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Solving Royalty Problem Through a New Modified Shooting Method

Wan Noor Afifah Wan Ahmad, Suliadi Firdaus Sufahani, Alan Zinober

Abstract: ***Optimal Control (OC) study in the recent year has been an interest of many young researchers, especially in the nonstandard OC context. The same goes for this research's intention. The nonstandard problem dealing with the free final state value $y(T)$ where the integrand of the performance index relies on the $y(T)$ value. In addition, the final costate value $p(T)$ is said to be a nonzero solution. Contrastly, in the standard setting, the boundary condition $p(T)=0$ and the integrand are independent to the known final state value $y(T)$. This paper is interested in solving the nonstandard OC problem through modified shooting method with the involvement of 4-stages piecewise constant integrand system for the royalty function, ρ. This system is then being converted into a continuous approximation of hyperbolic tangent (tanh) function in order to allow the system to become differentiable at any process. The outcome of the modified shooting method will be obtained by running the program in C++ programming language and will be compared with the discretised values as a validation procedure.***

Index Terms: ***Discretisation, Hyperbolic tangent function, Optimal Control, Royalty problem, Shooting method.***

I. INTRODUCTION

Optimal Control (OC) is a study in determining the control $u(t)$ of a certain system in order to achieve an optimality term. The OC study has been applied by thousands of researchers in different fields, for example, in the medical application where the research is conducted by [11] and [13], the study on diet planning is done by [6], [19], [20] and [21] including the study in aerospace field that have been done by [7] and [22], besides economics application [9], [25]. Other than that, [17] and [24] also have a very high interest in OC application where they have researched the royalty payment problem with the implementation of OC theory. In addition, [1], [2], [3], [4] and [5] also have a high research interest in economical application. They solved the economic problem through mathematical method and OC theory.

Based on [15] and [23], the definition of OC is expressed as in the following Definition 1.

Definition 1. [15], [23] *The OC problem is to find an admissible control $u^*(t)$ which causes the system*

$$\dot{y}(t)=f\left(t,y(t),u(t)\right) \tag{1}$$

to follow an admissible optimal trajectory $y^*(t)$ that extremises (minimises or maximises) the performance measure

$$J=h\left(T,y(T)\right)+\int_{t_0}^{T} g\left(t,y(t),u(t)\right)dt. \tag{2}$$

$u^*(t)$ is called the optimal control and $y^*(t)$ is the optimal trajectory.

OC is an elongation of the issue in the Calculus of Variations (CoV) subject. CoV deals with the problem that utilises variations in function including the functional. This is to discover the maximum and minimum of a given functional. Based on [8], [15] and [23], the CoV can be defined as in the following Definition 2.

Definition 2. [8], [15], [23] *Determine the function $y(t)$ which minimizes or maximizes the functional, i.e. the integral*

$$J\left[y(t)\right]=\int_{t_0}^{T} g\left(t,y(t),\dot{y}(t)\right)dt \tag{3}$$

where t_0 and T are given constants and the boundary conditions $y(t_0)=y_0$ and $y(T)$ can be specified or unspecified, subject to

$$\dot{y}(t)=\frac{dy}{dt} \tag{4}$$

where $J\left[y(t)\right]$ is denoted as a functional with the integral function which is denoted as $g(\cdot)$ and $(\cdot)=\left(t,y(t),\dot{y}(t)\right)$, a function of variables t, $y(t)$, and $\dot{y}(t)$.

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***** Correspondence Author
Wan Noor Afifah Wan Ahmad*, Department of Mathematics and Statistic, Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia, Pagoh Educational Hub, 84600 Pagoh, Muar, Johor.
Suliadi Firdaus Sufahani, Department of Mathematics and Statistic, Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia, Pagoh Educational Hub, 84600 Pagoh, Muar, Johor.
Alan Zinober, School of Mathematics and Statistics, University of Sheffield, Sheffield S3 7RH, United Kingdom.

