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Inclusion Relation Among Permutation Representations of S_6

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***Abstract:** Permutation representations arise from the action of groups on sets. Every conjugacy class C of a finite group G has a transitive action on G:g.x=gxg^{-1} for xεC and gεG. In this paper we consider the group S_6, group of all permutations on 6 symbols and completely describe the inter relationship among permutation representations arising from all its conjugacy classes. We have achieved a complete specification of these permutation representations: how they are mutually disposed to each other. For this purpose, we define subordination relationship between two conjugacy classes.Given two conjugacy classes C_1 and C_2, we say that C_1 is subordinate to C_2 if the permutation representation corresponding to C_1 is a subrepresentation included in that of C_2. As in whole of Mathematics relationship that are expressed in the form of functions will have special significance when these functions turn out to be injective. Our subordinate relationship naturally captures these significant cases. This relation is reflexive and transitive. Though not antisymmetric it has many desirable properties. The Hasse diagram has a very nice visual symmetry. In this paper we determine which conjugacy classes are subordinate to which for the case of the group G=S_6. Our method involves averaging process as used in Reynold's operator. Another important feature employed in our methodology is using duality which converts the problem of constructing an injective homomorphism to constructing a surjective homomorphism in the opposite direction.*

Index Terms: Symmetric groups, Permutation representations, Equivariant map, Global Conjugacy class

I. INTRODUCTION

For a finite group G the most fundamental representation is the regular representation of G. It is the permutation representation of G arising from the action of G on itself by left translation. This representation contains all the irreducible representations of G with multiplicity equal to its degree. There are other natural permutation representations available for all finite groups G. Our focus will be the permutation representations arising from the conjugation action of G on each of its conjugacy classes.

The conjugation action of S_n on the whole of S_n and the corresponding permutation representation is studied by A. Frumkin [1].

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Later G. Heide and A.E. Zalesski [2] studied the representations of alternating groups A_n arising from single conjugacy classes and Sheila Sundaram [4] did the same for the symmetric groups S_n. Their achievement is that they were able to determine which conjugacy classes lead to permutation representations containing all the irreducible representations of A_n and S_n respectively. Such conjugacy classes are defined to be *Global Conjugacy classes* [4]. So it becomes important to study permutation representations arising from conjugacy actions.

Sheila Sundaram's work covers the cases for n≥8 [4]. So we wanted to investigate the case S_6 which is not covered. We decomposed the permutation characters to find the multiplicity of all the irreducible representations in it. We used SAGE software to find the permutation characters and find the multiplicity of all the irreducible representations in it which are given in the appendix A and B. We found that the conjugacy class 5+1(5-cycles) is a global conjugacy class. This is evident from the second column from the right of the table included in Appendix C.

Further from the visual inspection of that table, we found that the multiplicities of all the irreducible representations in certain pair of permutation characters is such that one is always above the other. This paper arose of our attempt to understand why this happens in some cases and not in the others. We studied about the inter-relationship among the conjugacy classes in terms of their associated permutation representations. This relationship is depicted in the form of a 'Hasse diagram', though it is not actually a partial order (anti-symmetry does not hold). The theoretical proof of the inter-relationship is given by actually constructing the S_6-equivariant maps.

After this introduction, in Section 2 we provide definitions and preliminaries. In section 3, we describe all the S_6-equivariant maps corresponding to every edge in the diagram giving proofs about the properties of those maps in places where it is not obvious.

II. DEFINITIONS AND PRELIMINARIES

Definition 1: For a finite group G and two conjugacy classes C_1 and C_2 we say that C_1 is subordinate to C_2, denoted by $C_1 < C_2$ if the permutation representation corresponding to C_1 is a subrepresentation of that of C_2 .

The conjugacy classes corresponding to the partitions 6=4+2 and 6=4+1+1 lead to the same permutation representation: The centraliser of the 4-cycle (1234) ε S_6 coincides with that of (1234)(56) ε S_6. In S_6 this pair of conjugacy classes is the only exception.

Remark: For two representations V,W, there exists a G-equivariant injective linear transformation W → V iff there exists a G-equivariant surjective linear map of their duals V* →W*.

