(n-1)+1}{2}\right)}}{2}$$

$\left(\frac{n}{2}-1\right)$ times and

$$\frac{\left(\frac{(n-2)^2}{2}+1\right)\pm\sqrt{\left(\frac{(n-2)^2}{2}+1\right)^2-4\left(\frac{(n-2)^3}{64}(3n(n-2)(n-8)+16\right)}}{2}.$$

Hence $GE_C(X_{2p})$ is

$$\left(\frac{n}{2}-1\right)\sqrt{(n-3)^2(n-1)^2+4\left(\frac{(n-3)^2(n-1)+1}{2}\right)}+\left(\frac{(n-2)^3}{2}+1\right)$$

(p≠ 3).

Case 4: n is odd but not prime.

Let C = $\{v_i; i\in U_n\}$

Here,for $v_i \neq v_j$,

$$\begin{cases} d(v_i,v_j)=1 \text{ when } \gcd(i-j,n)=1 \\ d(v_i,v_j)=2 \text{ when } \gcd(i-j,n)\neq 1 \end{cases}$$

Therefore, the minimum covering Gutman matrix $A_{Cg}(G)$ follows:

$$(g_{ij}) = \begin{cases} 1, \text{if } v_i = v_j \text{ and } v_i \in C \\ 0, \text{if } v_i = v_j \text{ and } v_i \notin C \\ \phi(n)^2, \text{if } v_i \neq v_j \text{ and } d_{ij}=1 \\ 2.\phi(n)^2, \text{if } v_i \neq v_j \text{ and } d_{ij}=2 \end{cases}$$

Particularly , let us consider two conditions here (a) $n = p^2$ and (b) n = 3p.

(a)For $n = p^2$, the spectrum is $\begin{pmatrix} -2\phi(n)^2 \\ p-1 \end{pmatrix}, \begin{pmatrix} 1-2\phi(n)^2 \\ (p-1)^2 \end{pmatrix}$, $\begin{pmatrix} (p-2)\phi(n)^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1+(p-2)\phi(n)^2 \\ p-2 \end{pmatrix}$ and $\begin{pmatrix} 1+(p+2)(p-1)\phi(n)^2 \\ 1 \end{pmatrix}$.

(b) Similarly when n = 3p, the spectrum is

$\begin{pmatrix} -3\phi(n)^2 \\ p \end{pmatrix}, \begin{pmatrix} 1-3\phi(n)^2 \\ (p-2)^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ p-2 \end{pmatrix}, \begin{pmatrix} (p-3)\phi(n)^2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1+(p-3)\phi(n)^2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} (n+p)\phi(n)^2+1 \\ 1 \end{pmatrix}$..

Finally, the minimum covering Gutman energy, $GE_C(X_{p^2})$is

$\left|-2\phi(n)^2\right|$ (p-1)+$(p-1)^2$ |(1-2 $\phi(n)^2$)| + |(p-2)| $\phi(n)^2$ | +(p-2)|(1+(p-2) $\phi(n)^2$ | + (1+(p+2)(p-1) $\phi(n)^2$ |

= 4p(p-1) $\phi(n)^2$ - p^2 +3p -2 and analogously, $GE_C(X_{3p})$ is $\phi(n)^2$ (9p -12+n) +2.

III. CONCLUSION

In this paper, we observed minimum covering Gutman matrix and minimum covering Gutman energy which we presented in a conference. With these ideas, wedetermined here the minimum covering Gutman energy of Unitary Cayley graphs $X_n$for possible values of n.

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