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Fixed Point Theorems for Semi and Weak Compatible Mappings in Symmetric Spaces

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Abstract: The objective of our research paper is to establish a fixed point theorem for Symmetric space with the notion of weak and semi compatible mappings. The main result of our research paper enlarges previous ones in symmetric spaces.

Keywords: Symmetric spaces, compatibility, Semi compatibility, Weak compatibility Common fixed point.

I. INTRODUCTION:

The theory of fixed points has wide applications in several areas of mathematics like topology, functional analysis, ordinary and partial differential equations as well as in science and Engineering.It helps as a valuable approach in in the study of nonlinear phenomena arising in diverse field like mathematical economics, game theory, biology, engineering and physics. Several researchers generalized the results on metric space and study their applications. [3,4,10,11]. In 2017,Ahmed H. Soliman [2]generalized the result of Almeida, Roldan- Lopez-de-Hierroand Sadarangani [1] generalizedfewfixed point theorems.Here in our research article, we generalized the result of Ahmed H. Soliman [2] for S-complete symmetric space with the concept of compatibility, semi compatibility and weak compatibility of four self-mappings.

II. PRELIMINARIES

A. *Definition*[8] Let$X$ *is* a non-empty set &S:$X \times X \to [0, \infty)$ is a function satisfies:

(i) $S(x, y) = 0 \Leftrightarrow x = y$

(ii) $S(x, y) = S(y, x)$,where$x$, $y \in X$ Then the couple (X, S) is known as symmetric space.

B. *Definition*[8,9]Assume (X, S) is a symmetric space.

(a) [8] A sequence $\{x_n\}$ in X is S-Cauchy when $\lim_{n\to\infty} S(x_n, x_{n+r}) = 0, r \in N$.

(b) [8] (X, S) be S-complete when for each S-Cauchy $\{x_n\}$, we havein X such as $\lim_{n\to\infty} S(x_n, x) = 0$.

(c) [8] f:$X \to X$ iscalled S-continuous when $\lim_{n\to\infty} S(x_n, x) = 0$ implies $\lim_{n\to\infty} S(fx_n, fx) = 0$.

(d)W-3[9] For $\{x_n\}$, y &x in X, $\lim_{n\to\infty} S(x_n, x) = 0$ & $\lim_{n\to\infty} S(x_n, y) = 0$ imply that $x = y$.

(e) $W4$ [9] For $\{x_n\}$, $\{y_n\}$ &x in X , $\lim_{n\to\infty} S(x_n, x) = 0$ with $\lim_{n\to\infty} S(x_n, y_n) = 0$ then $\lim_{n\to\infty} S(y_n, x) = 0$.

(f) (1C) [4] A function S is said to be 1-continuous if $\lim_{n\to\infty} S(x_n, x) = 0 \Rightarrow \lim_{n\to\infty} S(x_n, y) = S(x, y)$.

(g)[9] $(W4) \Rightarrow (W3)$.

C. *Definition*Let (X,S) is a space which is symmetric.

(a) [5] (A,B) on X is called compatible if whenever is a sequence with $\lim_{n\to\infty} S(ABx_n, BAx_n) = 0$ whenever $\{x_n\}$

(b) is a sequence in X which satisfies $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x \in X$

(c) [5] A couple of self-mappings (A,B) on X is called weak compatible if fx=gx implies that fgx=gfx.