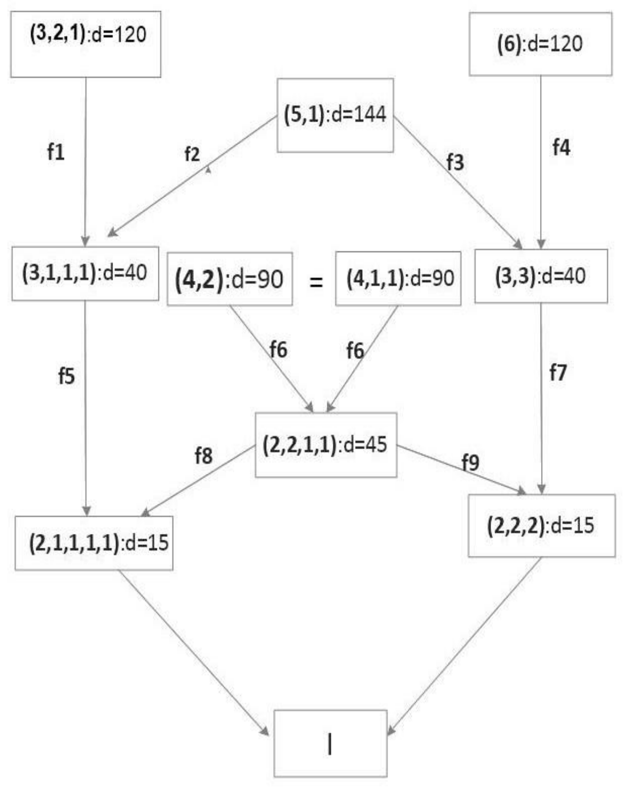
By this remark to show subordination relationship, we can exhibit a G-equivariant surjective map in the opposite direction. (Note that for the symmetric group S_n all representations are self-dual.)

Now we are ready to state the main result of the paper.

Theorem:

The diagram below (see Figure 1) depicts the subordination relationship among the conjugacy classes of S_6. In the diagram each node denotes the permutation representation corresponding to the conjugacy class. As we use duality the arrows in our diagram represents surjective S_6- equivariant maps.

Figure 1: Each node shown as a rectangle is a permutation representation corresponding to a conjugacy class, a partition. A partition such as 3+2+1 is shown here as (3,2,1) and the dimension of the representation is shown as d= \. (It is the number of elements of that conjugacy class). The labels on each arrow of the diagram denote specific equivariant maps which are explicitly given below.



III. EQUIVARIANT SURJECTIVE MAPS

In this section we define all the maps corresponding to every edge of the diagram. We need to show them to be S_6-equivariant and surjective. We provide proof for the first two maps and the proofs for the rest of them are omitted as they follow the same line of arguments.

Description of the map $f_1: 3+2+1 \to 3+1^3$**:**

The map f_1 is defined from 120-dimensional representation to 40-dimensional representation.

For any σ, the map is defined as $f_1(\sigma) = \sigma^2$.

More precisely, $f_1[(x_1x_2x_3)(x_4x_5)(x_6)] = (x_1x_3x_2)$

Equivariance of f_1:

Now by the definition of the map

$f_1(g\sigma g^{-1}) = (g\sigma g^{-1})^2 = g\sigma g^{-1}g\sigma g^{-1} = g\sigma^2 g^{-1} = g f_1(\sigma) g^{-1}$

(ie)

$f_1(g\sigma g^{-1}) = g f_1(\sigma) g^{-1}$

Hence f_1 is equivariant.

Surjectivity of f_1:

For a given 3-cycle τ, we need to find a 3+2+1-type σ such that $f_1(\sigma) = \tau$. This is easy: For example consider $\tau = (1\ 3\ 4)$, then for $\sigma = (1\ 4\ 3)(2\ 5)(6)$, we have $f_1(\sigma) = \sigma^2 = \tau$ proving surjectivity of f_1.

Description of map $f_2: 5+1 \to 3+1^3$ **:**

The map f_2 is defined from 144-dimensional representation to 40-dimensional representation.

$$f_2[(x_1x_2x_3x_4x_5)(x_6)] = (x_1x_2x_3) + (x_2x_3x_4) + (x_3x_4x_5) + (x_4x_5x_1) + (x_5x_1x_2)$$

(Here we consider the 3-cycle formed by the first 3 symbols in the given 5-cycle. As there are 5 ways of writing a 5-cycle with different starting point, we take the sum of all those to make it well-defined.)

Equivariance of f_2 :

$f_2(g\sigma g^{-1}) = f_2[g(x_1x_2x_3x_4x_5)(x_6)g^{-1}]$

$= f_2[(g(x_1)g(x_2)g(x_3)g(x_4)g(x_5))(g(x_6))]$

$= (g(x_1)g(x_2)g(x_3)) + (g(x_2)g(x_3)g(x_4)) + (g(x_3)g(x_4)g(x_5))$
$+ (g(x_4)g(x_5)g(x_1)) + (g(x_5)g(x_1)g(x_2))$

(by definition of f_2)

$= g(x_1x_2x_3)g^{-1} + g(x_2x_3x_4)g^{-1} + g(x_3x_4x_5)g^{-1}$

$= g(x_1x_2x_3)g^{-1} + g(x_2x_3x_4)g^{-1} + g(x_3x_4x_5)g^{-1}$
$+ g(x_4x_5x_1)g^{-1} + g(x_5x_1x_2)g^{-1}$

$= g[(x_1x_2x_3) + (x_2x_3x_4) + (x_3x_4x_5) + (x_4x_5x_1) + (x_5x_1x_2)]g^{-1}$

Hence the Equivariance.

