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International Journal of Recent Technology and Engineering (IJRTE)
ISSN: 2277-3878 (Online), Volume-8, Issue-1, May 2019

# On Certain Class of S-Hyperbolic Polynomials

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***Abstract:** This article considers the following problem: for a polynomial P, the principal part of which is hyperbolic with respect to the vector $\eta^0$, and for weight function of hyperbolicity g of special kinds, it is required to determine terms for lower part of the polynomial P (in terms of comparison) at which the polynomial P becomes g hyperbolic with respect to any vector $\eta$ from a certain neighborhood of $\eta^0$.*

***Index Terms:** comparison of polynomials, weight hyperbolicity, hyperbolicity function.*

## I. INTRODUCTION AND FORMULATION OF THE PROBLEM

Let $R^n$ is the $n$ dimensional Euclidean space of points (vectors), respectively, $\xi = (\xi_1, \xi_2 \ldots, \xi_n)$, $C$ is the set of complex numbers, $R^{n,+} := \{\xi \in R, \xi_j \geq 0, j = 1, \ldots n\}$, $N$ is the set of natural numbers, $N_0 = N \cup \{0\}$, $N_0^n = N_0 \times N_0 \ldots \times N_0$ is the set of multi-indices, that is, the set of points with nonnegative components: $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \in N_0$, $j = 1,2, \ldots, n$

For $\xi \in R^n$, $\alpha \in R_0^n$ let us denote $|\xi| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2}$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$, $t\xi = (t\xi_1, t\xi_2, \ldots, t\xi_n)$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_j = \frac{\partial}{\partial \xi_j}$, $j = 1,2, \ldots, n$. Continuous positive function $g$, determined over $R^n$, is called the weight function of hyperbolicity if :

1. $\inf_{\xi \in R^n} g(\xi) > 0$

$$\lim_{\xi \to \infty} \frac{g(\xi)}{|\xi|} = 0$$

$$\sup_{\xi, \eta \in R^n} \frac{g(\xi + \eta)}{g(\xi) + |\eta|)} < \infty$$

2. For any $\theta > 0$ there exists such $c = c(\theta) > 0$ that $c^{-1} g(\xi) \leq g(\theta\xi) \leq c g(\xi)$.

Let $G(n)$ denote the set of all weight functions of hyperbolicity determined over $R^n$, and for $s > 0$ $g_s(\xi)$ will denote the following function: $g_s(\xi) := 1 + |\xi|^{\frac{1}{s}}$. Obviously, $g_s \in G(n)$ at $s > 1$.

For $e = (e_1, e_2, \ldots, e_n)$, where $e_j = 1$ or $e_j = 0, 1 \leq j \leq n$, $supp e$ and $\xi^e$ will denote $supp e = \{j, 1 \leq j \leq n, e_j = 1\}$, $\xi^e = (\xi_1^{e_1} \ldots \xi_n^{e_n})$, respectively, where $\xi_j^{e_j} = \xi_j$ at $j \in supp e$ and $\xi_j^{e_j} = 0$ at $j \notin supp e$.

It can be seen that at $s > 1$, the function $g_{1,s}(\xi) := g_s(\xi^e) \in G(n)$. For the function $g_s$ and $\eta \in R^n$, $g_{s,\eta}$ will denote $g_{s,\eta}(\xi) := \min_{t \in R} g_s(\xi - t\eta)$.

It can be seen that for any $\eta \in R^n$ at $s > 1$

1) $g_{s,\eta}(\xi - t\eta) = g_{s,\eta}(\xi) \forall (\xi, t) \in R^{n+1}$,

2) If $h$ is such a function that with certain constant $c > 0$ $c^{-1} g_s(\xi) \leq h(\xi) \leq c g_s(\xi)$ $\xi \in R^n$, then $h \in G(n)$.