The classic set-up in OC is that the zero outcomes for the final costate value $p(T)$ will be obtained. This necessary boundary condition is yielded from the final state value $y(T)$. In addition, the integral of the performance index does not depend on $y(T)$. Contrastly, the integrand function in the non-classical setting is depending on the free state variable at the finishing time $y(T)$. In extension, the costate variable at the final time $p(T)$ is the nonzero result.

This study is concerned with the second setting (non-classical control problem) as the main problem in continuing the research. The setting disallowed the problem to be solved by using Pontryagin's Minimum Principle together with the boundary condition in the standard setting. The final state value $y(T)$ will be equal to another continuous function $y(T)=z$. This is because the integral system is the component of the final state value.

Therefore, the research will be continued in solving royalty payment problem as the application of OC theory. The system considers 4-stage constant piecewise function for ρ value but the problem arises where the system cannot be differentiated in certain processes. Thusly, this research approaches another suitable methodology where the system will be converted into a continuous form of hyperbolic tangent (tanh) function to ensure that it is differentiable in every process. The problem will use C++ programming language for the modified shooting method. Later on, the result obtained will be compared with the discretised values as a validation process. The discretisation methods include the Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations which will be run by AMPL program language.

II. COMBINATION OF NEWTON WITH GOLDEN SECTION SEARCH METHOD

This study combined the Newton and a one-dimensional minimisation technique in shooting method to solve the nonstandard royalty problem. According to Press et al. in 2007 [16], the Golden Section Search method is considered as a one-dimensional minimisation technique and these combinations are applied in C++ program language where the highly accurate algorithm is referred from the Numerical Recipe in [16].

The very first step is to assume that there are two scalar functions of f_1 and f_2. The Golden Section Search method is applied in determining the best final state value y_T and the Newton method is used for the root iteration while "Odeint" (ODE solver) runs the program with initial guess values. As for the result, the Golden Section Search will generate various possible final state value y_T and these values will be sent to Newton iteration. In this process, the values will be used at each iteration to make sure that the scalar function is close enough to zero and to adjust the freely selected guess value. This is to zero the error at the terminal time. At the terminal point, the program will check whether the scalar function is close to zero and the optimal solution is obtained. If not, the minimisation technique will run the program to find another possible value and the cycle continues until the optimal solution is achieved. The process terminates once the program generates the same results four times and there is no other possible value anymore. Since the optimal result is obtained by using the minimisation method, therefore, to maximise, the performance index J function needs to be multiplied by a negative one [16].

The algorithm for these combination is as shown in the following Algorithm 1.

Algorithm 1. Newton with Golden in shooting method

Input : t_0 (initial time), T (final time), $p(0)$ (initial value), y_T (guessed value), $y(0)=0$ (boundary condition), ODE's

Output : The approximation to $t=$ time, $y[0]=y(t)$, $y[1]=p(t)$ and $y[2]=\eta(T)$

Step 1: Initialization

(a) Define the number of ODEs and the number of guess(ed) values.

(b) Set the initial time, final time, boundary condition, initial values, guess(ed) values.

Step 2: Calculation

(c) Call Golden with three range values of y_T :
$a=r_1, b=r_2$ and $c=r_3$
(Golden will calculate and generate y_T value within the range (r_1, r_2, r_3) and transmit it to Newton) which maximizing the system.

(d) Call Newton solver with two scalar functions f_1 and f_2: $f_1=\left(y(T)-y_T\right)$, $f_2=\left(p(T)-\eta(T)\right)$

(e) Run the ODE solver with initial guess v say, $p(0)=v$.

(f) At T (final time), check whether the f_1 and f_2 become small enough.
(At the final time T, $y(T)$ is equal to y_T, $p(t)$ is equal to $\eta(T)$).

if yes then
Go to (g).
else
Go to (c) and update f_1 and f_2.

end if

(g) Check whether Golden generates the best value of y_T that maximizes the system.

If yes then

Go to (h).

Else

Go to (c) and update (r_1, r_2, r_3). This will update the y_T value.