(d) [5] A couple of self-mappings (A,B) on X is called semi compatible if whenever is a sequence in such that $\lim_{n\to\infty} S(ABx_n, Bx) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = x \in X$

III. KEY RESULT

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A. *Theorem*: Assume that (X,S) is a symmetric space which is s comlete, with W4 & 1-C. Suppose A,B, Rand T be self- mapping of (X,S) such that:

i. $A(X) \subseteq T(X), B(X) \subseteq R(X)$ (3.1)

ii. The couple (A,R) is semi compatible (3.2)

iii. The couple (B,T) is weak compatible (3.2)

iv. R is S-continuous.

v. $$S(Ax,By) \le \varphi(M(x,y)) + CMin\begin{Bmatrix} S(Rx,Tx), S(By,Ty), \\ S(Ax,Ry), S(By,Tx) \end{Bmatrix}$$

vi. (3.3) $\forall$x, y $\in$ X, C $\geq$0 , where M (x , y) is

$$M(x,y) = \max\begin{Bmatrix} S(x,y), \\ \frac{S(Rx,Tx)(S(By,Ty)+1)}{1+S(x,y)}, \frac{S(Bx,Tx)(S(Rx,Tx)+1)}{1+S(x,y)} \end{Bmatrix} \quad (3.4)$$

& φ: [0, ∞) → [0, ∞) is a continuous, non-decreasing with $\lim_{n\to\infty} \phi^n(t) = 0, \forall t > 0$

Then all four mappings have a unique and common fixed point.

Proof: Assume that $x_0 \in X$ is arandom point as A(X) ⊆ T(X) and B(X) ⊆ R(X), then x_1, $x_2 \in$ X are defined as $Ax_0=Tx_1$, $Bx_1=Rx_2$. Using Induction method, let us create sequences $\{y_n\}$ & $\{x_n\}$ in X with property $y_{n+1} = Ax_n = Tx_{n+1} = Bx_n = Rx_{n+1}$, for n=0,1,2.....

From equation (3.3), we have:

$$S(Ax_n, Bx_{n+1}) \le \varphi(M(x_n, x_{n+1})) + CMin\{S(Rx_n,Tx_n), S(Bx_{n+1},Tx_{n+1}), S(Ax_n,Rx_{n+1}), S(Bx_{n+1,}Tx_n\}$$

$$S(y_{n+1}, y_{n+2}) = \varphi(M(x_n, x_{n+1})) \quad (3.5)$$

Where

$$M(x_n,x_{n+1}) = \max\left\{ S(Rx_n,Tx_{n+1}), \frac{S(Rx_n,Tx_n)(S(Bx_{n+1},Tx_{n+1})+1)}{1+S(Rx_n,Tx_{n+1})}, \frac{S(Bx_n,Tx_n)(S(Rx_n,Tx_n)+1)}{1+S(Rx_n,Tx_{n+1})} \right\}$$

$$M(x_n,x_{n+1}) = \max\left\{ S(y_n,y_{n+1}), \frac{S(y_n,y_n)(S(y_{n+2},y_{n+1})+1)}{1+S(y_n,y_{n+1})}, \frac{S(y_{n+1},y_n)(S(y_n,y_n)+1)}{1+S(y_n,y_{n+1})} \right\}$$

$$M(x_n,x_{n+1}) = \max\left\{ S(y_n,y_{n+1}), 0, \frac{S(y_n,y_{n+1})}{1+S(y_n,y_{n+1})} \right\}$$

We think through the following cases, if

a. $M(x_n,x_{n+1}) = S(y_n,y_{n+1})$

$$S(y_{n+1},y_{n+2}) \le \varphi(M(x_n,x_{n+1})) \le \varphi(S(y_n,y_{n+1}) < S(y_n,y_{n+1}) \quad (3.6)$$

Using (3.5)

b. $M(x_n,x_{n+1}) = \frac{S(y_n,y_{n+1})}{1+S(y_n,y_{n+1})}$

$$S(y_{n+1},y_{n+2})\le\varphi(M(x_n,x_{n+1}))\le\varphi(\frac{S(y_n,y_{n+1})}{1+S(y_n,y_{n+1})})<\frac{S(y_n,y_{n+1})}{1+S(y_n,y_{n+1})}<S(y_{n.},y_{n+1})$$