Let $R$ is the polynomial of $n$ variables. We introduce the following Hörmander function (see [1],(10.4.2))

$$\tilde{R}(\xi, \eta) := \sum_{\alpha \in N_0^n} |D^\alpha R(\xi)||\tau|^{|\alpha|} := \sum_{\alpha \in N_0^n} |R^\alpha(\xi)||\tau|^{|\alpha|}, \quad (\xi, \tau) \in R^{n+1}.$$

Let $P(D) = \sum_\alpha \gamma_\alpha D^\alpha$ is the linear differential operator with constant coefficients where the sum is distributed over finite set $(P) = \{\alpha \in N_0^n, \gamma_\alpha \neq 0\}$, and $P(\xi) = \sum_{\alpha \in P} \gamma_\alpha \xi^\alpha$ is the complete symbol. Let us denote $m := \max\{|\alpha|, \alpha \in P\}$ and preset

$$P_j(\xi) = \sum_{\substack{|\alpha| = j \\ j \in (P)}} \gamma_\alpha \xi^\alpha$$

Then the polynomial $P$ can be represented as follows:

$P(\xi) = \sum_{j=0}^m P_j(\xi) = \sum_{j=0}^m \sum_{|\alpha|=j} \gamma_\alpha \xi^\alpha$ (1.1)

Definition 1.1: (see [2] or [1]:Definition 12.3.3 and Theorem 12.4.1)A polynomial $P$ is said to be hyperbolic (according to Gårding) with respect to the vector $\eta \in E^n$, if $P_m(\eta) \neq 0$ and there exists such a number $\tau_0 > 0$ that

$$P(\xi + i\tau\eta) \neq 0 \ \ \forall (\xi, \tau) \in R^{n+1}, |\tau| \geq \tau_0$$

Definition 1.2: (see [3])Let $1 < s < \infty$. A polynomial $P$ represented in the form (1.7) is referred as s-hyperbolic with respect to the vector $\eta \in E^n$, if $P_m(\eta) \neq 0$ and there exists such a number $c > 0$ that

$P(\xi + i\tau\eta) \neq 0 \ \ \forall (\xi, \tau) \in R^{n+1}, |\tau| \geq c\left(1 + |\xi|^{\frac{1}{s}}\right)$. (1.2)

Definition 1.3: Let $g \in G(n)$. We state that a polynomial $P$ is $g$–hyperbolic with respect to the vector $\eta \in E^n$, if $P_m(\eta) \neq 0$ and there exists such a number $c_1 > 0$ that

$$P(\xi + i\tau\eta) \neq 0 \ \ \forall (\xi, \tau) \in R^{n+1}, |\tau| \geq c_1 g(\xi).$$

Remark 1.2: It can be seen that

1)If the function $g \in G(n)$ is bounded, then any $g$ – hyperbolic polynomial is also hyperbolic according to Gårding with respect to the same vector.

2)If for certain $c > 0$ and $s > 1$

$$c^{-1}(1 + |\xi|)^{\frac{1}{s}} \leq g(\xi) \leq c(1 + |\xi|)^{\frac{1}{s}} \ \forall \xi \in R^n,$$

then any g – hyperbolic polynomial $P$ with respect to the vector $\eta$ is also s – hyperbolic with respect to the same vector.

Definition 1.4: (see, for instance, [3], [4], [5])The polynomial $P$ is weakly hyperbolic with respect to the vector $\eta$, if $P_m(\eta) \neq 0$ and

Revised Manuscript Received on 30 May 2019.
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1)For any $\xi \in R^n$the roots of polynomial $P_m(\xi + \tau\eta)$ (with to$\tau$) are real.
2)Among the roots there exists a root with multiplicity higher than one.

Definition 1.5 (see [6]):Let $g \in G(n)$and $P$and$Q$are polynomials of $n$ variables.