End if

(h) End and printout solution.

III. DISCRETIZATION METHOD

Discretisation methods involve the Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations. The discretisation and shooting results will be compared and this acts as a validation procedure that the shooting method produces a very accurate solution. The program is run by AMPL program language with MINOS solver [10]. In the program, AMPL program language sends the required data to MINOS solver in computing the values and partial derivatives in the objectives and constraints. After that, AMPL program language with MINOS solver interprets the problem from the associated dual variables to the associated Lagrange Multiplier [10]. This study sets up the discrete notation for the variables; $y_i(t)$ is the discrete state variable, $p_i(t)$ is the discrete costate variable, and $u_i(t)$ is the discrete control variable. The initial discrete costate value $p_i(t)$ is used as an initial guess value for the shooting method.

IV. NONSTANDARD OPTIMAL CONTROL PROBLEM

The integral for the objective function of the problem is depending on the state value $y(t)$ at the final time T, which is free and unknown. Additionally, the integrand of the objective function is in the form of a piecewise constant function which contains the unknown final state value $y(T)$. At the same time, the final value of costate $p(T)$ is not equal to zero and this situation needs to be expressed in the new boundary condition. This problem is considered as a nonstandard OC problem where the problem is unsolvable by using the Pontryagin Minimum Principle with the standard boundary condition at the final time. The proof for the new boundary condition has been discovered by Malinowska and Torres in 2010 [12].

Theorem 1. [12] *Suppose that s and S are real numbers where a<T. If $y(t)$ is the solution of the following problem*

$$J\left[y(t)\right] = \int_s^S g\left(t, y(t), \dot{y}(t), z\right) dt$$
$$y(s) = \alpha\,,\ y(T) \text{ is free and unknown} \tag{5}$$
$$y(t) \in C^1$$

Then

$$\frac{d}{dt} g_{\dot{y}}\left(t, y(t), \dot{y}(t), z\right) = g_z\left(t, y(t), \dot{y}(t), z\right) \tag{6}$$

For all $t \in [s, S]$ and hence,

$$g_{\dot{y}}\left(t, y(t), \dot{y}(t), z\right) = -\int_s^S g_z\left(t, y(t), \dot{y}(t), z\right) dt \tag{7}$$

The theorem above figures out that the necessary optimal condition does not have zero final costate value, $p(T) = 0$ and from the optimal control view, the final costate variable $p(T)$ is equivalent to $g_{\dot{y}}\left(t, y(t), \dot{y}(t), z\right)$ where the integrand function g is differentiable with respect to z. Thus,

$$p(T) = -\int_s^S g_z\left(t, y(t), \dot{y}(t), y(T)\right) dt \tag{8}$$

Equation (8) is considered as a new boundary condition which is also equal to $\eta(T)$. The next section will discuss in detail regarding the problem formulation.

V. ROYALTY MODEL

According to the above discussion, let us consider the following model [17]

$$\begin{aligned} J\left[u(t)\right] &= \int_{t_0}^{T} g\left(t, y(t), u(t)\right) dt \\ &= \int_{t_0}^{T} \left(a(t)u^{1-\alpha} - \left(\rho + m_0 + c_0 e^{-\lambda y}\right)u(t)\right)e^{-rt} dt \end{aligned} \tag{9}$$

where

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$\Rightarrow a(t) = e^{0.025t}$

$\Rightarrow \alpha = 0.5(\text{price elasticity of demand})$

$\Rightarrow \rho = \rho(\text{royalty})$

$\Rightarrow m_0 = 1.0(\text{asymptote of the learning curve})$

$\Rightarrow c_0 = 1.0(\text{component of unit cost that is subject to learning})$

$\Rightarrow \lambda = 0.12(\text{a parameter that defines the speed of learning})$

$\Rightarrow r = 0.1(\text{discount factor})$

The model illustrates the objective function or performance index which was proposed by previous researchers including [17] and [24]. Below is the ordinary differential equation (ODE) that has been involved in order to optimise the performance index.

$$\dot{y}(t) = u(t) \tag{10}$$

The royalty function, ρ will consider a 4-steps constant piecewise system which will then modify into a continuous hyperbolic tangent (tanh) approximation. This is to make sure that the system can implement the differentiation process everywhere. The settings that will be utilised are; zero initial time, the terminal time is equal to 10, the initial state value is zero and the free and unknown final state value.