Surjectivity: The matrix of this map is easy to describe with respect to obvious bases. Image of the basis vector of the domain space is the sum of five basis elements in the codomain as the RHS above is a sum of 5 terms. So each column of the matrix has five entries as 1, and all other entries zero. This matrix is checked to have rank 40 by SAGE, thereby showing surjectivity.

Description of the map for $f_3: 5+1 \to 3^2$**:**

The map f_3 is defined from 144 dimensional representation to 40 dimensional representation.

$$f_3[(x_1x_2x_3x_4x_5)(x_6)] = (x_1x_2x_3)(x_4x_5x_6) + (x_2x_3x_4)(x_5x_1x_6) + (x_3x_4x_5)(x_1x_2x_6) + (x_4x_5x_1)(x_2x_3x_6) + (x_5x_1x_2)(x_3x_4x_6)$$

The matrix of this map is easy to describe with respect to obvious bases. Image of the basis vector of the domain space is the sum of five basis elements in the codomain; so each column of the matrix has five entries as 1, and all other entries zero.

This matrix is checked to have rank 40 by SAGE, thereby showing surjectivity.

Description of the map $f_4: 6\ cycles \to 3^2$ **:**

The map f_4 is defined from 120-dimensional vector space to 40-dimensional vector space.

For any σ, the map is defined as $f_4(\sigma) = \sigma^2$

For example:

$f_4[(x_1x_2x_3x_4x_5x_6)] = (x_1x_3x_5)(x_2x_4x_6)$

Equivariance of f_4: The proof is similar to the proof given for the map f_1 and is omitted.

Surjectivity of f_4: For a given 3+3-type τ, we need to find a 6-cycle σ such that $f_1(\sigma)=\tau$.
This is easy: For example consider $\tau = (1\ 3\ 5)(2\ 4\ 6)$, then for $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$,
we have $f_4(\sigma) = \sigma^2 = \tau$ proving surjectivity of f_4.
Description of the map for $f_5 : 3+1^3 \to 2+1^4$ **:**
The map f_5 is defined from 40-dimensional vector space to 15-dimensional vector space.
$f_5[(x_1x_2x_3)(x_4)(x_5)(x_6)] = (x_1x_2)+(x_2x_3)+(x_3x_1)$
The matrix for f_5 is checked to have rank 15 by SAGE, thereby showing surjectivity.
Description of the map for $f_6 : 4+2 \to 2^2+1^2$ **:**
The map f_6 is defined from 90-dimensional vector space to 45-dimensional vector space.
For any σ, the map is defined as $f(\sigma) = \sigma^2$
For example: $f_6[(x_1x_2x_3x_4)(x_5x_6)] = (x_1x_3)(x_2x_4)(x_5)(x_6)$
Equivariance of f_6: Similar to f_1.
Surjectivity of f_6: For a given $2+2+1^2$-type τ , we need to find a $4+2$- cycle σ such that
$f_6(\sigma) = \tau$.
This is easy: For example consider $\tau = (1\ 3)(2\ 4)(5)(6)$, then for $\sigma = (1\ 2\ 3\ 4)(5\ 6)$, we have
$f_6(\sigma) = \sigma^2 = \tau$ proving surjectivity of f_6.
Description of the map $f_7 : 3^2 \to 2^3$**:**
The map f_7 is defined from 40-dimensional vector space to 15-dimensional vector space.