We state that the polynomial $P$is $g$ – stronger than the polynomial $Q$ ($Q$ is $g$ – weaker than $P$), and we write $Q \prec^g P$ ($P \succ^g Q$), if there exists sucha number $c > 0$that

$$\tilde{Q}(\xi, g(\xi)) \leq c\tilde{P}(\xi, g(\xi)) \ \forall \xi \in R^n.$$

If $g(\xi) \equiv const > 0$, then we saythat $Q$is weaker than $P$ (according Hörmander) and it is written as $Q \prec P$.
The following results are known (see [3], [1]):
1. Any weakly hyperbolic with respect to the vector $\eta$polynomial $P$is $s$hyperbolic $(1 < s < \frac{l}{l-1})$with respect to this vector where $l := \max_{|\xi|=1} l(\xi)$, and $l(\xi)$is the maximum multiplicity of zero of the polynomial $P_m(\xi + \tau\eta)$over $\tau$in the point $\xi \in R^n$,
2. If $g(\xi) \equiv const$,$Q \prec P_m$and $ordQ < m$, then $P_m + Q$is hyperbolic according to Gårding.
Our aim in this article for homogeneous hyperbolic according to Gårding with respect to the vector $0 \neq \eta^0$polynomial $P_m$and weight function of hyperbolicity $g_s \in G(n)$is to determine terms for the polynomial $Q$ at which the polynomial $P_m + Q$will be $g_s$ – hyperbolic with respect to any vector $\eta$from certain neighborhood of $\eta^0$.

## II. PRELIMINARY RESULTS

Statement 2.1:Let $s > 0$,and for the function $h$there exists suchconstant $c > 0$that
$g_s(\xi) \leq ch(\xi)$.(2.1)
Then for any $\eta \in R^n$
$g_{s,\eta}(\xi) \leq ch_\eta(\xi) \ \ \forall \xi \in R^n$.(2.2)
Proof:Let for any $\xi \in R^n$the number $\theta(\xi)$is determined by the term $g_s(\xi - \theta(\xi)\eta) = g_\eta(\xi)$ $(= \min_{t \in R^n} g(\xi - t\eta))$ . Then, for any $t \in R$determined by Eq. (2.1)due to definition of the number $\theta(\xi)$ we have that

$$g_{s,\eta}(\xi) = g(\xi - \theta\eta) \leq g(\xi - t\eta) \leq ch(\xi - t\eta). \ \forall \xi \in R^n$$

which directly leads to Eq. (2.2)
Consequence 2.1:Let $s > 0$and $h \in G(n)$is such a function for which with certain constant $c > 0$

$$c^{-1}g_s(\xi) \leq h(\xi) \leq cg_s(\xi) \ \ \forall \xi \in R^n.$$

Then, for any $\eta \in R^n$
$c^{-1}g_{s,\eta}(\xi) \leq h_\eta(\xi) \leq cg_{s,\eta}(\xi) \ \ \forall \xi \in R^n$.
Proof:Follows directly from statement 2.1.
Statement 2.2:Let $2 \leq k \in N$and $\{\eta^j\}_{j=1}^k \subset R^n$,$|\eta^j| = 1$is the set of linearly independent vectors. Then, for any $\delta > 0$the following is valid:

$$\min_{|\xi|=1} \sum_{j=1}^{k} \left|\xi - (\xi, \eta^j)\eta^j\right|^\delta := H_\delta > 0.$$

Proof:Let usprove by contradiction, that is, let us assume that for certain$\delta > 0$there exists $\xi^0 \in R^n$, $|\xi| = 1$for which$\sum_{j=1}^k \left|\xi^0 - (\xi^0, \eta^j)\eta^j\right|^\delta = 0$.It means that $\xi^0$is collinear to all vectors $\eta^j, j = 1,2, \ldots, k$and, hence, the vectors $\{\eta^j\}_{j=1}^k$are collinear to each other. This contradicts to the term of statement and, thus, proves the validity of the statement.