However, there will be a few necessary set-ups that need to meet the satisfaction in continuing the process; the state and costate equation with the stationarity term, the initial requirement of state and costate variable are defined, the integral boundary condition is satisfied at terminal time T and also, the value of z needs to be equivalent with the final state value $y(T)$ in order to obtain the function f_i that is near to zero. The conditions are satisfied whenever the costate system converges. After that, the optimal solution will be attained. The featured control problem will be stated and discussed more in the following section.

VI. ROYALTY EXAMPLE

The proposed model will use the following royalty ρ value which is equal to the 4-stage piecewise constant integrand function.

$$\rho(y(t)) = \begin{cases} 0.1 & \text{for } 0 \le y(t) \le 0.25z \\ 1.2 & \text{for } 0.25z < y(t) \le 0.5z \\ 0.24 & \text{for } 0.5z < y(t) \le 0.75z \\ 0.12 & \text{for } 0.75z < y(t) \le z \end{cases} \tag{11}$$

The $\rho(y)$ can be converted into the hyperbolic tangent (tanh) function

$$\begin{aligned}\rho(y) = {} & \frac{11}{100} + \frac{11}{20}\tanh(k(y-0.25z)) - \\ & \frac{12}{25}\tanh(k(y-0.5z)) - \\ & \frac{3}{50}\tanh(k(y-0.75z))\end{aligned} \tag{12}$$

and in this study, the values $k = 50$ and $k = 250$ are chosen in order to approximate (12). The larger the smoothing value k is, the smoother the ρ plot will be. The continuous integrand function g is

$$g = \left(e^{0.025t}u^{0.5} - \left(\rho + 1 + e^{-0.12y}\right)u\right)e^{-0.1t} \tag{13}$$

where $\rho = \rho(y)$. The Hamiltonian function is $H = g + pu$ where

$$H = \left(e^{0.025t}u^{0.5} - \left(\rho + 1 + e^{-0.12y}\right)u\right)e^{-0.1t} + pu \tag{14}$$

The state equation satisfies the Hamiltonian system where

$$\begin{aligned}\dot{y}(t) &= H_p \\ &= \left(e^{0.025t}u^{0.5} - \left(\rho + 1 + e^{-0.12y}\right)u\right)e^{-0.1t} + u\end{aligned} \tag{15}$$

Function g depends on $y(t)$ and ρ, thus, the costate satisfies

$$\begin{aligned}\dot{p} &= -H_y \\ &= \left(\frac{109}{100}k - \frac{11}{20}k\tanh(k(y-0.25z))^2 + \right. \\ & \frac{12}{25}k\tanh(k(y-0.5z))^2 + \\ & \left. \frac{3}{50}k\tanh(k(y-0.75z))^2 - 0.12e^{-0.12y}\right)ue^{-0.1t}\end{aligned} \tag{16}$$

The stationarity condition is $H_u = 0$ where

$$H_u = \left(\frac{0.5e^{0.025t}}{\sqrt{u}} - \rho - e^{-0.12y} - 1\right)e^{-0.1t} + p \tag{17}$$

This produced

$$u(t) = \frac{0.25\left(e^{0.025t}\right)^2\left(e^{-0.1t}\right)^2}{\left(\rho e^{-0.1t} + e^{-0.12y}e^{-0.1t} + e^{-0.1t} - p\right)^2} \tag{18}$$

and the integral yields as

$$p(T) = \int_0^{10} \left(\frac{59}{400}k + \frac{11}{80}k \tanh\left(k\left(y - \frac{1}{4}z\right)\right)^2 - \frac{6}{25}k \tanh\left(k\left(y - \frac{1}{2}z\right)\right)^2 - \frac{9}{200}k \tanh\left(k\left(y - \frac{3}{4}z\right)\right)^2 \right) ue^{-0.1t} dt = \eta(T) \tag{19}$$

The results for the 4-stage piecewise ρ with $k = 50$ and $k = 250$ for the shooting method and the nonlinear programming validations are now presented in two types of k values.