Hence (3.6) satisfies.In any of the above instance, we proved (3.6) holds. As $S(y_{n+1}, y_{n+2})$ is decreasing, therefore it is convergent to a non-negative k. If k>0, at that time take n→∞ in equation (3.5), we will get

$$k<\phi\left\{\max\left(k,0,\frac{k}{1+k}\right)\right\}<\phi(k)<k$$

Which is not possible, so we must have k=0, that is $\lim_{n\to\infty} S(y_{n+1},y_{n+2})=0$, but from W-4 and for r(integer), we have $\lim_{n\to\infty} S(y_n,y_{n+r})=0$ which implies that $\{y_n\}$ is a S-Cauchy in X, but X is space which is complete. So $\{y_n\}$ and its subsequences are convergent to u∈ X, that is $\{Ax_{2n}\}\to u$, $\{B_{2n+1}\}\to u$, $\{Rx_{2n}\}\to u$, $\{Tx_{2n+1}\}\to u$.

AsR is continuous: We have $RAx_{2n}\to Ru$, $RRx_{2n}\to Ru$as $n\to\infty$ (3.7)

As the couple (A,R) is semi compatible, therefore we must have

$\lim_{n\to\infty} ARx_{2n} = Ru$. (3.8)

Step 1: Substitute $x = Rx_{2n}$, $y = x_{2n+1}$ in equation (3.4) and Choose n→ ∞

$$S(Ru,u)\le\varphi(M(u,x_{2n+1}))+CMin\left\{\begin{matrix}S(Ru,Tu),S(u,u),\\ S(Ru,Ru),S(u,Tu)\end{matrix}\right\}$$

Where

$$M(u,x_{2n+1})=\max\left\{S(u,x_{2n+1}),\frac{S(Ru,Tu)(S(Bx_{2n+1},Tx_{2n+1})+1)}{1+S(u,x_{2n+1})},\frac{S(Bu,Tu)(S(Ru,Tu)+1)}{1+S(u,x_{2n+1})}\right\}=0$$

Hence $S(Ru,u)\le\phi(0)=0$. Which is possible only if Ru=u.

Step 2: Substitute x =u, y = x_{2n+1} in (3.4),

$$S(Au,Bx_{2n+1})\le\phi(M(u,y))+CMin\{S(Rx,Tx),S(By,Ty),S(Au,Rx_{2n}),S(Bx_{2n+1},Tu\}$$
$$S(Au,u)=0(\text{ when n}\to\infty\ as\ above)$$

Step 3: Now u= Tw and A(X) ⊆ T(X). Take $x=x_{2n}$ and y= w in equation (3.4)

$$S(Ax_{2n},Bw)\le\varphi(M(x_{2n},w))+CMin\{S(Rx_{2n},Tx_{2n}),S(Bw,Tw),S(Ax_{2n},Rw),S(Bw,Tx_{2n})\}$$

Taking limit as n → ∞

$$S(u,Bw)\le\varphi(M(x_{2n},w))+CMin\{S(u,u),S(Bw,Tw),S(u,Rw),S(Bw,u)\}=0$$

That is Bw=u. This gives Bw=Tw=u. Byweak compatibility of (B,T) is, TBw=BTw i.e. Bu=Tu

Step 4: Take x =u, y = u in equation (3.4),

$$S(Au,Bu)\le\varphi(M(u,u))+CMin\{S(Ru,Tu),S(Bu,Tu),S(Au,Ru),S(Bu,Tu)\}$$

$$S(u,Bu) \le \varphi(M(u,u)) + CMin\{S(u,Bu), S(Bu,Bu), S(u,u), S(Bu,Bu)\}$$

Where

$$M(u,u) = \max\left\{S(u,u), \frac{S(Ru,Tu)(S(Bu,Tu)+1)}{1+S(u,u)}, \frac{S(Bu,Tu)(S(Ru,Tu)+1)}{1+S(u,u)}\right\} = \text{S(u,Bu)}$$

Hence

$$S(u,Bu) \le \phi(S(u,Bu) < S(u,Bu)$$

Which is possible only if Bu=u

For uniqueness, assume z is also a common fixed point so z = Az = Bz = Sz = Tz.