Consequence 2.2:Under the terms of statement 2.2the following inequality is valid
$\sum_{j=1}^k \left|\xi - (\xi, \eta^j)\eta^j\right|^\delta \geq H_\delta |\xi|^\delta \ \forall \xi \in R^n$.(2.3)
Proof:Due to homogeneity of multiplicity $\delta$of the function $\left|\xi - (\xi, \eta^j)\eta^j\right|^\delta j = 1,2, \ldots, k$on the basis of statement 2.2at all $0 \neq \xi \in R^n$, we have the following inequality:

$$\sum_{j=1}^{k} \left|\xi - (\xi, \eta^j)\eta^j\right|^\delta = |\xi|^\delta \sum_{j=1}^{k} \left|\frac{\xi}{|\xi|} - \left(\frac{\xi}{|\xi|}, \eta^j\right)\eta^j\right|^\delta \geq H_\delta |\xi|^\delta.$$

Since Eq. (2.3) is obviously valid at $\xi = 0$, then Eq. (2.3)is proven.
Lemma 2.1:Let $s > 0.2 \leq k \in N$,a$\{\eta^j\}_{j=1}^k \subset R^n$are linearly independent vectors. Then, at certain constant $c = c(s) > 0$
$c^{-1}g_s(\xi) \leq \sum_{j=1}^k g_{s,\eta^j}(\xi) \leq cg_s(\xi)$. (2.4)
Proof:Since for any $\eta \in R^n, g_{s,\eta}(\xi) \leq g_s(\xi), \xi \in R^n$, then the right side of Eq. (2.4)is evident. Let us prove the left side of Eq. (2.4).Since for any $\xi, \eta \in R^n \xi - \frac{(\xi,\eta)}{(\eta,\eta)}\eta$is orthogonal to $\eta$, then at all $\xi, \eta \in R^n, \eta \neq 0$
$g_{s,\eta}(\xi) = 1 + \min_{t \in R} |\xi - t\eta|^{\frac{1}{s}} = 1 + \min_{t \in R} \Big|\xi - (\xi, \eta)\eta +$
$\xi,\eta-t\eta|1s = 1+mint\in R\xi-\xi,\eta(\eta,\eta)\eta+|(\xi,\eta(\eta,\eta)-t,\eta)|1s$
$= 1 + |\xi - (\xi, \eta)\eta|^{\frac{1}{s}}$(2.5)

From the lemma term, using Eq. (2.5), on the basis of statement 2.2we have that

$$\sum_{j=1}^{k} g_{s,\eta^j}(\xi) = k + \sum_{j=1}^{k} \left|\xi - (\xi, \eta^j)\eta^j\right|^{\frac{1}{s}} \geq H_{\frac{1}{s}} \cdot \left(1 + |\xi|^{\frac{1}{s}}\right) = H_{\frac{1}{s}} \cdot g_s(\xi) \ \forall \xi \in R^n$$

This is the left side of Eq. (2.5), therefore, lemma 2.1 itself is proven.
Statement 2.1:It follows directly from Eq. (2.5)that at $s > 1$ $g_{s,\eta} \in G(n)$for $\forall \eta \in R^n$.
Lemma 2.2:Let $g$is the function determined over $R^n$and satisfying the term 1),page. 1, the homogenous polynomial $P_m$of multiplicity $m$is hyperbolic with respect to the vector $\eta \in R^n$, and $P_k$is the homogeneous polynomial of multiplicity $k (k < m)$. If $P_k \prec^g P_m$, then for any $\epsilon > 0$there exists such $\theta_0 = \theta_0(\epsilon) > 0$that at $\theta \geq \theta_0$

$$|P_k(\xi \pm i\theta g(\xi/\theta)\eta| \leq \epsilon |P_m(\xi \pm i\theta g(\xi/\theta)\eta| \ \forall \xi \in R^n.$$