$$f_7[(x_1x_2x_3)(x_4x_5x_6)] = (x_1x_4)(x_2x_5)(x_3x_6)$$

$$+(x_1x_5)(x_2x_6)(x_3x_4) + (x_1x_6)(x_2x_4)(x_3x_5)$$

The matrix for f_7 is checked to have rank 15 by SAGE, thereby showing surjectivity.
Description of the map $f_8 : 2^2+1^2 \to 2+1^4$:
The map f_8 is defined from 45-dimensional vector space to 15-dimensional vector space.
$f_8[(x_1x_2)(x_3x_4)] = (x_5x_6)$
Equivariance of f_8:
Now by the definition of the map
$f_8(g\sigma g^{-1}) = (g\sigma g^{-1})^2 = g\sigma g^{-1}g\sigma g^{-1} = g\sigma^2 g^{-1} = g f_8(\sigma) g^{-1}$
(ie) $f_8(g\sigma g^{-1}) = g f_8(\sigma) g^{-1}$
Hence f_8 is equivariant.
Surjectivity of f_8:
For a given $2+1^4$-type τ , we need to find a 2^2+1^2-cycle σ such that $f_8(\sigma) = \tau$.
This is easy: For example consider $\tau = (5\ 6)$, then for $\sigma = (1\ 2\ )(3\ 4)(5\ 6)$, we have $f_8(\sigma) = \sigma^2 = \tau$
proving surjectivity of f_8 .
Description of the map $f_9 : 2^2+1^2 \to 2^3$:
The map f_9 is defined from 45 dimensional vector space to 15 dimensional vector space.
$f_9[(x_1x_2)(x_3x_4)] = (x_1x_2)(x_3x_4)(x_5x_6)$
The matrix for f_9 checked to have rank 15 by SAGE, thereby showing surjectivity.

IV. CONCLUSION

Classically representation theory guarantees existence of one irreducible representation for every conjugacy class of the group. Explicit connection is available only for the permutation groups and cyclic groups. Our theorem for the group S_6 shows that the reducible permutation representation for each conjugacy class is rich enough to contain all its irreducible representations. Classically one has character tables of a group. By our method we have alternative to character tables which are multiplicities of irreducible representations in each conjugacy class. Being a table of same indexing set as character table this might provide an alternative source of information. Further the entries always being non-negative integers to complex numbersin classical character tables. So the problem that remains to be investigated in determining of groups admitting conjugacy classes.

APPENDIX A

Python/SAGE code for finding the permutation characters obtained by the action of the symmetric group S_6 on its conjugacy classes.

```
n=6
G=SymmetricGroup(n)
clist=G.conjugacy_classes()
ctype=G.conjugacy_classes_representative
s()
sizes=[]
for cl in clist:
  print cl[0], ":", len(cl)
  sizes.append(len(cl))
chcount = len(sizes)
permchars = [ ]
for cl in clist[:]:
  cl_rep = cl[0]
row=[]
for cc in clist:
  fix_pts=0
  for g in cc[:]:
    if cl_rep*g == g*cl_rep:
      fix_pts += 1
    row.append(fix_pts)
permchars.append(row)
for row in permchars: print row
ct=G.character_table()
print ct
```

APPENDIX B

Python/SAGE code for finding the multiplicity of the irreducible characters in the permutation characters

```
split_perms=[]
for i in range(chcount): ## i index for
 permchar
 row =[]
 for j in range(chcount): ## j index for  irr chars
   inn_prod=0
   for k in range(chcount): ## k index for


         conj classes
     inn_prod += ct[j][k] *
      permchars[k][i]*sizes[k]
```

```
    row.append(inn_prod/factorial(n))
    print inn_prod/factorial(n),
  print
  split_perms.append(row)
snum=0
for i in range(chcount):
     for j in range(i+1,chcount):
    i_score = 0
    j_score = 0
    snum +=1
    for r in range(11):
      if split_perms[i][r] <
        split_perms[j][r]:
        j_score += 1
      elif split_perms[i][r] >
        split_perms[j][r]:
        i_score += 1
      print snum,":",
      if i_score + j_score ==0:
        print ctype[i], "=", ctype[j]
      elif i_score == 0:
        print ctype[i], " <", ctype[j]
      elif j_score == 0:
        print ctype[j], "<", ctype[i]
      elif i_score*j_score > 0:
        print ctype[j], "#", ctype[i]
```

V. APPENDIX C

Table 1: Multiplicities of the irreducible representations of S_6 in the permutation characters

Per Rep	I	21^4	$2^2 1^2$	2^3	31^3	321	3^2	41^2	42	51	6
Size	1	15	45	15	40	120	40	90	90	144	120
1-dim	0	0	0	0	0	0	0	0	0	1	0
5-dim	0	0	0	0	0	1	1	0	0	1	1
9-dim	0	0	0	0	0	1	0	1	1	1	1
5-dim	0	0	1	1	0	0	1	1	1	1	2
10-dim	0	0	0	0	1	2	1	1	1	2	2
16-dim	0	0	1	0	0	2	0	2	2	4	2
5-dim	0	0	0	0	1	1	0	0	0	1	1
10-dim	0	0	0	0	1	2	1	1	1	2	2
9-dim	0	1	2	1	1	2	1	2	2	1	2
5-dim	0	1	1	0	1	2	0	1	1	1	0
1-dim	1	1	1	1	1	1	1	1	1	1	1

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