Take x = u and y = z in (3.4),

$$S(Au,Bz) \le \varphi(M(u,z)) + CMin\{S(Ru,Tu), S(Bz,Tz), S(Au,Rz), S(Bz,Tu)\}$$

Wherever

$$M(u,z) = \max\left\{S(u,z), \frac{S(Ru,Tu)(S(Bz,Tz)+1)}{1+S(u,z)}, \frac{S(Bu,Tu)(S(Ru,Tu)+1)}{1+S(u,z)}\right\} = \text{S(u,z)}$$

Hence

$$S(u,z) \le \phi(S(u,z) < S(u,z)$$

u=z. Hence u is exclusive fixed point.

Example: Let $X = [1,\infty)$. Define the mapping A, B, R and T from X to X as $Ax = x, Rx = x^2, Bx = 3x-2, Tx = x^2$.Then clearly we seen that $A(X) \subset T(X)$and $B(X) \subset R(X)$.

Define $x_n = \left(1+\frac{1}{n}\right)$.

Then by definition of semi-compatible of pair (A,R)

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1and\lim_{n\to\infty} Rx_n = \lim_{n\to\infty}(x_n)^2 = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1$ $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Rx_n = 1$ for $1\epsilon[1,\infty)$.

Then by definition $\lim_{n\to\infty} ARx_n = Rt$ for some $t \,\epsilon\, X$. Then the pair (A,R) is semi-compatible. Also to show pair (B,T) is occasionally weakly compatible.

Consider for $x = 1. Bx = 3x-2 \Rightarrow B1 = 1$ and $Tx = x^2 \Longrightarrow T1 = 1$.

Also $BT(1) = B(1) = 1 \; and \; TB(1) = T(1) = 1^2 = 1$

i.e.$BT(1) = T(B)1 \; iff \; B1 = T1$.

Also we see that at $x = 2$, $B2 = 4$ and $T2 = 4$,

but$(BT)(2) = B(4) = 10$and $T(B(2)) = T(4) = 16$i.e. $BT(2) \neq TB(2)$

Hence the pair $(B,T)$are occasionally weakly compatible.Now applying the inequality

$$S(Ax,By) \le \varphi(M(x,y)) + CMin\{S(Rx,Tx), S(By,Ty), S(Ax,Ry), S(By,Tx)\}$$

∀x, y ∈ X, C ≥0 , whereas

$$M(x,y) = \max\left\{S(x,y), \frac{S(Rx,Tx)(S(By,Ty)+1)}{1+S(x,y)}, \frac{S(Bx,Tx)(S(Rx,Tx)+1)}{1+S(x,y)}\right\}$$

$Ax = x, Rx = x^2, Bx = 3x-2, Tx = x^2$

$$S(A1,B1) \le \varphi(M(1,1)) + CMin\{S(R1,T1), S(B1,T1), S(A1,R1), S(B1,T1)\}$$

Where,

$$M(1,1) = \max\left\{S(1,1), \frac{S(R1,T1)(S(B1,T1)+1)}{1+S(1,1)}, \frac{S(B1,T1)(S(R1,T1)+1)}{1+S(1,1)}\right\}$$

$$S(A1,B1) \le \varphi(\max\left\{S(1,1), \frac{S(1,1)(S(1,1)+1)}{1+S(1,1)}, \frac{S(1,1)(S(1,1)+1)}{1+S(1,1)}\right\}) + CMin\{S(1,1), S(1,1), S(1,1), S(B,1)\}$$

$$S(A1,B1) \le \varphi\{0,0,0\} + CMin\{0,0,0,0\} = 0 \Rightarrow A1 = B1$$

Thus it is a common fixed point of all four mappings.

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