VII. RESULTS

The illustrated example will be solved in two different k values. The next section will show the result when the smoothing values are 50 and 250.

A. Case I: *k*=50

The first case considers the smoothing value that is equal to 50 and the outcome is tabled as shown in Table 1.

Table 1 The result generated by the shooting and discretization methods for *k*=50. (NG=Newton & Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

	Results				
	$y(T)$	$p(0)$	$p(T)$	$\eta(T)$	$J(T)$
	Shooting method				
NG	0.585406	-0.278096	-0.108470	-0.108470	0.820501
	Discretization method				
EU	0.580630	-0.281408	-	-	0.824692
RK	0.583067	-0.278355	-	-	0.825111
TR	0.578117	-0.278567	-	-	0.825874
HS	0.593281	-0.273071	-	-	0.825276

Based on Table 1, the results generated by the shooting method for $y(T)$, $p(0)$ and $J(T)$ are slightly different when compared with the discretised values. The Euler and Runge-Kutta results for the final state value $y(T)$ are similar up to two decimal places when compared with shooting outcome. Meanwhile, both the Trapezoidal and Hermite-Simpson methods give the result for $y(T)$ which is comparative up to one decimal place when contrasted with the shooting result.

At the initial time, the costate value $p(0)$ for the Runge-Kutta and Trapezoidal results are similar up to three decimal places when compared with the shooting method. Both the Euler and Hermite-Simpson give the initial costate value $p(0)$ which is similar up to one and two decimal places with the shooting result, respectively.

As for the objective function $J(T)$ value, all methods give a comparative answer up to two decimal places.

The result is then transformed into a graphical idea as shown in Fig. 1 where the optimal solution for state variable $y(t)$ and $y_i(t)$, costate variable $p(t)$ and $p_i(t)$, and control variable $u(t)$ and $u_i(t)$ are plotted together with the performance index $J(t)$.

Fig. 1 illustrates the optimal solution generated by the shooting and discretisation methods. The shooting method produces a smoother plot when compared with the discretised plot. The optimal plot curves are similar for all variables except for the control plot where the discretised values tend to differ from the shooting values at a certain time. This is due to the discretisation error that occurs during the process [14], [18]. This can be concluded that the C++ routine in Numerical Recipe [16] produces a solution that is high in accuracy.

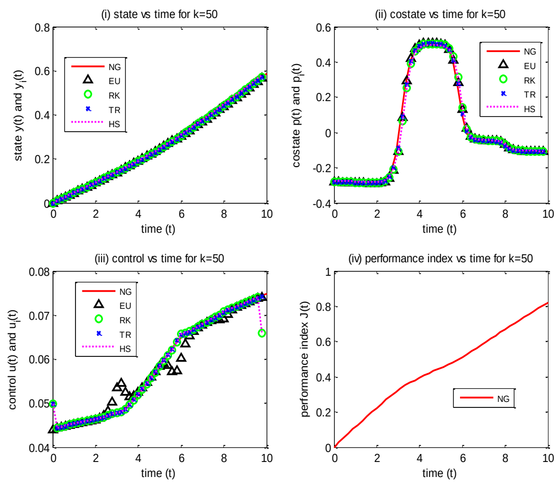

Fig. 1 The optimal plot for *k* = 50 generated by the shootingand discretization techniques (NG=Newton and Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson).