Proof:Due to homogeneity of the polynomial $P_k$, using the Taylor series with certain constant $c_1 = c_1(\eta, k) > 0$, we have that

$$\left|P_k\left(\xi \pm i\theta g\left(\frac{\xi}{\theta}\right)\eta\right| = \theta^k \left|P_k\left(\frac{\xi}{\theta} \pm ig\left(\frac{\xi}{\theta}\right)\eta\right| = \right.\right.$$
$$= \theta^k \left| \sum_\alpha \frac{1}{\alpha!} P_k^\alpha\left(\frac{\xi}{\theta}\right)\left(\pm ig\left(\frac{\xi}{\theta}\right)\eta\right)^\alpha \right|$$
$$\leq c_1\theta^k \widetilde{P_k}\left(\frac{\xi}{\theta}, g\left(\frac{\xi}{\theta}\right)\right), \forall \xi \in R^n, \theta > 0.$$

By virtue of the term $P_k \prec^g P_m$of lemma 2.2with certain constant $c_2 = c_2(\eta, P_k, P_m) > 0$we have that
$\left|P_k\left(\xi \pm i\theta g\left(\frac{\xi}{\theta}\right)\eta\right)\right| \leq c_2\theta^k \widetilde{P_m}\left(\frac{\xi}{\theta}, g\left(\frac{\xi}{\theta}\right)\right), \forall \xi \in R^n, \theta > 0$.(2.6)
Since the homogenous polynomial $P_m$is hyperbolic with respect to the vector $\eta$, then ([6] lemma 3.2) with certain constant $c_3 = c_3(P_m) > 0$

$$|P_m(\xi + i\tau)| \geq c_3\widetilde{P_m}(\xi, \tau), \forall (\xi, \tau) \in R^{n+1}$$

Using this from Eq. (2.6)with certain constant $c_4 = c_4(P_m, P_k, \tau) > 0$we obtain that

$$\left|P_k\left(\xi \pm i\theta g\left(\frac{\xi}{\theta}\right)\eta\right)\right| \leq c_4\theta^k \left|P_m\left(\frac{\xi}{\theta} \pm ig\left(\frac{\xi}{\theta}\right)\eta\right)\right|, \forall \xi \in R^n, \theta > 0.$$

By virtue of homogeneity of the polynomial $P_m$we have that:

$\left|P_k\left(\xi \pm i\theta g\left(\frac{\xi}{\theta}\right)\eta\right)\right| \le c_4\theta^{k-m}\left|P_m\left(\xi \pm i\theta g\left(\frac{\xi}{\theta}\right)\eta\right)\right|, \xi \in R^n, \theta >$ 0.(2.7)

Since $k-m<0$, then for any $\epsilon>0$there exists such $\theta_0=\theta_0(\epsilon)$that $c_4\theta_0^{k-m}=\epsilon$.Then, from Eq. (2.7) we obtain the statement of lemma at $\theta \ge \theta_0$.

The lemma is proven.

Lemma 2.3(see [6]):Let $g$and $P_m$are the same as in lemma 2.2, and $Q$is the polynomial of multiplicity $k(k<m)$, presented as:

$Q(\xi)=\sum_{j=0}^{k} Q_j(\xi)$,(2.8)

where $Q_j$is a homogeneous polynomial of multiplicity $j, j=0,1,\dots,k$.

The polynomial $P_m$is $g$stronger than the polynomial $Q$if and only if $Q_j \prec^{g} P_m, j=0,1,\dots,k$.

Theorem 2.1:Let $s>1$, the homogeneous polynomial $P_m$of multiplicity $m$is hyperbolic with respect to the vector $\eta \in R^n$, and $Q$is the polynomial of multiplicity$k(k<m)$ presented in the form of Eq. (2.3). If $Q \prec^{g_s} P_m$, then $P_m+Q$is s–hyperbolic with respect to the vector $\eta$.

Proof:By virtue of lemma 2.2 on the basis of lemma 2.3there exists such $\theta_0>0$that at all $\xi \in R^n, \theta>\theta_0$

$$\sum_{j=0}^{k}\left|Q_j\left(\xi \pm i\theta g_s\left(\frac{\xi}{\theta}\right)\eta\right)\right| \le \frac{1}{2}\left|P_m\left(\xi \pm i\theta g_s\left(\frac{\xi}{\theta}\right)\eta\right)\right|.$$

Therefore,

$\left|[P_m+Q]\left(\xi \pm i\theta g_s\left(\frac{\xi}{\theta}\right)\eta\right)\right| \ge \frac{1}{2}\left|P_m\left(\xi \pm i\theta g_s\left(\frac{\xi}{\theta}\right)\eta\right)\right|, \xi \in R^n, \theta \ge \theta_0.$

Since from the term of hyperbolicity of homogeneous polynomial $P_m$ with respect to the vector $\eta$it follows that $P_m(\xi+i\tau\eta) \neq 0, (\xi,\eta) \in R^n$at $\tau \neq 0$, we have that

$[P_m+Q]\left(\xi \pm i\theta g_s\left(\frac{\xi}{\theta}\right)\eta\right) \neq 0, (\xi,\theta) \in R^{n+1}, \theta \ge \theta_0$.(2.9)

Let $(\xi,\tau) \in R^{n+1}$and $|\tau| \ge \theta_0 g_s\left(\frac{\xi}{\theta_0}\right)$.Let us demonstratethat $(P_m+Q)(\xi+i\tau\eta) \neq 0$.

Since $t g_s\left(\frac{\xi}{t}\right) \to \infty$at $t \to \infty$, then at all $(\xi,\tau) \in R^{n+1}|\tau| \ge \theta_0 g_s\left(\frac{\xi}{\theta_0}\right)$there exists such $\theta \ge \theta_0$that $|\tau|=\theta g_s\left(\frac{\xi}{\theta}\right)$. Hence, by virtue of Eq. (2.9):

$(P_m+Q)(\xi+i\tau\eta)=(P_m+Q)\left(\xi+i sgn\tau \cdot \theta_0 g_s\left(\frac{\xi}{\theta}\right)\eta\right) \neq 0$at $(\xi,\tau) \in R^{n+1}, |\tau| \ge \theta_0 g_s\left(\frac{\xi}{\theta}\right)$.

Since with certain constant $c_1>0$

$$\theta_0 g_s\left(\frac{\xi}{\theta_0}\right) \le c_1 g_s(\xi)\ \forall \xi \in R^n,$$

then we have that:

$$(P_m+Q)(\xi+i\tau\eta) \neq 0\ (\xi,\tau) \in R^{n+1}, |\tau| \ge c_1 g_s(\xi),$$

that is, the polynomial $(P_m+Q)$is s–hyperbolic with respect to the vector $\eta$.

The lemma is proven.

## III. MAIN RESULT

Theorem 3.1:Let the homogeneous polynomial $P_m$of multiplicity $m$is hyperbolic with respect to the vector $\eta^0$, $\delta>0$is such a number that $P_m(\eta) \neq 0$ for all $\eta \in U_\delta(\eta^0)=\{\xi, |\xi-\eta^0|<\delta\}$. Let $Q$is a polynomial of multiplicity $r(r<m)$, and $\{\eta^j\}_{j=1}^{k} \subset U_\delta(\eta^0), k \ge 2$are linearly independent vectors. If $Q \prec^{g_{s,\eta^j}} P_m, j=1,2,\dots,k$, then the polynomial $(P_m+Q)$is s–hyperbolic with respect to any vector $\eta \in U_\delta(\eta^0)$.

Proof:It is easy to show that $Q \prec^{g_{s,\eta^j}} P_m, j=1,2,\dots,k$if and only if $Q \prec^{\sum_{j=1}^{k} g_{s,\eta^j}} P_m$, then by virtue of lemma 2.1 we have that under the terms of the theorem $Q \prec^{g_s} P_m$. On the other hand, (see [1]consequence 12.4.5) from the term of the theorem we have that $P_m$is hyperbolic with respect to any vector $\eta \in U_\delta(\eta^0)$. Then, by virtue of theorem 2.1 we obtain that the polynomial $(P_m+Q)$is s–hyperbolic with respect to any vector $\eta \in U_\delta(\eta^0)$.

The theorem is proven.

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