B. Case II: *k*=250

The second case considers the smoothing value which is equal to 250 and the generated result is tabled as shown in Table 2.Table 2 The result generated by the shooting and discretization methods for *k*=250. (NG=Newton & Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson)

	Results				
	$y(T)$	$p(0)$	$p(T)$	$\eta(T)$	$J(T)$
Shooting method					
NG	0.585314	-0.277648	-0.108443	-0.108443	0.820619
Discretization method					
EU	0.543349	0.082824	-	-	0.814081
RK	0.588223	-0.246044	-	-	0.826710
TR	0.575365	-0.198810	-	-	0.826766
HS	0.593381	-0.249868	-	-	0.826571

In light of Table 2, the final state $y(T)$ result for Runge-Kutta is similar up to two decimal places when compared with the shooting result, while, the other discretisation methods give similar answer up to one decimal place with the shooting outcome.

As for the initial costate value $p(0)$, the generated solution by the Runge-Kutta and Hermite-Simpson methods are comparative up to one decimal place when compared with the shooting output.

The performance index $J(T)$ gives the result which is comparative up to two decimal places for the Runge-Kutta, Trapezoidal and Hermite-Simpson when contrasted with the shooting. Meanwhile, the use of the Euler method resulting in the performance index $J(T)$ which is similar up to one decimal place with the shooting method.

The next step and discussion are similar to the first case where the generated result will be transformed into a graphical form.

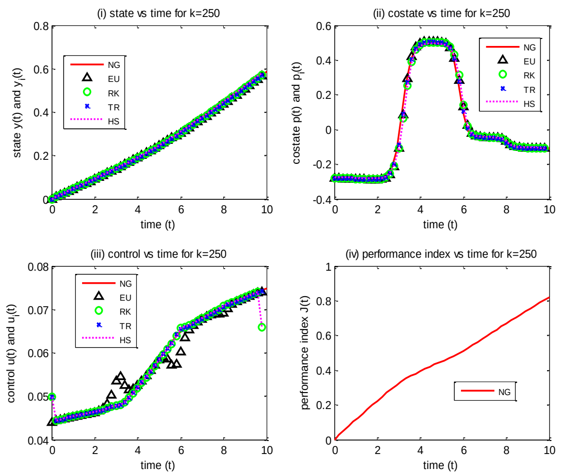

Fig. 2 The optimal plot for $k = 250$ generated from the shooting method and discretization techniques (NG=Newton and Golden; EU=Euler; RK=Runge-Kutta; TR=Trapezoidal; HS=Hermite-Simpson).

Fig. 2 shows the optimal solution in a graphical form for the second case where the smoothing value is equal to 250. Based on the illustration, the curves are similar with the first case where $k = 50$. However, the plot with k =250 is smoother when compared to the plot for the previous illustration in previous section.

VIII. CONCLUSION

This paper solved the nonstandard OC problem which involved the constant royalty system. In this study, the system was then converted into a continuous form of hyperbolic tangent (tanh) function. The problem applied the nonlinear shooting method (the combination of the Newton and the Golden Section Search methods) which was then compared with the discretisation techniques (the Euler, Runge-Kutta, Trapezoidal and Hermite-Simpson approximations) as a validation process. The problem was run by C++ and AMPL program language with MINOS solver in order to obtain an optimal solution.

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AUTHORS PROFILE

Wan Noor Afifah Wan Ahmad graduated from Universiti Tun Hussein Onn Malaysia in Master of Science in 2018 and Bachelor Degree of Science (Mathematics Technology) with Honours in 2017. Currently, the author pursue her study in Doctor of Philosophy in Science in the same institution. Her areas of interest are Optimal Control and Optimization.

Ts Dr Suliadi Sufahani is a Senior Lecturer and the Head Panel of Statistics in the Faculty of Applied Sciences and Technology, Universiti Tun Hussein Onn Malaysia. His areas of interest are Statistical Analysis, Operational Research, Mathematical Programming, Statistical Modeling, Optimization and Data Mining.

Prof Dr Alan Zinober is an Emeritus Professor at the School of Mathematics and Statistics, University of Sheffield, United Kingdom. His areas of interest is on Nonlinear control, optimisation.